Math 185

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1 The Dual Nature of the Complex Plane

A complex number is usually written $z = x + iy$, where $i^2 = -1$, and x, y are real numbers. On the one hand, you can think of it as a point on the plane, which therefore has two coordinates. On the other hand, we like to think of z as a single number. This duality has a lot to do with what makes complex analysis different from real analysis.

Addition of complex numbers is just the same as addition of real two-dimensional vectors. In this sense, we like to say that $\mathbb{C} \cong \mathbb{R}^2$. We may even write $z = x + iy = (x, y)$. However, $\mathbb{C}$ has something important that $\mathbb{R}^2$ doesn't have; it has a funny multiplication. This multiplication is defined so that $i^2 = -1$. If we write the complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, then we get the funny formula:

$$z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1).$$

This formula implies in particular that $(0, 1)^2 = -1 = (-1, 0)$, and it's what makes the complex numbers different from just ordinary vectors in the real plane. Of course, we still like to think of complex numbers as vectors in the plane; we just have to remember that they're more than that.

Now, you might say that we can multiply elements of $\mathbb{R}^2$ by setting $(x_1, y_1)(x_2, y_2) = (x_1 x_2, y_1 y_2)$. However, this type of multiplication does not have the nice property that every nonzero element is invertible. In other words, for multiplication of complex numbers, whenever $z \neq 0$, there is a complex number z^{-1} such that $zz^{-1} = z^{-1}z = 1$.

In terms of limits and topology, we also think of the complex numbers as no different from $\mathbb{R}^2$. We define $|z| = \sqrt{x^2 + y^2}$, and we define the open ball

$$D_r(z_0) \coloneqq \{z \in \mathbb{C} \mid |z - z_0| < r\}$$

A subset $\Omega \subseteq \mathbb{C}$ is said to be *open* if it is the union of open balls of the form $D_r(z_0)$ (possibly for infinitely many $z_0 \in \mathbb{C}$ and $r \in \mathbb{R}_{>0}$). An open subset Ω of the complex plane is often called a *(complex) domain*.

If f is a function defined on an open neighborhood of z_0, then we say $\lim_{z \to z_0} f(z) = w$ if for all $\epsilon > 0$, there is $\delta > 0$ such that $f(z) \in D_\epsilon(w)$ for all $z \in D_\delta(z_0)$.

Another way to put all of this is that the "$\cong$" in $\mathbb{C} \cong \mathbb{R}^2$ preserves everything to do with addition and with geometry, but it does not preserve the multiplication of complex numbers. Or in other words, this $\cong$ *forgets* the structure of multiplication in $\mathbb{C}$. The use of the word "forgets" is a common one in mathematics.

2 Complex Differentiation

If $\Omega \subseteq \mathbb{C}$ is an open subset of $\mathbb{C}$ (equivalently, of $\mathbb{R}^2$), and $f \colon \Omega \to \mathbb{C}$ is a function, we can try to say what it means to differentiate f.

It is natural to define

$$f'(z) \coloneqq \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}, \tag{1}$$

if the limit exists, in which case we say that f is *complex differentiable* or *holomorphic*.

Notice that the expression $f(z+h) - f(z)$ only uses addition and subtraction, both of which are simply vector operations. The thing that makes this definition fundamentally about $\mathbb{C}$ and not about $\mathbb{R}^2$ is the *division* of the top by the bottom.

Now, we can of course forget the structure of $\mathbb{C}$ and just think of f as a function from $\Omega \subseteq \mathbb{R}^2$ to $\mathbb{R}^2$. When we do this, it's best to write $f(x + iy) = u(x, y) + iv(x, y)$, where u and v are real-valued functions on $\mathbb{R}^2$. For functions on $\mathbb{R}^2$, we have the notion of partial derivative from multivariable calculus. In particular, we have

$$\frac{\partial u}{\partial x}(x,y) = \lim_{h\to 0} \frac{u(x+h,y)-u(x,y)}{h}$$
$$\frac{\partial u}{\partial y}(x,y) = \lim_{h\to 0} \frac{u(x,y+h)-u(x,y)}{h}$$
$$\frac{\partial v}{\partial x}(x,y) = \lim_{h\to 0} \frac{v(x+h,y)-v(x,y)}{h}$$
$$\frac{\partial v}{\partial y}(x,y) = \lim_{h\to 0} \frac{v(x,y+h)-v(x,y)}{h}.$$

Remark 2.1. Notice a key difference between the limits here and 1 is that the h in 1 is a *complex number*, whereas the limits just above are only real numbers. In particular, the limit in 1 must be the same *regardless of which direction h approaches zero from.*

More precisely, in the definition of limit, we require that for all $\epsilon > 0$, the expression is within ϵ of the limit *for all h* within δ of 0. But the key question is what "for all" means. The set of all *real h* within δ of 0 is a line segment of length 2δ, while the set of all *complex h* within δ of 0 is a circle around zero. For the former, we sometimes write $\lim_{h\to 0\, h\in\mathbb{R}}$ instead of just $\lim_{h\to 0}$.

Therefore, it is harder for a complex limit to exist than for a real limit to exist; i.e., if $\lim_{h\to 0}$ of something exists, then $\lim_{\substack{h\to 0\\ h\in\mathbb{R}}}$ automatically exists and equals the former limit. It is because of this that *it is harder for a function to be complex differentiable than for it to be a (two-variable) real differentiable function.*

Despite how hard it is, we will soon see many examples of complex-differentiable functions. On the other hand, as I will explain in Section 3, the property of being complex-differentiable implies a lot of nice properties that don't hold for real-valued functions.

2.1 Cauchy-Riemann Equations

Let's say $f(z) = u(x,y)+iv(x,y)$ is holomorphic in a domain $\Omega \subseteq \mathbb{C}$, and that u and v have all partial derivatives in Ω. We want to understand how the notion of being holomorphic relates to partial derivatives.

Following the idea of Remark 2.1, let's see what happens as h approaches 0 along the real axis versus what happens as h approaches 0 along the imaginary axis. If the function is holomorphic, then both of these limits should exist, and they should be equal to each other.

First, we consider the limit

$$\lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{f(z+h)-f(z)}{h}.$$

Writing in terms of real and imaginary parts, we get

$$\begin{aligned} f'(z) &= \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{u(x+h,y)+iv(x+h,y)-u(x,y)-iv(x,y)}{h} \\ &= \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{u(x+h,y)-u(x,y)}{h} + i \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{v(x+h,y)-v(x,y)}{h} \\ &= \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) \end{aligned}$$

It follows that

$$f'(z) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y).$$

Now let's consider the limit from the imaginary direction. To do this, we consider the limit

$$\lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{f(z+ih)-f(z)}{ih}.$$

This limit should also equal $f'(z)$.

Writing in terms of real and imaginary parts, we get

$$\begin{aligned} f'(z) &= \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{u(x,y+h)+iv(x,y+h)-u(x,y)-iv(x,y)}{ih} \\ &= \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{v(x,y+h)-v(x,y)}{h} - i \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{u(x,y+h)-u(x,y+h)}{h} \\ &= \frac{\partial v}{\partial y}(x,y) - i\frac{\partial u}{\partial y}(x,y) \end{aligned}$$

It follows that

$$f'(z) = \frac{\partial v}{\partial y}(x,y) - i\frac{\partial u}{\partial y}(x,y).$$

But this is strange - we have two different expressions for $f'(z)$! In order for $f'(z)$ to be well-defined (i.e., for f to be holomorphic), these expressions have to be equal. So we find that if f is holomorphic, then

$$\frac{\partial v}{\partial y}(x,y) - i\frac{\partial u}{\partial y}(x,y) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y).$$

Since an equality of complex numbers implies an equality of the corresponding real and imaginary parts, we find that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

at all points of Ω.

These are the celebrated *Cauchy-Riemann Equations.*

Note that we haven't proven that any differentiable function satisfying the Cauchy-Riemann equations is holomorphic; we've simply shown that in order for a function to be holomorphic, it must satisfy these equations. Soon, we'll show that the converse is true, i.e., that these equations imply holomorphicity. That's Ch 1 Theorem 2.4 in [S-S].

2.2 Geometric Interpretation

As we said before, the key structure that distinguishes $\mathbb{C}$ from $\mathbb{R}^2$ is the method of multiplying complex numbers. Let's fix a complex number $w = a + ib$, for $a, b \in \mathbb{R}$, and let's consider the function from $\mathbb{C}$ to $\mathbb{C}$ sending

$$z \mapsto wz.$$

Identifying $\mathbb{C}$ with $\mathbb{R}^2$, this can be written as

$$(x,y) \mapsto (ax - by, ay + bx).$$

Alternatively, we may represent this by the matrix

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}. \tag{2}$$

Representing it by a matrix recalls for us that it's a linear transformation. Geometrically, this linear transformation corresponds to a rotation by θ and scaling by r, where $w = re^{i\theta}$, or equivalently, $r = \sqrt{a^2 + b^2}$ and $\theta = \arctan(b/a)$.

Conversely, any linear transformation consisting of a rotation and a scaling can be written as a matrix of the form 2 and corresponds to multiplication by a complex number. In this sense, we can consider $\mathbb{C}$ as a subset of the set $M_2(\mathbb{R})$ of all 2×2 real matrices, where the complex number $a + bi$ corresponds to the matrix 2.

Now, let's go back to calculus. If we have an open set $\Omega \subseteq \mathbb{R}^2$ and a differentiable function $f(x, y) = (u(x, y), v(x, y)) \colon \Omega \to \mathbb{R}^2$, then at each point $(x, y) \in \Omega$, there is the Jacobian matrix

$$J(x, y) = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix}$$

The Jacobian has a nice interpretation in terms of linear approximations of functions. In ordinary calculus, if $g(x)$ is a single-variable differentiable real-valued function, then g' can be defined by saying that for any $x_0 \in \mathbb{R}$, the linear function

$$g'(x_0)(x - x_0) + g(x_0)$$

is the best linear approximation to the function $g(x)$ near the point $x = x_0$. Similarly, around a point (x_0, y_0), the best linear approximation to $f(x, y)$ is the function

$$J(x_0, y_0) \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + \begin{pmatrix} u(x_0, y_0) \\ v(x_0, y_0) \end{pmatrix},$$

where we view $\mathbb{R}^2$ as the space of 2×1 column vectors.

The Cauchy-Riemann equations are in fact equivalent to the statement that $J(x, y)$ is of the form 2, i.e. that $J(x, y)$ lies in $\mathbb{C} \subseteq M_2(\mathbb{R})$.

Thus, geometrically, we can think of a holomorphic function as a function that locally acts like multiplication by a complex number (and in fact, this complex number is precisely the derivative of that function); equivalently, a function that locally acts by rotation and scaling (but not *skewing*). One important consequence of this is that holomorphic functions are *conformal*, which is a fancy way of saying that they preserve angles.

3 Properties Enjoyed by Holomorphic Functions

If f is a function from a complex domain Ω to $\mathbb{C}$, then the property of being holomorphic on Ω implies a lot of nice properties of f that are not implied by real differentiability. We list some of these properties, as a series of "Facts," interspersed with comments about how these differ from real analysis.

Fact 3.1 (Ch 2 Corollary 4.2 of S-S)**.** If f is holomorphic on Ω, then f' is also holomorphic on Ω. In fact, letting $f(x+iy) = u(x,y) + iv(x,y)$, if f is holomorphic, then u and v are infinitely differentiable in either variable.

Note that the corresponding statement is very much false if we simply assume that f is differentiable in the variables x or y. A simple example would be to set $u(x,y) = x^2$ for $x \geq 0$ and $-x^2$ for $x \leq 0$, and $v(x,y) = 0$.

Fact 3.2 (Ch 2 Theorem 4.4 of S-S)**.** If f is holomorphic on Ω, and $D_r(z_0) \subseteq \Omega$ for some $z_0 \in \Omega$ and $r \in \mathbb{R}_{>0}$, then on $D_r(z_0)$, we have

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k.$$

Again, the corresponding statement is false in real analysis. There are real differentiable functions that are infinitely differentiable but not equal to the sum of their Taylor series. For example, consider the function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ for $x \leq 0$ and $f(x) = e^{-1/x}$ for $x > 0$.

Fact 3.3 (Ch 2 Corollary 2.3 of S-S)**.** If f is holomorphic on Ω, and $D_r(z_0)$ is an open ball with boundary γ, such that $D_r(z_0)$ and γ are both contained in Ω, then

$$\int_\gamma f(z)dz = 0.$$

In fact, if γ is *any* curve (not just a circle) contained in $D_r(z_0)$, then $\displaystyle\int_\gamma f(z)dz = 0$. See Ch 2 Theorem 2.2 of [S-S].

If f is an arbitrary differentiable function from Ω to $\mathbb{R}^2$, then this is not guaranteed to hold. But I actually claim that we can understand this statement in terms of ordinary multivariable calculus.

To see this, we write $f(z) = u(x,y) + iv(x,y)$, and we note that $dz = d(x+iy) = dx + idy$. Then we write

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_\gamma (u(x,y) + iv(x,y))(dx + idy) \\
&= \int_\gamma (u(x,y)dx - v(x,y)dy) + i(u(x,y)dy + v(x,y)dx) \\
&= \int_\gamma u(x,y)dx - v(x,y)dy + i\int_\gamma u(x,y)dy + v(x,y)dx.
\end{aligned}$$

Now, assuming that all the partial derivatives of u and v exist and are continuous, Green's Theorem from multivariable calculus tells us that

$$\int_\gamma F dx + G dy = \int_{D_r(z_0)} \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} dx dy.$$

Applying Green's Theorem to the real and imaginary parts of the previous equations, we find that

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_\gamma u(x,y)dx - v(x,y)dy + i\int_\gamma u(x,y)dy + v(x,y)dx \\
&= \int_{D_r(z_0)} \frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y} dxdy + i\int_{D_r(z_0)} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} dxdy \\
&= \int_{D_r(z_0)} 0 dxdy + i\int_{D_r(z_0)} 0 dxdy \\
&= 0,
\end{aligned}$$

where the second to last step holds because of the Cauchy-Riemann equations.

Related to Fact 3.3, it turns out that every holomorphic functions $f(z)$ on a disc has a primitive, i.e., a holomorphic function $F(z)$ such that $F'(z) = f(z)$ for all z in the disc (see Ch 2 Theorem 2.1 in [S-S]). You can try to define a primitive for a function f by defining $F(z) = \int_{z_0}^{z} f(z)dz$. The problem with this definition is that it a priori depends on *which path you choose from z to z_0!* But Fact 3.3 can be used to show that this integral does not depend on the path! On the flip side, if you think of γ as starting and ending at $z_0 + r$ (the rightmost point of the circle γ), then applying the fundamental theorem of calculus would tell you

$$\int_\gamma f(z)dz = [F(z)]_{z_0+r}^{z_0+r} = F(z_0 + r) - F(z_0 + r) = 0. \tag{3}$$

Note that in Fact 3.3, we had to assume that the function $f(z)$ was holomorphic on all of Ω, and in particular, in the entire interior of the circle γ. The following fact shows that this assumption was necessary:

Fact 3.4. If γ is the circle as above, then

$$\int_\gamma \frac{1}{z - z_0} dz = 2\pi i.$$

This fact is described near the bottom of p.23 of [S-S].

Proof. To prove this, let's parametrize γ by $\gamma(t) = z_0 + re^{2\pi it}$ for $t \in [0,1]$. While I haven't defined complex line integrals yet, you can imagine that $dz = \frac{d\gamma(t)}{dt}dt = 2\pi ire^{2\pi it}$, so we have

$$\begin{aligned}
\int_\gamma \frac{1}{z-z_0}dz &= \int_0^1 \frac{1}{\gamma(t)-z_0}\frac{d\gamma(t)}{dt}dt \\
&= \int_0^1 \frac{1}{re^{2\pi it}}2\pi ire^{2\pi it}dt \\
&= \int_0^1 \frac{2\pi ire^{2\pi it}}{re^{2\pi it}}dt \\
&= \int_0^1 2\pi idt \\
&= 2\pi i.
\end{aligned}$$

□

That was a rigorous derivation; but more importantly, this has a nice intuitive explanation in terms of the complex logarithm, as follows.

Recall that $e^{i\theta} = \cos\theta + i\sin\theta$, which implies de Moivre's formula $e^{2\pi i} = e^0 = 1$. Assume for simplicity that $z_0 = 0$. Then we should think of $\int_\gamma \frac{1}{z}dz$ as an integral from $z = 1$ to $z = 1$, and since the antiderivative of $\frac{1}{z}$ is the logarithm, it should be $\log(1) - \log(1)$. But the fact that $e^{2\pi i} = e^0 = 1$ suggests that $\log(1)$ has more than one possible value, and so it makes sense (some *weird* sense) to say that $\log(1) - \log(1) = 2\pi i - 0 = 2\pi i$.

In other words, the fact that the integral of $\frac{1}{z}$ along a path around the origin is nonzero is related to the fact that the complex logarithm is ambiguous, at least up to an integer multiple of $2\pi i$. This is an important idea that will come up a lot in complex analysis.

In fact, an important corollary of Fact 3.4 is the following, known as the Cauchy Integral Formula:

Fact 3.5 (Ch 2 Theorem 4.1 of S-S)**.** Let $f(z)$ be a holomorphic function in a domain Ω, and suppose $D_r(z_0)$ is an open ball whose boundary γ is also contained in Ω. Then

$$\int_\gamma \frac{f(z)}{z-z_0}dz = 2\pi if(z_0).$$

Here's a simple intuitive reason to believe Fact 3.5. By Fact 3.2, we can write $f(z)$ as a power series $\sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$ in that ball. Assuming we can interchange integration and infinite summation, we have

$$\begin{aligned}
\int_\gamma \frac{f(z)}{z-z_0} dz &= \int_\gamma \frac{\sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k}{z-z_0} dz \\
&= \int_\gamma \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^{k-1} dz \\
&= \sum_{k=0}^{\infty} \int_\gamma \frac{f^{(k)}(z_0)}{k!}(z-z_0)^{k-1} dz \\
&= \int_\gamma \frac{f^{(0)}(z_0)}{0!}(z-z_0)^{-1} dz \\
&= \int_\gamma \frac{f(z_0)}{z-z_0} dz \\
&= 2\pi i f(z_0).
\end{aligned}$$

The third-to-last step, which gets rid of all the terms for $k > 0$, follows by Fact 3.3 because $\frac{f^{(k)}(z_0)}{k!}(z-z_0)^{k-1}$ is holomorphic on all of Ω when $k > 0$. The last step follows from Fact 3.4.

3.1 The Principle of Analytic Continuation

The amazing thing about the Cauchy Integral Formula is that it tells you that if f is holomorphic, then the value of f at z_0 is determined by the values of f on any circle around z_0. This fact should come as quite a surprise; we shouldn't normally be able to determine the value of the function in terms of some far away values!

In fact, this relates to a very important principle in complex analysis: *if you know that a function is holomorphic and you know some of its values, then you know what the function is.* This is a principle known as *analytic continuation.*

More precisely:

Fact 3.6 (Ch 2 Corollary 4.9 of S-S)**.** Let $f(z)$ and $g(z)$ be holomorphic functions on a connected domain Ω. If $g(z) = f(z)$ on a subset $S \subseteq \Omega$ with a limit point in Ω (for example, S contains an open subset or even an interval), then $g(z) = f(z)$ for all $z \in \Omega$.

3.2 Meromorphic Functions and the Residue Theorem

If Ω is a domain, $z_0 \in \Omega$, and f holomorphic on Ω, then we can always write $f(z) = (z - z_0)^m g(z)$, where g is a holomorphic function that does not vanish at $z = z_0$. The integer m is unique and is known as the *order* of f at z_0.

More generally, if f is a function defined on $\Omega \setminus z_0$, and $f(z) = (z - z_0)^m g(z)$ for a holomorphic function g in a neighborhood of z_0 that does not vanish at $z = z_0$, then we say that f is *meromorphic* at $z = z_0$. Then m is unique, and if $m < 0$, we say that f has a *pole* at $z = z_0$. The positive integer $-m$ is called the *order* of the pole. Note that if $m \geq 0$, then $f(z)$ can be defined as a holomorphic function on all of Ω, by Riemann's theorem on removable singularities (Theorem 3.1 of Ch 3 of S-S).

Note that while most functions defined on a punctured disc are meromorphic, the function $f(z) = e^{1/z}$ is *not meromorphic*. It has what's known as an *essential singularity* at $z = 0$.

Facts 3.3 and 3.5 seem to suggest that the only reason for the integral of a holomorphic function around a loop to be nonzero is if the holomorphic function has some sort of singularity in the region bounded by the loop. The residue formula (Ch 3 Corollary 2.2 of S-S) gives a formula for the integral of any meromorphic function f along a closed loop as a sum over the poles of f. Then Fact 3.5, the Cauchy integral formula, is a special case of this formula.

The residue formula is important for many applications, as discussed below for zeta functions and elliptic functions. It also allows us to evaluate certain real integrals, such as the beautiful formula:

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 1} dx = \frac{\pi}{e}.$$

4 Special Topics

Here are introductions to some (but not all) of the later topics in the book. In particular, I've left out Chapters 4 and 8 in this discussion, but I might still talk about them this semester.

4.1 The Gamma Function

One consequence of the principle of analytic continuation is the following:

Fact 4.1. This is a unique function $f(z)$ defined and holomorphic on $\Omega = \mathbb{C} \setminus \{-1, -2, -3, \cdots\}$, such that $f(n) = n!$ for all positive integers n.

This allows us to give meaning to expressions like $i!$ and 1.5! and so on and so forth. Without this, we would know that whatever 1.5! is, it should be half of 0.5!, but we wouldn't know what that is. So the notion of being holomorphic can tell us the correct way to extend a function beyond its natural domain of definition.

Furthermore, it turns out that there is no way to extend the factorial function holomorphically to the whole complex plane. Somehow, the factorial function, as defined on the integers, *knows* that it has to have singularities at the negative integers.

For historical reasons, one writes $\Gamma(z) = f(z-1)$, where f is the function of Fact 4.1. This is the *gamma function*, which is studied in Chapter 6 of [S-S].

4.2 Product Formulas

Definition 4.2. A holomorphic on the entire complex plane is known as an *entire function*.

Polynomials give the most basic class of entire functions. It's a well-known fact that a polynomial $p(z)$ can be expressed as a product

$$p(0) \prod_{p(\alpha)=0} \left(1 - \frac{z}{\alpha}\right)$$

at least if $p(0) \neq 0$, where the zeroes are counted with multiplicity (and if $p(0) = 0$, we multiply by a power of z).

One might try to generalize this to an entire function like the sine function. The sine function has as its zeroes $n\pi$, for $n \in \mathbb{Z}$. These zeroes are in fact symmetric under negation, and, noting that $\left(1 - \frac{z}{n\pi}\right)\left(1 - \frac{z}{-n\pi}\right) = \left(1 - \frac{z^2}{n^2\pi^2}\right)$, we might hope that

$$\sin(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right).$$

It turns out that this is true, and it is part of a more general discussion about infinite products in Chapter 5 of S-S. As part of this discussion, one may prove Hadamard's factorization theorem, giving a general formula for a function with a prescribed set of zeroes, assuming those zeroes satisfy certain growth conditions.

4.3 The Riemann Zeta Function

If one compares the product formula $\sin(z) = z \prod_{n \in \mathbb{Z}_{>0}} \left(1 - \frac{z^2}{n^2\pi^2}\right)$ with the Taylor series

$$\sin(z) = z - z^3/6 + \cdots ,$$

one might hope that some infinite version of Viete's formulas hold. More specifically, comparing the coefficent of z^3 on either side, one might hope that $\sum_{n=1}^{\infty} -\frac{1}{n^2\pi^2} = -\frac{1}{6}$, or that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

This is actual a special case of the *Riemann zeta function*, defined originally by Riemann as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

This function is discussed in Chapter 6. The series converges absolutely for $\mathrm{Re}(s) > 1$, and since all the terms are holomorphic in s, the series defines a holomorphic function for all such s by Theorem 5.2 of Ch 2. In Chapter 6, it's shown that $\zeta(s)$ can be extended to a holomorphic function on $\mathbb{C} \setminus \{1\}$. Of course, by analytic continuation, such an extension is unique.

The real power of $\zeta(s)$ comes from the formula for $\zeta(s)$ in terms of the prime numbers:

$$\zeta(s) = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}.$$

In Chapter 7 of S-S, it is explained how this can be used to prove the prime number theorem, which determines the asymptotic distribution of the prime numbers. One key part of this proof uses the residue theorem to write the indefinite integral

$$\int_{-\infty}^{\infty} \frac{x^{c+ix+1}}{(c+ix)(c+ix+1)} \left(\frac{\zeta'(c+ix)}{\zeta(c+ix)}\right) dx$$

for a real number $c > 1$ as a certain sum over prime numbers.

4.4 Elliptic Functions

The function $f(z) = e^z$ satisfies the nice identity $f(z) = f(z + 2\pi i)$. We can visualize this by saying that the function "wraps around" as one goes up along the imaginary axis. More precisely, one could imagine taking the complex plane, like a sheet of paper, and rolling it up into a cylinder so that z and $z + 2\pi i$ line up with each other. Then one should think of the exponential function as being defined on this cylinder.

The thing about the complex plane is that it has two independent directions, and so we could imagine rolling up the cylinder in the other direction. This would give us a torus.

Algebraically, this would be asking for a function f with two periods ω_1 and ω_2 (i.e., $f(z) = f(z + \omega_1) = f(z + \omega_2)$ for all z) such that ω_1 and ω_2 are independent over $\mathbb{R}$.

The group generated by ω_1 and ω_2, i.e. the set of all complex numbers $m\omega_1 + n\omega_2$ for $m, n \in \mathbb{Z}$, is a lattice (see the picture on p.263 of S-S). When we look at $\mathbb{C}$ modulo this lattice, we get a torus. An *elliptic function* is a function with two such independent periods, and we should think of it as a function on a torus.

One first proves some general facts about elliptic functions, using the residue theorem and the symmetry of the period parallelogram (see the figure on p.264).

This leaves the question of finding a nonconstant example of an elliptic function. For simplicity, we may set $\omega_1 = 1$, and we write $\tau = \omega_2$. Next, one defines the Weierstrass $\wp$-function by

$$\wp(z) = \frac{1}{z^2} + \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(z+m+n\tau)^2} - \frac{1}{(m+n\tau)^2}.$$

This gives a nontrivial example of an elliptic function (it is in fact a meromorphic function, not a holomorphic function, meaning it has a pole).

The more amazing part is that $\wp$ satsifies a differential equation of the form

$$\wp'(z)^2 = 4\wp(z)^3 - g_2(\tau)\wp(z) + g_3(\tau),$$

where g_2 and g_3 are holomorphic functions of τ (but independent of z). This means that the map $\mathbb{C} \to \mathbb{C}^2$ defined by

$$z \mapsto (z_1, z_2) = (\wp(z), \wp'(z))$$

has image inside the graph of the equation $z_2^2 = 4z_1^3 - g_2 z_1 + g_3$. This equation defines an *elliptic curve*, a type of mathematical object that is very important in number theory and became famous from Wiles' proof of Fermat's Last Theorem.

5 How We're Going to Prove All of This

What I've described in Section 3 is what you might call the *phenomenology* of complex analysis. In other words, I've told you about what phenomena show up in complex analysis and how they're different from real analysis, but I haven't told you *why* these phenomena are true (except in a few cases). That's useful for getting an intuitive feeling for how complex analysis works, but we will still have to learn how to rigorously prove all of these facts.

The tl;dr I'm trying to get across here is that the order in which we prove the previous Facts is different from the order in which they are presented in this document.

More specifically, it turns out that we will first prove a version of Fact 3.3 for triangles rather than circles. This is known as *Goursat's Theorem*. While the argument I presented to you assumed that the partial derivatives existed and were continuous (because it used Green's Theorem), our proof will only assume that the function is holomorphic (i.e., that the limit defining $f'(z)$ exists for every $z \in \Omega$).

Once we prove Goursat's Theorem, we can use it to show that the integral $\int_{z_0}^{z} f(z)dz$ is well-defined (independent of path), which shows that $f(z)$ has a primitive in any disc. It then follows by the fundamental theorem of calculus that the integral along any closed curve inside a disc is zero.

This then allows us to prove the Cauchy-Integral formula. The Cauchy Integral formula is very powerful, and it implies other results like analyticity and analytic continuation.