REMARKS ON THE HODGE FILTRATION IN NON-ABELIAN CHABAUTY

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This is a brief note clarifying some statements about the Hodge filtration in [Kim09] and [Bro17].

Following notation in [Bro17], we use \( y \Pi^\text{dR}_x \) to refer to the de Rham torsor of paths from \( x \) to \( y \) and \( \mathcal{O}(y \Pi^\text{dR}_x) \) to refer to its coordinate ring.

Let us fix a basepoint (called \( b \) in [Kim09] and \( 0 \) in [Bro17]). For completeness, we express the notation of [Kim09] in terms of the notation of [Bro17]:

\[
\begin{align*}
U^\text{DR} &= b \Pi^\text{dR}_b \\
A^\text{DR} &= \mathcal{O}(b \Pi^\text{dR}_b) \\
P^\text{DR}(x) &= x \Pi^\text{dR}_b \\
\mathcal{P}^\text{DR}(x) &= \mathcal{O}(x \Pi^\text{dR}_b).
\end{align*}
\]

1. A Couple of Corrections

1.1. Definition of the Hodge Filtration in Kim and Brown. [Kim09] §1, p.103] and [Bro17] 11.5.1] both rightly state that the coordinate ring \( \mathcal{O}(\Pi^\text{dR}_0) \) has a Hodge filtration, with the former giving [Woj93] as a reference.

Kim states earlier in his paper (p.92) that there is a Hodge filtration on \( \Pi^\text{dR}_b \) itself (and as mentioned below in 1.2 he incorrectly states that it is a filtration by subgroups). There is indeed such a filtration, but the definitions given by both Kim and Brown are incorrect.

Modulo a missing negative sign, Formula (11.11) of [Bro17] defines

\[
F^n_{x \Pi^\text{dR}_0} := \text{Spec}(\mathcal{O}(x \Pi^\text{dR}_0)/F^{1-n}),
\]

and a similar formula appears later on p.103 of [Kim09] and again in [Bea17] Definition 2.27.

We claim, however, that this formula is incorrect, for the following reason. In the case that \( \Pi^\text{dR}_0 \) is abelian, it is just a vector group \( V \), and therefore it carries a Hodge filtration, such that \( \mathcal{O}(\Pi^\text{dR}_0) \) is \( \text{Sym}(V^\vee) \) as a filtered vector space. The Hodge filtration on \( \Pi^\text{dR}_0 \) should then be the same as this Hodge filtration.

Here is a concrete example that shows that this does not work properly for the definition in Kim and Brown. Let \( V \) be a Hodge filtered vector space with \( F^1V = 0 \), and let \( A = \text{Sym}(V^\vee) \) the coordinate ring on \( V \) (viewed as an abelian unipotent group). Let \( V^\vee \) be generated by \( x_0 \) and \( x_1, x_2 \) where \( x_1 \in F^1(V^\vee) \), and \( x_2 \in F^2(V^\vee) \). Then \( F^2A \) is indeed an ideal, and it has

\[\text{Date: February 17, 2021.}\]
both $x_2$ and $x_1^2$. That means that the (set-theoretic) vanishing locus of this ideal is actually $F^0V$, not $F^{-1}V$ as one would like.

The correct way to define the Hodge filtration is to define the Hodge filtration on the Lie algebra $\text{Lie}_0\Pi^\text{dR}_0$ and then declare that the exponential map (which is algebraic) preserves the Hodge filtration. This is in fact the definition of the Hodge filtration given in [Del89, §7.5].

**Remark 1.1.** Assuming that the coordinate ring is supported in non-negative Hodge weights, the definition of Brown and Kim is correct for $F^0\times \Pi^\text{dR}_0$, which is the only case that they actually use.

1.2. A Claim of Kim. Kim also claims in [Kim09, p.92] that the $F^n\Pi^\text{dR}_0$ are subgroups. As pointed out in [Bro17, 11.5.1], this is false when $n < 0$. To see this, note that being a subgroup is equivalent to the claim that $F^n\text{Lie}_0\Pi^\text{dR}_0$ is a Lie subalgebra. For $n = 0$, assuming the Lie algebra is non-positively graded, this is true. But for $n < 0$, the Lie bracket of two elements of $F^n\setminus F^0$ will not in general lie in $F^m$.

**Remark 1.2.** When $\Pi^\text{dR}_0$ is abelian, this is no longer a problem, as the Lie bracket of any two elements is always zero.

**Remark 1.3.** As $F^n\text{Lie}_0\Pi^\text{dR}_0$ is always a vector subspace, $F^n\Pi^\text{dR}_0$ is an affine subspace of $\Pi^\text{dR}_0$, just not necessarily a subgroup.

2. **Comparison of Kim and Brown**

In [Bro17, Definition 11.4], Brown defines a quotient $\times H_0$ of $\times \Pi_0^\text{dR}$, with coordinate ring $\times H_0$, a subalgebra of $\mathcal{O}(\times \Pi_0^\text{dR})$ defined as follows:

**Definition 2.1.** Let $\times H_0$ denote the largest subalgebra of $\mathcal{O}(\times \Pi_0^\text{dR})$ such that

(i) $W_{0x}H_0 = W_0\mathcal{O}(\times \Pi_0^\text{dR}) = k$
(ii) $\times H_0$ is stable under the action $\Delta: \times H_0 \to \mathcal{O}(\Pi_0^\text{dR}) \otimes \times H_0$
(iii) $\times H_0 \subseteq F^1\mathcal{O}(\times \Pi_0^\text{dR}) + W_{0x}H_0$

The usefulness of $\times H_0$ is that it has a canonical de Rham path, providing an isomorphism $$\times 1_0: \times H_0 \to \Pi_0^\text{dR}.$$

In this section, we show that $\Pi^\text{dR}_0$ is in fact the same as the right coset space $F^0\setminus \Pi^\text{dR}_0$. To simplify notation, we set

$$\Pi := \Pi^\text{dR}_0$$

and $$\mathcal{O} := \mathcal{O}(\Pi).$$

First, we clarify the definition of $F^0\setminus \Pi$. The coordinate ring $\mathcal{O}(F^0\setminus \Pi)$ is the set of functions on $\Pi$ that are invariant under the left action of $F^0\Pi$.

Noting that $F^0\Pi = \text{Spec}(\mathcal{O}/F^1\mathcal{O})$, the set of functions invariant under $F^0\Pi$ is the following equalizer:
\[ \mathcal{O}(F^0 \backslash \Pi) = \text{Eq} \left( \mathcal{O} \xrightarrow{\Delta} \mathcal{O} \otimes (\mathcal{O}/F^1) \right), \]

where \( \text{pr}_1 \) sends \( x \in \mathcal{O} \) to \( x \otimes 1 \in \mathcal{O} \otimes (\mathcal{O}/F^1) \), and \( \Delta \) is the coproduct modulo \( \mathcal{O} \otimes F^1 \mathcal{O} \).

**Remark 2.2.** One may check for the example of a cotensor algebra that [Bro17, Definition 11.10] is therefore not quite correct.

Similarly, the definition in [Bes12, Remark 4.3] for the quotient by \( F^0 \) on the right side, while closer, has a slight mistake. It should be replaced by the condition that

\[ \Delta(f) - 1 \otimes f \in F^1 K[G^\text{dR}_x] \otimes K[G^\text{dR}_x], \]

in the notation of loc.cit.

**Proposition 2.3.** \( \mathcal{O}(F^0 \backslash \Pi) \) as defined by the equalizer above coincides with the Hopf algebra \( _0H_0 \) of [Bro17, Definition 11.4].

**Proof.** We first show that \( _0H_0 \subseteq \mathcal{O}(F^0 \backslash \Pi) \).

Let \( I \) denote the augmentation ideal of \( \mathcal{O} \), i.e., the kernel of the counit map. Suppose \( x \in _0H_0 \cap I \). By Definition 2.1(iii), we have \( x = a + b \) where \( a \in k = W_0(\mathcal{O}) \), and \( b \in F^1 \mathcal{O} \). Then \( \epsilon(x) = a \), so \( a = 0 \), hence \( x = b \in F^1 \mathcal{O} \).

By the axioms for a Hopf algebra applied to \( \mathcal{O} \), we have that \( (\text{id} \otimes \epsilon)\Delta x = x \). As well, we have \( (\text{id} \otimes \epsilon)(x \otimes 1) = x \). Therefore,

\[(\text{id} \otimes \epsilon)(\Delta x - x \otimes 1) = 0.\]

Now by (ii), we know \( \Delta x \in \mathcal{O} \otimes _0H_0 \). We can therefore write \( \Delta x - 1 \otimes x \) as

\[ \sum_i c_i \otimes (a_i + b_i) \]

where \( c_i \in \mathcal{O} \), \( a_i \in k \), and \( b_i \in F^1 \mathcal{O} \) for each \( i \). Applying \( \text{id} \otimes \epsilon \) to this sum, we get

\[ \sum_i a_i c_i = 0. \]

But by the \( k \)-linearity of the tensor product, we have

\[ \Delta x - 1 \otimes x = \sum_i c_i \otimes (a_i + b_i) \]

\[ = \sum_i a_i c_i \otimes 1 + \sum_i c_i \otimes b_i \]

\[ = \left( \sum_i a_i c_i \right) \otimes 1 + \sum_i c_i \otimes b_i \]

\[ = 0 \otimes 1 + \sum_i c_i \otimes b_i \]

\[ = \sum_i c_i \otimes b_i. \]
But this implies that
\[ \Delta x - 1 \otimes x = \sum_i c_i \otimes b_i \]
is zero in \( \mathcal{O} \otimes (\mathcal{O}/F^1) \), which implies that \( x \in \mathcal{O}(F^0\backslash \Pi) \). Finally, note that a general element of \( _0H_0 \) is in \( k + (I \cap _0H_0) \), so it must also be in \( \mathcal{O}(F^0\backslash \Pi) \).

We now show that \( \mathcal{O}(F^0\backslash \Pi) \subseteq _0H_0 \).

Note that \( \mathcal{O}(F^0\backslash \Pi) \) is a subalgebra because it is the equalizer of two algebra homomorphisms. To show that \( \mathcal{O}(F^0\backslash \Pi) \) is in \( _0H_0 \), we simply need to show that it satisfies properties (i)-(iii) of Definition 2.1.

For (i), note that it is a subalgebra, so it contains 1, which is all that is required.

For (ii), note that the left action by \( F^0\Pi \) commutes with the right action by \( \Pi \), so the quotient \( F^0\Pi \) still has an induced right action by \( \Pi \). This implies that \( \mathcal{O}(F^0\backslash \Pi) \) is stable under the coaction by \( \mathcal{O} \).

For (iii), let \( x \in \mathcal{O}(F^0\backslash \Pi) \). We wish to show that \( y := x - \eta(\epsilon(x)) \) is in \( F^1\mathcal{O} \). For this, note that \( y \) vanishes at the identity and is in \( \mathcal{O}(F^0\backslash \Pi) \). Therefore, it is invariant under \( F^0\Pi \), which means it vanishes on all of \( F^0\Pi \), hence is in \( F^1\mathcal{O} \).

\[ \square \]

Remark 2.4. It seems that this statement might generalize to
\[ _xH_0 = \mathcal{O}(F^0_x\Pi_x^{\text{dR}}\backslash \Pi^{\text{dR}}_0). \]

3. Motivic Periods for the Quotient by \( F^0 \)

Let \( G^{\text{dR}} \) be the Tannakian Galois group with respect to the de Rham realization of a category of mixed motives containing all (the Lie algebras of finite quotients of) the path torsors \( \gamma^x_{\Pi^{\text{dR}}_x} \). For example, if \( X \) is a rational curve, then we may take the category of mixed Artin-Tate motives, and if \( X \) is a curve of genus \( g \geq 1 \), we may take the category of mixed Abelian motives generated by \( h^1(X) \).

Then the composition morphisms \( \gamma^x_{\Pi^{\text{dR}}_x} \times \gamma^x_{\Pi^{\text{dR}}_x} \to \gamma^x_{\Pi^{\text{dR}}_x} \) are \( G^{\text{dR}} \)-equivariant. However, the action of \( G^{\text{dR}} \) does not in general respect the Hodge filtration. Therefore, one cannot expect Brown’s \( _xH_0 \) have an action of \( G^{\text{dR}} \).

In particular, if \( \omega \in _xH_0 \), then one cannot simply use the canonical de Rham path \( _xc_0 : _xH_0 \to \mathbb{Q} \) to define a Tannakian matrix coefficient \( [_xH_0, _xc_0, \omega] \), because \( _xH_0 \) is not an object of the Tannakian category.

Another possible approach is to choose an arbitrary \( _xc_0 \in F^0_x\Pi^{\text{dR}}_0(\mathbb{Q}) \) and show that if \( \omega \in _xH_0 \), then \( [\mathcal{O}(x\Pi^{\text{dR}}_0), _xc_0, \omega] \) is independent of the choice of \( _xc_0 \). This also fails, because \( G^{\text{dR}} \) does not preserve the Hodge filtration on \( _x\Pi^{\text{dR}}_x \). In particular, if \( _xc_0 \) and \( _xc'_0 \) differ by an element of \( F^0_x\Pi^{\text{dR}}_x \), then there might be \( g \in G^{\text{dR}} \) for which \( g(_xc_0) \) and \( g(_xc'_0) \) are not the same in \( F^0_x\Pi^{\text{dR}}_x \backslash \Pi^{\text{dR}}_0 \).
Nonetheless, in [Bro17 §11.8], there is a slightly more canonical choice of $c_0 \in F^0 \Pi^0 (\mathbb{Q})$, defined by a splitting of the character $\chi: G^{\text{dR}} \to \mathbb{G}_m$. It would be interesting to check how much this actually depends on the splitting.

There are three solutions to this problem. Let $x_0 \in \Pi^0 (\mathbb{Q}_p)$ denote the unique Frobenius invariant path. Let $\omega \in \mathfrak{h}_0$ and $x \in \mathcal{X} (\mathbb{Q})$. We let $x^{-1}_0 \omega$ denote the corresponding element of $\mathfrak{h}_0$ under [Bro17 Lemma 11.8]. Then our three options for obtaining a period from $\omega$ and $x$ are

1. The ordinary approach giving the Tannakian period $[O(x_0 \Pi^0), x_0, x^{-1}_0 \omega]$. This might work even in the elliptic case by the approach of [Bro17 §11.8].
2. Brown’s approach in [Bro17 (11.15)] of doing $[O(x_0 \Pi^0), x_0, x^{-1}_0 \omega]$
3. Ishai’s idea of doing $[O(x_0 \Pi^0), x_0, x^{-1}_0 \omega]$

3.1. The Case of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. We let $y \Pi^0_x$ denote the polylogarithmic quotient for all $x, y \in \mathbb{P}^1 \setminus \{0, 1, \infty\}(\mathbb{Q}_p)$. We transport the polylogarithmic quotient to all path torsors using the canonical de Rham path.

For $\omega \in O(y \Pi^0_0)$ and $x \in X(\mathbb{Q})$, we have $\omega_x = x^{-1}_0 \omega \in O(x_0 \Pi^0_0)$ defined by $\omega(u) = \omega_x(x_0 u)$ for $u \in \Pi^0_0 (\mathbb{Q})$.

Note that $\text{Li}^p_\omega(x) = \omega_x(x_0)$. Therefore, if we let $u_x \in \Pi^0_0 (\mathbb{Q}_p)$ such that $x_0 u_x = x_0$, then for $\omega \in O(\Pi)$, we have $\text{Li}^p_\omega(x) = \omega(u_x)$.

Remark 3.1. Notice, in particular, that this is the inverse of $u_T$ defined in [Kim09 Proposition 1]. We believe this is related to the fact that Kim uses a quotient by $F^0$ on the right, while Brown uses a quotient by $F^0$ in the left.

We want to compute Brown’s and Ishai’s periods in this case. So for $g \in G^{\text{dR}}$, we wish to understand $\omega(x^{-1}_0 g(x_0))$ and $\omega(x_0^{-1} g(x_0))$. For this, we need to understand $g(x_0)$ for $g \in G^{\text{dR}} (\mathbb{Q})$. We recall that we have $\chi: G^{\text{dR}} \to \mathbb{G}_m$, and we note that in this case, the kernel of $\chi$ is pro-unipotent.

We let $y \Pi^0_x$ denote the polylogarithmic quotient for all $x, y \in \mathbb{P}^1 \setminus \{0, 1, \infty\}(\mathbb{Q}_p)$. We transport the polylogarithmic quotient to all path torsors using the canonical de Rham path.
To do this, we need to apply $g$ to the equation
\[ x^{\gamma_0} = x_{10}u_x \]
to get
\[ g(x^{\gamma_0}) = g(x_{10})g(u_x) \]

In fact, we only care about $g \in \pi_1^{un}(Z) = \ker \chi$. For such $g$, we have $g(u_x) = u_x$, because we are using the polylogarithmic quotient. We therefore get
\[ g(x^{\gamma_0}) = g(x_{10})u_x \]

To make these elements of $\pi_0^{dR}(\mathbb{Q}_p)$ rather than $\pi_0^{dR}(\mathbb{Q}_p)$, we write
\[ x_{10}^{-1}g(x^{\gamma_0}) = x_{10}^{-1}g(x_{10})u_x \]

To understand $u_x$, note that
\[ \text{Li}^u_n(u_x) = \text{Li}^p_n(x). \]

As well, we have
\[ \text{Li}^u_n(x_{10}^{-1}g(x_{10})) = \text{Li}^u_n(x)(g), \]
where $\text{Li}^u_n(x) \in A(Z)$.

We may therefore compute that
\[
\text{Li}^u_n(x_{10}^{-1}g(x^{\gamma_0})) = \text{Li}^u_n(x_{10}^{-1}g(x_{10})u_x)
\]
\[
= \text{Li}^u_n(u_x) + \sum_{i=0}^{n-1} \text{Li}^u_{n-i}(x_{10}^{-1}g(x_{10})) \frac{\log^u(u_x)^i}{i!}
\]
\[
= \text{Li}^p_n(x) + \sum_{i=0}^{n-1} \text{Li}^u_{n-i}(x)(g) \frac{\log^p(x)^i}{i!}.
\]

In other words, the version of $\text{Li}^u_n(x)$ coming from \cite[11.15]{Bro17} is
\[
\text{Li}^p_n(x) + \sum_{i=0}^{n-1} \frac{\log^p(x)^i}{i!} \text{Li}^u_{n-i}(x) \in A(Z) \otimes \mathbb{Q}_p
\]

For Ishai’s version, note
\[
x^{\gamma_0}^{-1}g(x^{\gamma_0}) = x_{10}^{-1}g(x_{10})u_x
\]
\[
= (x_{10}u_x)^{-1}g(x_{10})u_x
\]
\[
= u_x^{-1}x_{10}^{-1}g(x_{10})u_x.
\]

In other words, it’s conjugate of the ordinary thing by $u_x$ (whose coordinates are $p$-adic polylogarithms evaluated at $x$).
3.2. A Slight Error. [Bro17] §11.7] states that for an open affine $U \subset X$ containing 0, the fact that $W_n\mathcal{O}(\Pi_0)$ is trivial as a vector bundle over $U$ implies that there is a canonical isomorphism

$$x c_0 : W_n\mathcal{O}(x \Pi_0^\omega) \cong \Gamma(U, W_n\mathcal{O}(\Pi_0)) \cong W_n\mathcal{O}(0\Pi_0^\omega).$$

There are two problems with this:

1. The isomorphism between the left and right sides is not canonical; it depends on a trivialization of the vector bundle (or at least an appropriate subspace of $\Gamma(U, W_n\mathcal{O}(\Pi_0)))$

2. Even if one chooses a trivialization, the middle is not isomorphic to either side.

In fact, the middle is isomorphic to $W_n\mathcal{O}(0\Pi_0^\omega) \otimes \mathcal{O}(U)$. But $\mathcal{O}(U)$ is an infinite-dimensional vector space (in particular, it is not one-dimensional). As one meme puts it: [https://www.facebook.com/geometryofmemes/photos/a.2132782976771087/2350973868285329/?type=3&theater](https://www.facebook.com/geometryofmemes/photos/a.2132782976771087/2350973868285329/?type=3&theater)

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