Proof of the Residue Theorem

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Let D be an open disc bounded by a circle C, let $k \in \mathbb{Z}$ and $z_0 \in \mathbb{C}$. Then we recall that

$$\int_C (z - z_0)^k dz = \left\{ \begin{array}{ll} 0, & \text{for } k \neq -1 \\ 0, & \text{for } z_0 \notin D \\ 2\pi i, & \text{otherwise} \end{array} \right\}.$$

By Theorem 1.3 of Chapter 3 of S-S, if $f(z)$ has a singularity at z_0 that is either a pole or removable, there is a unique way to write $f(z)$ as

$$f(z) = G(z) + \sum_{k=\mathrm{ord}_{z_0}(f)}^{-1} a_k (z - z_0)^k$$

where $G(z)$ is holomorphic at z_0. In this efer to $\sum_{notation, we r k=\mathrm{ord}_{z_0}(f)}^{-1} a_k(z-z_0)^k$ as $P_{f,z_0}(z)$ or the *principal part* of $f(z)$ at z_0. We write $G_{f,z_0} = f(z) - P_{f,z_0}(z)$, which is the *holomorphic part* of $f(z)$ at z_0. If z_0 is understood, we write $P_f(z)$ and $G_f(z)$, respectively.

The function f is holomorphic at z_0 (or has a removable singularity at z_0) if any only if $P_{f,z_0}(z) = 0$.

If γ is a simple closed loop going around z_0, then

$$\int_\gamma P_{f,z_0}(z) dz = 2\pi i \mathrm{res}_{z_0}(f).$$

We then have:

Lemma. *Let $w_0 \in \mathbb{C} \setminus \{z_0\}$, and suppose $f(z)$ is either holomorphic at w_0, or holomorphic in a deleted neighboor of w_0 with a pole at w_0.*

Then the principal part of $f(z)$ at w_0 is the same as the principal part of $G_{f,z_0}(z)$ at w_0.

Proof. We have

$$\begin{aligned} G_{f,z_0}(z) &= f(z) - P_{f,z_0}(z) \\ &= P_{f,w_0} + f(z) - P_{f,z_0}(z) - P_{f,w_0}(z) \\ &= P_{f,w_0} + [f(z) - P_{f,w_0}(z)] - P_{f,z_0}(z). \end{aligned}$$

Now $[f(z) - P_{f,w_0}(z)]$ is holomorphic at w_0 by the definition of $P_{f,w_0}(z)$, and $P_{f,z_0}(z)$ is holomorphic at w_0 because $w_0 \neq z_0$. Thus $[f(z) - P_{f,w_0}(z)] - P_{f,z_0}(z)$ is holomorphic at w_0. It follows by the uniqueness statement in Theorem 1.3 of Chapter 3 that $P_{f,w_0}(z)$ is the principal part of $G_{f,z_0}(z)$ at w_0. □

Using this, we may now prove the Residue Theorem:

Theorem. *Let Ω be an open subset of the complex plane containing a simple closed curve γ and its interior U (i.e., the region it bounds). Suppose that f is a function that is holomorphic on Ω except for a finite set of distinct poles $z_1, \cdots, z_N$, all lying in U. Then*

$$\int_\gamma f(z)dz = \sum_{j=1}^{N} 2\pi i \operatorname{res}_{z_j}(f).$$

Proof. We prove this by induction on N.

For $N = 1$, we have

$$\begin{aligned} \int_\gamma f(z)dz &= \int_\gamma P_{f,z_1}(z) + G_{f,z_1}(z)dz \\ &= \int_\gamma P_{f,z_1}(z)dz + \int_\gamma G_{f,z_1}dz \\ &= 2\pi i \operatorname{res}_{z_1}(f) + \int_\gamma G_{f,z_1}dz \end{aligned}$$

By the lemma, $G_{f,z_1}(z)$ is holomorphic on all of Ω, so Cauchy's Theorem implies that $\int_\gamma G_{f,z_1}dz$. This means $\int_\gamma f(z)dz = 2\pi i \operatorname{res}_{z_1}(f)$, proving the Residue Theorem for $N = 1$.

Now suppose the Residue Theorem is true for $N \geq 1$ and all f. We prove it for $N+1$. That is, suppose that f is holomorphic except for poles $z_1, \cdots, z_N, z_{N+1}$.

Then by the lemma, $G_{f,z_{N+1}}(z)$ is holomorphic on all of Ω except for poles at $z_1, \cdots, z_N$ (but not at z_{N+1}), where its residues are the same as those of f. As we are assuming the Residue Theorem for N, we have

$$\int_\gamma G_{f,z_{N+1}}(z)dz = \sum_{j=1}^{N} 2\pi i \mathrm{res}_{z_j}(G_{f,z_{N+1}}(z)) = \sum_{j=1}^{N} 2\pi i \mathrm{res}_{z_j}(f).$$

We then have

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_\gamma P_{f,z_{N+1}}(z) + G_{f,z_{N+1}}(z)dz \\
&= \int_\gamma P_{f,z_{N+1}}(z)dz + \int_\gamma G_{f,z_{N+1}}dz \\
&= 2\pi i \mathrm{res}_{z_{N+1}}(f) + \sum_{j=1}^{N} 2\pi i \mathrm{res}_{z_j}(f) \\
&= \sum_{j=1}^{N+1} 2\pi i \mathrm{res}_{z_j}(f),
\end{aligned}$$

so the Residue Theorem is true.

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