Interiors and Simple Connectivity

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1 Interiors of Loops

Since I said a few things related to Chapter 2, Section 2 that weren't in the book, I'm writing a short note.

The general principle behind a lot of what we've done is the following:

Principle 1.1. If a function f is holomorphic at every point of a loop γ *and the interior of* γ, i.e., the region bounded by γ, then $\int_\gamma f(z)dz = 0$.

This principle can be made into a theorem, known as Cauchy's theorem. The problem is that rigorously defining the term "the interior of γ" in a way that would allow us to prove such a theorem would lead us into technical details that are tangential to what we're doing right now.

For now, we've made this result precise and proven it in the case of a triangle (Goursat's Theorem) and a circle (Corollary 2.3), and when γ is contained in a disc on which f is holomorphic (Theorem 2.2). I also proved in class that Theorem 2.2 holds more generally for any convex region:

Theorem 1.2. *If Ω is an open convex subset of $\mathbb{C}$, and f is defined on Ω, then f has a primitive on Ω, and hence $\int_\gamma f(z)dz = 0$ for any loop γ contained in Ω.*

Furthermore, you may use it for any of the "toy contours" on Figure 7 of p.42 of S-S, even if we haven't technically proven those cases. (Nonetheless, in the case of a sector, a parallelogram, and a semicircle, you can easily prove it by applying the argument of Corollary 2.3 to Theorem 1.2.)

2 Simple-Connectivity

If we assume we know what is meant by the term "region bounded by γ," then we can make the following definition:

Definition 2.1. Let Ω be a region in $\mathbb{C}$. We say that Ω is *simply-connected* if for any loop γ contained in Ω, the region bounded by γ is also in Ω.

Some standard examples are (1) any convex open set and (2) the interiors of any of the toy contours in Figure 7.

Some non-examples are:

1. an annulus of the form $\{z \in \mathbb{C}\colon\ a < |z| < b$ for some $0 \leq a < b$
2. the punctured complex plane $\mathbb{C} \setminus \{0\}$
3. more generally, $\mathbb{C}$ punctured at a finite nonempty set of points
4. even more generally, any nonempty open region with a nonempty finite set of points removed.

Intuitively, being simply connected means that Ω doesn't have any holes, i.e., doesn't wrap around any points of $\mathbb{C}$ not contained in Ω.

We then have the following principle:

Principle 2.2. If Ω is simply-connected, and f is holomorphic in Ω, then f has a primitive in Ω. Hence for any loop γ contained in Ω, we have $\displaystyle\int_\gamma f(z)dz = 0$.

Remark 2.3. In fact, the converse is true. I.e., if every holomorphic function on Ω has a primitive, then Ω is simply-connected.

In practice, you can only use this principle only if you know which regions are simply-connected. *You can use Principle 2.2 whenever Ω is convex* because we proved Theorem 1.2. Furthermore, I mentioned the following in class:

Fact 2.4. If R denotes a (closed) ray in $\mathbb{C}$, then the complement $\mathbb{C} \setminus R$ is simply-connected.

This is important for defining the logarithm. This says that if R is a ray containing the origin, then since the function $\dfrac{1}{z}$ is holomorphic on $\Omega := \mathbb{C} \setminus R$, it has a primitive on Ω. We will come back to this concept toward the end of Chapter 3.

3 A general criterion for having existence of primitives

Here is some material that I did NOT discuss in class. It's optional, because I already told you that you can use the fact that $\mathbb{C} \setminus R$ is simply-connected if you want to do so in calculations. But for those who don't like taking statements on faith, here's a simple proof in this case.

Definition 3.1. If Ω is an open subset of $\mathbb{C}$, we say that Ω is *holomorphically simply-connected* or *holomorphically sc* if every holomorphic function f on Ω has a primitve on Ω.

Remark 3.2. If you accept Principle 2.2 and Remark 2.3, then then being holomorphically sc is equivalent to being simply-connected. As well, Theorem 1.2 says that all convex open sets are holomorphically sc.

Theorem 3.3. *Suppose Ω_1 and Ω_2 be two holomorphically sc open sets such that $\Omega_1 \cap \Omega_2$ is connected. Then $\Omega := \Omega_1 \cup \Omega_2$ is holomorphically sc.*

Proof. Let f be an arbitrary holomorphic function on Ω, and let F_1 and F_2 be primitives on Ω_1 and Ω_2, respectively. If $\Omega_1 \cap \Omega_2$ is empty, then we are done. Otherwise, the function $F_1 - F_2$, defined on $\Omega_1 \cap \Omega_2$ has derivative 0, so it is constant, call it C (this is by Corollary 3.4 in Chapter 1 of S-S). It follows that $F_2 + C$ is a primitive of f on Ω_2 that agrees with F_1 on $\Omega_1 \cap \Omega_2$.

We therefore defined a function F on Ω by $F(z) = F_1(z)$ for $z \in \Omega_1$ and $F(z) = F_2(z) + C$ for $z \in \Omega_2$. The agreement on $\Omega_1 \cap \Omega_2$ ensures that F is well-defined. Furthermore, as derivatives can be taken locally, it follows that F is holomorphic and is a primitive of f. Thus we are done. □

Corollary 3.4. *If R is a ray in $\mathbb{C}$, then $\mathbb{C} \setminus R$ is holomorphically sc.*

Proof. By symmetry, the proof reduces to the case where R is the negative real axis. In this case, we define $\Omega_1 = \{z \in \mathbb{C} : \Im(z) > 0\}$, $\Omega_2 = \{z \in \mathbb{C} : \Re(z) > 0\}$, and $\Omega_3 = \{z \in \mathbb{C} \colon \Im(z) < 0\}$.

Now each Ω_i is convex, hence holomorphically sc. As $\Omega_1 \cap \Omega_2$ is convex and hence connected, we find that $\Omega_1 \cup \Omega_2$ is holomorphically sc. Finally, as $\Omega_3 \cap (\Omega_1 \cup \Omega_2) = \Omega_3 \cap \Omega_2$ is also connected, we find that $(\Omega_1 \cup \Omega_2) \cup \Omega_3 = \mathbb{C} \setminus R$ is holomorphically sc. □

4 One rigorous definition of interior

This section is also *optional*, but if you want things to be more precise, it could be helpful. For now, you really only need the Facts and Theorems mentioned above, and to understand *intuitively* the notions of interior and simple-connectedness.

Definition 4.1. If γ is a loop in $\mathbb{C}$, then the *interior* of γ or the *region bounded by* γ is the set of $z \in \mathbb{C} \setminus \gamma$ such that

$$\int_\gamma \frac{1}{w - z} dw \neq 0.$$

With this definition of interior and hence of simply-connected, Remark 2.3 is easy to prove:

Proposition 4.2. *If Ω is holomorphically sc, and γ is a loop in Ω, then the interior of γ (via Definition 4.1) is in Ω.*

Proof. Suppose otherwise, i.e., that z is in the interior of γ but not in Ω. Then $\frac{1}{w - z}$ is holomorphic on Ω as a function of w, so it must have a primitive on Ω. As γ is contained in Ω, we therefore have $\int_\gamma \frac{1}{w - z} dw = 0$, contradicting the claim that z was in the interior of γ. □

Furthermore, with Definition 4.1, Principle 1.1 implies Principle 2.2. The reason I didn't just use Definition 4.1 is because it's hard to rigorously prove Principle 1.1 using it.