REMARKS ON THE HODGE FILTRATION IN NON-ABELIAN CHABAUTY

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This is a brief note clarifying some statements about the Hodge filtration in [Kim09] and [Bro17].

Following notation in [Bro17], we use ${}_y\Pi_x^{\mathrm{dR}}$ to refer to the de Rham torsor of paths from x to y and $\mathcal{O}({}_y\Pi_x^{\mathrm{dR}})$ to refer to its coordinate ring.

Let us a fix a basepoint (called b in [Kim09] and 0 in [Bro17]). For completeness, we express the notation of [Kim09] in terms of the notation of [Bro17]:

$$U^{\mathrm{DR}} = {}_b\Pi_b^{\mathrm{dR}}$$
$$A^{\mathrm{DR}} = \mathcal{O}({}_b\Pi_b^{\mathrm{dR}})$$
$$P^{\mathrm{DR}}(x) = {}_x\Pi_b^{\mathrm{dR}}$$
$$\mathcal{P}^{\mathrm{DR}}(x) = \mathcal{O}({}_x\Pi_b^{\mathrm{dR}}).$$

1. A Couple of Corrections

1.1. **Definition of the Hodge Filtration in Kim and Brown.** [Kim09, §1, p.103] and [Bro17, 11.5.1] both rightly state that the coordinate ring $\mathcal{O}({}_x\Pi_0^{\mathrm{dR}})$ has a Hodge filtration, with the former giving [Woj93] as a reference.

Kim states earlier in his paper (p.92) that there is a Hodge filtration on ${}_b\Pi_b^{\mathrm{dR}}$ itself (and as mentioned below in 1.2, he incorrectly states that it is a filtration by subgroups). There is indeed such a filtration, but the definitions given by both Kim and Brown are incorrect.

Modulo a missing negative sign, Formula (11.11) of [Bro17] defines

$$F^n{}_x\Pi_0^{\mathrm{dR}} := \mathrm{Spec}(\mathcal{O}({}_x\Pi_0^{\mathrm{dR}})/F^{1-n}),$$

and a similar formula appears later on p.103 of [Kim09] and again in [Bea17, Definition 2.27].

We claim, however, that this formula is incorrect, for the following reason. In the case that ${}_0\Pi_0^{\mathrm{dR}}$ is abelian, it is just a vector group V, and therefore it carries a Hodge filtration, such that $\mathcal{O}({}_0\Pi_0^{\mathrm{dR}})$ is $\mathrm{Sym}(V^\vee)$ as a filtered vector space. The Hodge filtration on ${}_0\Pi_0^{\mathrm{dR}}$ should then be the same as this Hodge filtration

Here is a concrete example that shows that this does not work properly for the definition in Kim and Brown. Let V be a Hodge filtered vector space with $F^1V = 0$, and let $A = \mathrm{Sym}(V^\vee)$ the coordinate ring on V (viewed as an abelian unipotent group). Let $V^\vee$ be generated by x_0 and x_1, x_2 where $x_1 \in F^1(V^\vee)$, and $x_2 \in F^2(V^\vee)$. Then F^2A is indeed an ideal, and it has

Date: February 17, 2021.

both x_2 and x_1^2. That means that the (set-theoretic) vanishing locus of this ideal is actually F^0V, not $F^{-1}V$ as one would like.

The correct way to define the Hodge filtration is to define the Hodge filtration on the Lie algebra $\mathrm{Lie}_0\Pi_0^{\mathrm{dR}}$ and then declare that the exponential map (which is algebraic) preserves the Hodge filtration. This is in fact the definition of the Hodge filtration given in [Del89, §7.5].

Remark 1.1. Assuming that the coordinate ring is supported in non-negative Hodge weights, the definition of Brown and Kim is correct for $F^0{}_x\Pi_0^{\mathrm{dR}}$, which is the only case that they actually use.

1.2. **A Claim of Kim.** Kim also claims in [Kim09, p.92] that the $F^n{}_0\Pi_0^{\mathrm{dR}}$ are subgroups. As pointed out in [Bro17, 11.5.1], this is false when $n < 0$. To see this, note that being a subgroup is equivalent to the claim that $F^n\mathrm{Lie}_0\Pi_0^{\mathrm{dR}}$ is a Lie subalgebra. For $n = 0$, assuming the Lie algebra is non-positively graded, this is true. But for $n < 0$, the Lie bracket of two elements of $F^n \backslash F^0$ will not in general lie in F^n.

Remark 1.2. When ${}_0\Pi_0^{\mathrm{dR}}$ is abelian, this is no longer a problem, as the Lie bracket of any two elements is always zero.

Remark 1.3. As $F^n\mathrm{Lie}_0\Pi_0^{\mathrm{dR}}$ is always a vector subspace, $F^n{}_0\Pi_0^{\mathrm{dR}}$ is an affine subspace of ${}_0\Pi_0^{\mathrm{dR}}$, just not necessarily a subgroup.

2. Comparison of Kim and Brown

In [Bro17, Definition 11.4], Brown defines a quotient ${}_x\underline{\Pi}_0^{\mathrm{dR}}$ of ${}_x\Pi_0^{\mathrm{dR}}$, with coordinate ring ${}_xH_0$, a subalgebra of $\mathcal{O}({}_x\Pi_0^{\mathrm{dR}})$ defined as follows:

Definition 2.1. Let ${}_xH_0$ denote the largest subalgebra of $\mathcal{O}({}_x\Pi_0^{\mathrm{dR}})$ such that

(i) $W_0{}_xH_0 = W_0\mathcal{O}({}_x\Pi_0^{\mathrm{dR}}) = k$
(ii) ${}_xH_0$ is stable under the coaction $\Delta\colon {}_xH_0 \to \mathcal{O}({}_0\Pi_0^{\mathrm{dR}}) \otimes {}_xH_0$
(iii) ${}_xH_0 \subseteq F^1\mathcal{O}({}_x\Pi_0^{\mathrm{dR}}) + W_0{}_xH_0$

The usefulness of ${}_xH_0$ is that it has a canonical de Rham path, providing an isomorphism

$$ {}_x1_0\colon {}_xH_0 \to {}_0H_0. $$

In this section, we show that ${}_0\underline{\Pi}_0^{\mathrm{dR}}$ is in fact the same as the right coset space $F^0\backslash{}_0\Pi_0^{\mathrm{dR}}$. To simplify notation, we set

$$ \Pi \coloneqq {}_0\Pi_0^{\mathrm{dR}} $$

and

$$ \mathcal{O} \coloneqq \mathcal{O}(\Pi). $$

First, we clarify the definition of $F^0\backslash\Pi$. The coordinate ring $\mathcal{O}(F^0\backslash\Pi)$ is the set of functions on Π that are invariant under the left action of $F^0\Pi$.

Noting that $F^0\Pi = \mathrm{Spec}(\mathcal{O}/F^1\mathcal{O})$, the set of functions invariant under $F^0\Pi$ is the following equalizer:

$$\mathcal{O}(F^0\backslash\Pi) = \mathrm{Eq}\left(\mathcal{O} \underset{\mathrm{pr}_1}{\overset{\Delta}{\rightrightarrows}} \mathcal{O}\otimes(\mathcal{O}/F^1)\right),$$

where pr_1 sends $x \in \mathcal{O}$ to $x \otimes 1 \in \mathcal{O}\otimes(\mathcal{O}/F^1)$, and Δ is the coproduct modulo $\mathcal{O}\otimes F^1\mathcal{O}$.

Remark 2.2. One may check for the example of a cotensor algebra that [Bro17, Definition 11.10] is therefore not quite correct.

Similarly, the definition in [Bes12, Remark 4.3] for the quotient by F^0 on the right side, while closer, has a slight mistake. It should be replaced by the condition that

$$\Delta(f) - 1\otimes f \in \mathrm{F}^1\mathrm{K}[\mathrm{G}_x^{\mathrm{dR}}]\otimes\mathrm{K}[\mathrm{G}_x^{\mathrm{dR}}],$$

in the notation of loc.cit.

Proposition 2.3. $\mathcal{O}(F^0\backslash\Pi)$ *as defined by the equalizer above coincides with the Hopf algebra* ${}_0H_0$ *of* [Bro17, Definition 11.4].

Proof. We first show that ${}_0H_0 \subseteq \mathcal{O}(F^0\backslash\Pi)$.

Let I denote the augmentation ideal of $\mathcal{O}$, i.e., the kernel of the counit map. Suppose $x \in {}_0H_0 \cap I$. By Definition 2.1(iii), we have $x = a + b$ where $a \in k = W_0(\mathcal{O})$, and $b \in F^1\mathcal{O}$. Then $\epsilon(x) = a$, so $a = 0$, hence $x = b \in F^1\mathcal{O}$.

By the axioms for a Hopf algebra applied to $\mathcal{O}$, we have that $(\mathrm{id}\otimes\epsilon)\Delta x = x$. As well, we have $(\mathrm{id}\otimes\epsilon)(x\otimes 1) = x$. Therefore,

$$(\mathrm{id}\otimes\epsilon)(\Delta x - x\otimes 1) = 0.$$

Now by (ii), we know $\Delta x \in \mathcal{O}\otimes {}_0H_0$. We can therefore write $\Delta x - 1\otimes x$ as

$$\sum_i c_i \otimes (a_i + b_i)$$

where $c_i \in \mathcal{O}$, $a_i \in k$, and $b_i \in F^1\mathcal{O}$ for each i. Applying $id\otimes\epsilon$ to this sum, we get

$$\sum_i a_i c_i = 0.$$

But by the k-linearity of the tensor product, we have

$$\begin{aligned}
\Delta x - 1\otimes x &= \sum_i c_i\otimes(a_i+b_i)\\
&= \sum_i a_i c_i \otimes 1 + \sum_i c_i\otimes b_i\\
&= \left(\sum_i a_i c_i\right)\otimes 1 + \sum_i c_i\otimes b_i\\
&= 0\otimes 1 + \sum_i c_i\otimes b_i\\
&= \sum_i c_i\otimes b_i.
\end{aligned}$$

But this implies that

$$\Delta x - 1 \otimes x = \sum_i c_i \otimes b_i$$

is zero in $\mathcal{O} \otimes (\mathcal{O}/F^1)$, which implies that $x \in \mathcal{O}(F^0\backslash\Pi)$. Finally, note that a general element of ${}_0H_0$ is in $k + (I \cap {}_0H_0)$, so it must also be in $\mathcal{O}(F^0\backslash\Pi)$.

We now show that $\mathcal{O}(F^0\backslash\Pi) \subseteq {}_0H_0$.

Note that $\mathcal{O}(F^0\backslash\Pi)$ is a subalgebra because it is the equalizer of two algebra homomorphisms. To show that $\mathcal{O}(F^0\backslash\Pi)$ is in ${}_0H_0$, we simply need to show that it satisfies properties (i)-(iii) of Definition 2.1.

For (i), note that it is a subalgebra, so it contains 1, which is all that is required.

For (ii), note that the left action by $F^0\Pi$ commutes with the right action by Π, so the quotient $F^0\backslash\Pi$ still has an induced right action by Π. This implies that $\mathcal{O}(F^0\backslash\Pi)$ is stable under the coaction by $\mathcal{O}$.

For (iii), let $x \in \mathcal{O}(F^0\backslash\Pi)$. We wish to show that $y \coloneqq x - \eta(\epsilon(x))$ is in $F^1\mathcal{O}$. For this, note that y vanishes at the identity and is in $\mathcal{O}(F^0\backslash\Pi)$. Therefore, it is invariant under $F^0\Pi$, which means it vanishes on all of $F^0\Pi$, hence is in $F^1\mathcal{O}$.

□

Remark 2.4. It seems that this statement might generalize to

$${}_xH_0 = \mathcal{O}(F^0{}_x\Pi_x^{\mathrm{dR}}\backslash{}_x\Pi_0^{\mathrm{dR}}).$$

3. Motivic Periods for the Quotient by F^0

Let G^{dR} be the Tannakian Galois group with respect to the de Rham realization of a category of mixed motives containing all (the Lie algebras of finite quotients of) the path torsors ${}_y\Pi_x^{\mathrm{dR}}$. For example, if X is a rational curve, then we may take the category of mixed Artin-Tate motives, and if X is a curve of genus $g \geq 1$, we may take the category of mixed Abelian motives generated by $h^1(X)$.

Then the composition morphisms ${}_z\Pi_y^{\mathrm{dR}} \times {}_y\Pi_x^{\mathrm{dR}} \to {}_z\Pi_x^{\mathrm{dR}}$ are G^{dR}-equivariant. However, the action of G^{dR} does not in general respect the Hodge filtration. Therefore, one cannot expect Brown's ${}_xH_0$ have an action of G^{dR}.

In particular, if $\omega \in {}_xH_0$, then one cannot simply use the canonical de Rham path ${}_xc_0 \colon {}_xH_0 \to \mathbb{Q}$ to define a Tannakian matrix coefficient $[{}_xH_0, {}_xc_0, \omega]$, because ${}_xH_0$ is not an object of the Tannakian category.

Another possible approach is to choose an arbitrary ${}_xc_0 \in F^0{}_x\Pi_0^{\mathrm{dR}}(\mathbb{Q})$ and show that if $\omega \in {}_xH_0$, then $[\mathcal{O}({}_x\Pi_0^{\mathrm{dR}}), {}_xc_0, \omega]$ is independent of the choice of ${}_xc_0$. This also fails, because G^{dR} does not preserve the Hodge filtration on ${}_x\Pi_x^{\mathrm{dR}}$. In particular, if ${}_xc_0$ and ${}_xc_0'$ differ by an element of $F^0{}_x\Pi_x^{\mathrm{dR}}$, then there might be $g \in G^{\mathrm{dR}}$ for which $g({}_xc_0)$ and $g({}_xc_0')$ are not the same in $F^0{}_x\Pi_x^{\mathrm{dR}}\backslash{}_x\Pi_0^{\mathrm{dR}}$.

Nonetheless, in [Bro17, §11.8], there is a slightly more canonical choice of ${}_xc_0 \in F^0\,{}_x\Pi_0^{\mathrm{dR}}(\mathbb{Q})$, defined by a splitting of the character $\chi\colon G^{\mathrm{dR}} \to \mathbb{G}_m$. It would be interesting to check how much this actually depends on the splitting.

There are three solutions to this problem. Let ${}_x\gamma_0 \in {}_x\Pi_0^{\mathrm{dR}}(\mathbb{Q}_p)$ denote the unique Frobenius invariant path. Let $\omega \in {}_0H_0$ and $x \in X(\mathbb{Q})$. We let ${}_x1_0^{-1}\omega$ denote the corresponding element of ${}_xH_0$ under [Bro17, Lemma 11.8]. Then our three options for obtaining a period from ω and x are

(1) The ordinary approach giving the Tannakian period $[\mathcal{O}({}_x\Pi_0^{\mathrm{dR}}), {}_xc_0, {}_x1_0^{-1}\omega]$. This might work even in the elliptic case by the approach of [Bro17, §11.8].
(2) Brown's approach in [Bro17, (11.15)] of doing
$$[\mathcal{O}({}_x\Pi_0^{\mathrm{dR}}), {}_x\gamma_0, {}_x1_0^{-1}\omega]$$
(3) Ishai's idea of doing
$$[\mathcal{O}({}_x\Pi_0^{\mathrm{dR}}), {}_x\gamma_0, {}_x\gamma_0^{-1}\omega]$$

3.1. **The Case of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.** We let ${}_y\Pi_x^{\mathrm{dR}}$ denote the *polylogarithmic quotient* for all $x, y \in \mathbb{P}^1 \setminus \{0, 1, \infty\}(\mathbb{Q}_p)$. We transport the polylogarithmic quotient to all path torsors using the canonical de Rham path.

For $\omega \in \mathcal{O}({}_0\Pi_0^{\mathrm{dR}})$ and $x \in X(\mathbb{Q})$, we have $\omega_x = {}_x1_0^{-1}\omega \in \mathcal{O}({}_x\Pi_0^{\mathrm{dR}})$ defined by
$$\omega(u) = \omega_x({}_x1_0u)$$
for $u \in {}_0\Pi_0^{\mathrm{dR}}(\mathbb{Q})$.

Note that
$$\mathrm{Li}_\omega^p(x) = \omega_x({}_x\gamma_0).$$
Therefore, if we let $u_x \in {}_0\Pi_0^{\mathrm{dR}}(\mathbb{Q}_p)$ such that
$${}_x1_0u_x = {}_x\gamma_0,$$
then for $\omega \in \mathcal{O}(\Pi)$, we have
$$\mathrm{Li}_\omega^p(x) = \omega(u_x).$$
Remark 3.1. Notice, in particular, that this is the inverse of u_T defined in [Kim09, Proposition 1]. We believe this is related to the fact that Kim uses a quotient by F^0 on the right, while Brown uses a quotient by F^0 in the left.

We want to compute Brown's and Ishai's periods in this case. So for $g \in G^{\mathrm{dR}}$, we wish to understand
$$\omega({}_x1_0^{-1}g({}_x\gamma_0))$$
and
$$\omega({}_x\gamma_0^{-1}g({}_x\gamma_0)).$$
For this, we need to understand $g({}_x\gamma_0)$ for $g \in G^{\mathrm{dR}}(\mathbb{Q})$. We recall that we have $\chi\colon G^{\mathrm{dR}} \to \mathbb{G}_m$, and we note that in this case, the kernel of χ is pro-unipotent.

We let ${}_y\Pi_x^{\mathrm{dR}}$ denote the *polylogarithmic quotient* for all $x, y \in \mathbb{P}^1 \setminus \{0, 1, \infty\}(\mathbb{Q}_p)$. We transport the polylogarithmic quotient to all path torsors using the canonical de Rham path.

To do this, we need to apply g to the equation

$$ {}_x\gamma_0 = {}_x1_0 u_x $$

to get

$$ g({}_x\gamma_0) = g({}_x1_0)g(u_x) $$

In fact, we only care about $g \in \pi_1^{\mathrm{un}}(Z) = \ker \chi$. For such g, we have $g(u_x) = u_x$, because we are using the polylogarithmic quotient. We therefore get

$$ g({}_x\gamma_0) = g({}_x1_0)u_x $$

To make these elements of ${}_0\Pi_0^{\mathrm{dR}}(\mathbb{Q}_p)$ rather than ${}_x\Pi_0^{\mathrm{dR}}(\mathbb{Q}_p)$, we write

$$ {}_x1_0^{-1} g({}_x\gamma_0) = {}_x1_0^{-1} g({}_x1_0)u_x $$

To understand u_x, note that

$$ \mathrm{Li}_n^{\mathfrak{u}}(u_x) = \mathrm{Li}_n^p(x). $$

As well, we have

$$ \mathrm{Li}_n^{\mathfrak{u}}({}_x1_0^{-1} g({}_x1_0)) = \mathrm{Li}_n^{\mathfrak{u}}(x)(g), $$

where $\mathrm{Li}_n^{\mathfrak{u}}(x) \in A(Z)$.

We may therefore compute that

$$ \begin{aligned} \mathrm{Li}_n^{\mathfrak{u}}({}_x1_0^{-1} g({}_x\gamma_0)) &= \mathrm{Li}_n^{\mathfrak{u}}({}_x1_0^{-1} g({}_x1_0)u_x) \\ &= \mathrm{Li}_n^{\mathfrak{u}}(u_x) + \sum_{i=0}^{n-1} \mathrm{Li}_{n-i}^{\mathfrak{u}}({}_x1_0^{-1} g({}_x1_0)) \frac{\log^{\mathfrak{u}}(u_x)^i}{i!} \\ &= \mathrm{Li}_n^p(x) + \sum_{i=0}^{n-1} \mathrm{Li}_{n-i}^{\mathfrak{u}}(x)(g) \frac{\log^p(x)^i}{i!}. \end{aligned} $$

In other words, the version of $\mathrm{Li}_n^{\mathfrak{u}}(x)$ coming from [Bro17, (11.15)] is

$$ \mathrm{Li}_n^p(x) + \sum_{i=0}^{n-1} \frac{\log^p(x)^i}{i!} \mathrm{Li}_{n-i}^{\mathfrak{u}}(x) \in A(Z) \otimes \mathbb{Q}_p $$

For Ishai's version, note

$$ \begin{aligned} {}_x\gamma_0^{-1} g({}_x\gamma_0) &= {}_x\gamma_0^{-1} g({}_x1_0)u_x \\ &= ({}_x1_0 u_x)^{-1} g({}_x1_0)u_x \\ &= u_x^{-1}\, {}_x1_0^{-1} g({}_x1_0)u_x. \end{aligned} $$

In other words, it's conjugate of the ordinary thing by u_x (whose coordinates are p-adic polylogarithms evaluated at x).

3.2. **A Slight Error.** [Bro17, §11.7] states that for an open affine $U \subset X$ containing 0, the fact that $W_n\mathcal{O}(\Pi_0)$ is trivial as a vector bundle over U implies that there is a canonical isomorphism

$$ {}_xc_0 \colon W_n\mathcal{O}({}_x\Pi_0^\omega) \cong \Gamma(U, W_n\mathcal{O}(\Pi_0)) \cong W_n\mathcal{O}({}_0\Pi_0^\omega). $$

There are two problems with this:

(1) The isomorphism between the left and right sides is not canonical; it depends on a trivialization of the vector bundle (or at least an appropriate subspace of $\Gamma(U, W_n\mathcal{O}(\Pi_0))$)
(2) Even if one chooses a trivialization, the middle is not isomorphic to either side.

In fact, the middle is isomorphic to $W_n\mathcal{O}({}_0\Pi_0^\omega) \otimes \mathcal{O}(U)$. But $\mathcal{O}(U)$ is an infinite-dimensional vector space (in particular, it is not one-dimensional). As one meme puts it: `https://www.facebook.com/geometryofmemes/photos/a.2132782976771087/2350973868285329/?type=3&theater`.

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