

Centralizers and Normalizers

David Corwin

March 11, 2019

1 Centralizers and Conjugacy Classes

Recall that for $g \in G$, the map

$$[g]: G \rightarrow G$$

defined by

$$[g](x) = gxg^{-1}$$

for $x \in G$ is called *conjugation by g* . For a fixed g , this gives an automorphism from G to itself (to see that it is bijection, notice that $[g^{-1}]$ gives the inverse automorphism). As we let g vary, this defines an action of G on itself.

Given $x \in G$, define the *centralizer* of x in G by

$$C_G(x) = \{g \in G \mid gxg^{-1} = x\}.$$

It is clear from the definition that this is the stabilizer of x under the conjugation action, so $C_G(x)$ must be a subgroup. It is also simply the set of $g \in G$ that commute with x .

So that tells us that $g \in C_G(x)$ if and only if g commutes with x . But that's the same thing as saying that x commutes with g , so it's equivalent to saying that $x \in C_g(x)$. In fact, by that reasoning, we have

$$C_G(x) = \{g \in G \mid x \in C_G(g)\}.$$

Then, since $Z(G)$ is defined to be the set of elements of G that commute with everything, it is clear that $Z(G)$ is the intersection of all centralizers, i.e.,

$$Z(G) = \cap_{x \in G} C_G(x).$$

By the Orbit-Stabilizer Theorem, we know that the size of the conjugacy class of x times the size of $C_G(x)$ is $|G|$ (at least assuming these are finite). (If this is confusing to you, it's really just restating the definitions and the Orbit-Stabilizer Theorem in this case.)

The previous fact is *very important* for computing the centralizer of an element. If you know $|C_G(x)|$, and you've found that many elements that commute with x , then you know you've found all of $C_G(x)$. See, for example, Example 1.3.

Remark 1.1. Since x commutes with all of powers, we always have

$$\langle x \rangle \subseteq C_G(x).$$

Assuming G is finite, note that $|\langle x \rangle|$ is just the order of x . It follows that $|C_G(x)| \geq \text{ord}(x)$, and thus the conjugacy class of x has size at most $\frac{|G|}{\text{ord}(x)}$.

Example 1.2. If G is abelian (for example, $\mathbb{Z}/m\mathbb{Z}$), then $C_G(x) = G$ for every element x , and every conjugacy class has one element.

Example 1.3. As an example, consider $x = (123) \in G = \text{Sym}_5$. Then the conjugacy class of x is the set of all 3-cycles. To count the number of three-cycles, notice that there are $\binom{5}{3} = 10$ ways to choose the three elements that are cycled, and there are two cycles for each triple (think about how (123) and (132) are different elements of Sym_5), so there are 20 three-cycles. Since $|\text{Sym}_5| = 5! = 120$, the size of $C_G(x)$ must be $120/20 = 6$.

What are these six elements? We can take the subgroup generated by $(123)(45)$. Notice that this element has order 6 (as it's the LCM of 2 and 3) and is in $C_G(x)$, so it must in fact generate all of $C_G(x)$.

Example 1.4. For $G = \text{Sym}_3$, the conjugacy classes are $\{\text{id}\}$, $\{(12), (23), (31)\}$, and $\{(123), (132)\}$. Since $|G| = 6$, the stabilizer of id is G , the centralizer of (12) has two elements, and the centralizer of (123) has three elements. In fact, in the latter two cases, the centralizer of the element is just the subgroup it generates (so the inclusion in Remark 1.1 is in fact an equality of subgroups in these cases).

Example 1.5. For $G = \text{Sym}_4$, the conjugacy classes are $\{\text{id}\}$, $\{(12), (23), (34), (41), (13), (24)\}$, $\{(123), (132), (124), (142), (134), (143), (234), (243)\}$, $\{(1234), (1243), (1324), (1342), (1423), (1432)\}$, and $\{(12)(34), (13)(24), (14)(23)\}$.

The centralizer of id is G , as always.

The centralizer of (12) must have four elements, as its conjugacy class has $24/4 = 6$ elements. Recall that disjoint cycles don't commute, so (34) is in

$C_G((12))$. As well, by Remark 1.1, we know that $(12) \in C_G((12))$. So we have $C_G((12)) = \{\text{id}, (12), (34), (12)(34)\}$.

Notice that the conjugacy class of (123) has eight elements, so its centralizer has $24/8 = 3$ elements. In fact, it has order 3, so its centralizer is just the subgroup it generates. Similarly, the conjugacy class of (1234) has six elements, so its centralizer has 4 elements, and it has order 4, so it must generate its centralizer.

Finally, note that the conjugacy class of $(12)(34)$ has three elements, so its centralizer must have eight elements. Recall that $(12)(34)$ commutes with the cycles (12) and (34) , so it commutes with the subgroup they generate, i.e., $\{\text{id}, (12), (34), (12)(34)\}$. Finally, more subtle is the fact that $(13)(24)$ also commutes with $(12)(34)$. Note carefully that (13) and (24) do NOT commute with it. We can then take the subgroup generated by $(13)(24)$ and $\{\text{id}, (12), (34), (12)(34)\}$, and this indeed has eight elements so it is the centralizer.

2 Normalizers

Let G be a group and $H \subseteq G$ a subgroup. Given $g \in G$, let

$$gHg^{-1} := \{ghg^{-1} \mid h \in H\}.$$

Since conjugation by g is a group automorphism of G , and the image of a subgroup under a homomorphism is also a subgroup, we find that gHg^{-1} is a subgroup of G . Letting $\text{Sub}(G)$ denote the set of all subgroups of G , this construction defines an action of G on $\text{Sub}(G)$.

The stabilizer of a subgroup H is known as the *normalizer* of H and denoted $N_G(H)$. In other words,

$$N_G(H) = \{g \in G \mid gHg^{-1} = H\}.$$

It follows obviously from the definition of normal that H is normal if and only if $N_G(H) = G$.

In fact, more generally, we have that H is a normal subgroup of $N_G(H)$ (but not necessarily of G).

One of your homework problems asks you to compute a normalizer in a simple case.