Thm (Ax–Grothendieck)

Say we have
$f_i(x_1, \dots, x_n)$ giving $f : \mathbb{C}^n \to \mathbb{C}^n$

Example $n=1$ $f_1(x_1) = x_1^2$ surjective, not injective
so converse is false

$n=2$ $f_1(x_1, x_2) = x_1$
$f_2(x_1, x_2) = 0$ not surj ((1,1) not in image)
also ~~not~~ not injective (does not depend on x_2)

n invertible $n \times n$ matrix (f_j all linear in x_i)

$n=2$ $f_1 = x_1 - x_2^2$
$f_2 = x_2$

$\mathbb{C}$ is an alg. closed field – add, mult, everything but 0 has multiplicative inverse

and any polyn. has a soln $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$
$n \geq 1$

p.2 · $\mathbb{Q}$ is not e.g. $x^2-2=0$ $x^2+1=0$ $x^3+x+1=0$

now "$\mathbb{R}$ has soln to first two" | get things of form $a+b\sqrt{2}$, $a,b\in\mathbb{Q}$

<u>but</u> not $x^2+1=0$ "so we just create this thing i and declare $i^2+1=0$"

"formally adjoin a root to $x^2+1=0$"

<u>char 0</u> + <u>write</u> i.e. $1+\cdots+1\neq 0$ as opposed to $\mathbb{Z}/n\mathbb{Z}$

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now consider $\mathbb{F}_p=\mathbb{Z}/p\mathbb{Z}$ <u>is a field</u> p prime

say $f_{\cancel{\phi}}(x_1,\dots,x_n)$ map $\mathbb{F}_p^n\to\mathbb{F}_p^n$

f_m

if injective ... then surjective!

now $\mathbb{F}_p$ <u>not</u> alg. closed. ~~But~~ Can

e.g. $x^2-2=0$ has no soln in $\mathbb{F}_3$

(can cook up higher deg examples)

can do $\mathbb{F}_9=\{a+b\sqrt{2}\mid a,b\in\mathbb{F}_3\}$

$\mathbb{F}_{27}$ ("if you adjoin roots of two irred. poly. of same degree get same field")

$\mathbb{F}_{3^{45}}$ etc. ~~deg~~ <u>finite so thm is true</u>

adjoining all roots of all polynomials get "$\overline{\mathbb{F}_p}$"

$\mathbb{F}_p \subseteq \overline{\mathbb{F}_p}$

so $\underbrace{1+1+\cdots+1}_{p \text{ times}}=0$ in $\overline{\mathbb{F}_p}$

and every poly has soln.

now $\overline{\mathbb{F}_p}$ is union of $\mathbb{F}_{p^K}$ over $K \in \mathbb{N}$

<u>wts</u> true for $\overline{\mathbb{F}_p}$ in place of $\mathbb{C}$, $\mathbb{F}_p$

say $f_1(x_1,\dots,x_n)$ $\quad$ <u>injective</u> $f: \overline{\mathbb{F}_p}^n \to \overline{\mathbb{F}_p}^n$

$\vdots$

$f_n(x_1,\dots,x_n)$ $\quad$ has all coeff in $\mathbb{F}_{p^K}$

Then if $\mathbb{F}_{p^{K'}} \supseteq \mathbb{F}_{p^K}$ defines map

$$\mathbb{F}_{p^{K'}}^n \to \mathbb{F}_{p^{K'}}^n$$

inj as inj for $\overline{\mathbb{F}_p}$

<u>so</u> surj as $\mathbb{F}_{p^{K'}}$ is finite

<u>now every</u> $(y_1,\dots,y_n) \in \overline{\mathbb{F}_p}^n$ lies in $\mathbb{F}_{p^{K'}}$ for some K'

so in image of f. <u>so</u> f surjective

<u>so</u> done for $\overline{\mathbb{F}_p}$ instead of $\mathbb{C}$

p. 4 "proof sketch" if $\exists$ counterexample then $\exists$ proof of counterexample using axioms of alg. closed fields of char 0

i.e uses $1+1\neq 0$ ("not char 2") $1+1+1\neq 0$ ("not char 3") $1+1+1+1+1\neq 0$ (... 5) ~~1+1+1~~ $7\neq 0$ (not char 7) etc

"I'm writing $1+\dots+1$ b/c in any abstract field we have a unit elt.)

guess what? a proof is finite!

so uses finitely many of those axioms

so $\exists$ p s.t. does not ~~hold~~ use the axiom $p\neq 0$

so $\exists$ counterexample for $\mathbb{F}_p$ but, we already proved... $\square$

Model Theory (~~Lecture I~~ Lecture I

What is a Mathematical structure?

$\mathbb{Z} \sim (0, 1, +, \cdot, <)$

$\mathbb{R} \sim (0, 1, +, \cdot, <)$

$\mathbb{Q}$ ...

one board

Defn A structure is a set

I_0 fcns $\{f_i \mid i\in I_0\}$ $f_i : M^{n_i} \to M$ $n_i \geq 1$

I_1 relations $\{R_i \mid i \in I_1\}$ $R_i : M^{m_i} \to \{\text{true}, \text{false}\}$ or $R_i \subseteq M^{m_i}$

I_2 "constants" $\{c_i \mid i\in I_2\}$ $c_i \in M$

another

now theory of groups ~~xxx~~ "some set" w/operation (or operations for rings)

and w/certain axioms

a model of the theory is a set w/ an operation and an id ~~...~~

~~satisfying~~

satisfying some axioms

(be clear I say "axiom" for intuition)

so to talk in general:

Defn A language $\mathcal{I}$ is three sets of "symbols"

I_0 "$\{\hat{f}_i\}$" n: fcn $I_0 \to \mathbb{N}$

I_1 "$\{\hat{R}_i\}$" m: fcn $I_1 \to \mathbb{N}$

I_2 "$\{c_i\}$"

Language

e.g. groups $|I_0| = 1$, $|I_1| = 0$, $|I_2| = 2$

ordered fields $|I_1| = 1$

rings $|I_0| = 2$, $|I_1| = 0$, $|I_2| = 2$

fields same language

Defn If $\mathcal{I}$ is a language, an $\mathcal{I}$-structure M is a set M

along with a fcn $f_i : M^{n_i} \to M$ for each $i \in I_0$

relation $R_i \subseteq M^{m_i}$ $\forall\ i \in I_1$

an element $c_i \in M$ $\forall\ i \in I_2$

p. 6

Formulas

say $\mathcal{L} = (\hat{+}, \hat{\times}, \hat{0}, \hat{1})$

$x_2 \times x_2$

$\hat{1} \times x_1$ terms

constants and fcns

$\hat{0} \times x_2$

$x_2 \times \hat{0} = x_1$ atomic formula use $=$ or an R_i

use $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$

$x_2 + x_2 = x_1$ formula

$(x_2 + x_1) \times x_3 = \hat{0}$ formula

"$\forall x_1 \forall x_2$" $(x_1 = x_2 \vee \neg(f(x_1) = f(x_2)))$

$x_1 < x_2$

$\neg(x_1 = x_2)$

$\forall x_1 \exists x_2 (x_2 \times x_2 = x_1)$ sentence

$\ldots\ldots (f_i(x_2) = x_1)$ f_i is surjective $(n_i = 1)$

(f_i some fcn in some language)

$\forall x_1 (x_1 = 0 \vee (\exists x_2 (x_2 \times x_1 = 1)))$

a formula $(\exists x_2 \; x_2 \times x_2 = x_1) \vee (x_1 < 0)$

if we append $\forall x_1$ we get a sentence in $\mathcal{L} = (\hat{\times}, \hat{<}, \hat{0})$

if ϕ is an $\mathcal{L}$-sentence and M an $\mathcal{L}$-structure

we write $M \models \phi$ to mean "ϕ holds in M"

else $M \models \neg\phi$

<u>Defn</u> An $\mathcal{L}$-theory T is a ~~set~~ collection of sentences in $\mathcal{L}$ ("possibly infinite")

we say $M \models T$ if M is an $\mathcal{L}$-struct. s.t. $M \models \phi \ \forall \phi \in T$

we say M is a <u>model</u> of T

<u>Examp</u> $\mathcal{L} = (\hat{+}, \hat{\times}, \hat{0}, \hat{1})$

<u>theory</u> of rings

$$T = \begin{cases} \forall x_1 \ x_1 + \hat{0} = x_1 \\ \forall x_1 \forall x_2 \ x_1 + x_2 = x_2 + x_1 \\ \forall x_1 \forall x_2 \forall x_3 \ x_1 + (x_2 + x_3) = (x_1 + x_2) + x_3 \\ \forall x_1 \exists x_2 \ x_1 + x_2 = \hat{0} \\ \text{id, assoc for mult.} \ \forall x_1 (\hat{1} \times x_1 = x_1) \\ \forall x_1 \forall x_2 \forall x_3 \ x_1 \times (x_2 + x_3) = x_1 \times x_2 + x_1 \times x_3 \end{cases}$$

so a model of T "i.e. an $\mathcal{L}$-struct. in which all sentences in T are true" ("i.e. 0, 1, two bin. operations")

is . . . a <u>ring</u>

say we add these axioms (<u>sentences</u>) too in T:

$$\forall x_1 (\exists x_2 (x_1 \times x_2 = \hat{1})) \vee (x_1 = \hat{0})$$

$$\forall x_1 \neg (x_1 = 0) \to (\exists x_2 \ x_2 \times x_1 = \hat{1})$$

"φ_3" $\forall x_1 \forall x_2 \forall x_3 \exists y \ (y^3 + x_1 \times y^2 + x_2 \times y + x_3 = \hat{0})$ "φ_n" in general p.8 We get ACF a theory "collection of sentences in $\mathcal{L}$"

now $\phi_3 \; \neg(\hat{1}+\hat{1}+\hat{1}=0)$

$\phi_2 \; \neg(\hat{1}+\hat{1}=0)$

etc $\phi_5 \; \phi_7 \; \cdots$

get ACF_0

taking $\neg\phi_3, \phi_2, \phi_5, \phi_7, \cdots$ get ACF_3 or ACF_p in general

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Defn A theory T is

satisfiable if $\exists$ a model M

consistent if cannot derive a contradiction from T

Clearly satisfiable $\Rightarrow$ consistent

Thm (Gödel's Completeness) if T consistent, satisfiable

$\forall \; K \geq \aleph_0 + |T| \; \exists$ model M s.t. $|M| = K$

So say

$T_0 = ACF_0 + \{\gamma_{n,d}\}_{n,d \in \mathbb{N}}$ → Φ in Marker

$\gamma_{2,2}$ (see next page)

T is consistent. why? if we could derive a contradiction then it would use finitely many ϕ_p. so it would be true for ACF_p (T_p) for some p. but we showed this is not the case.

SO T is satisfiable (by Gödel completeness) by a model w/ cardinality $|\mathbb{C}|$ $(=|\mathbb{R}|)$

"so some alg. closed field of char. 0 w/ # of elts $|\mathbb{C}|$"

algebra tells us all such things are isom to $\mathbb{C}$ so ~~true~~ for $\mathbb{C}$

$$\forall a_{0,0} \forall a_{0,1} \forall a_{0,2} \forall a_{1,0} \forall a_{1,1} \forall a_{2,0} \forall b_{0,0} \forall b_{0,1} \forall b_{0,2} \forall b_{1,0} \forall b_{1,1} \forall b_{2,0} \Big[\Big(\forall x_1 \forall y_1 \forall x_2 \forall y_2 \Big(\Big(\sum a_{i,j} x_1^i y_1^j = \sum a_{i,j} x_2^i y_2^j \wedge \sum b_{i,j} x_1^i y_1^j = \sum b_{i,j} x_2^i y_2^j \Big) \rightarrow (x_1 = x_2 \wedge y_1 = y_2) \Big) \Big)$$

$$\rightarrow \forall u \forall v \exists x \exists y \sum a_{i,j} x^i y^i = u \wedge \sum b_{i,j} x^i y^j = v \Big]$$

This is $\gamma_{2,2}$

(or $\Phi_{2,2}$ in Marker's notes)