Math 113 Homework 8

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There are five problems due Wednesday, April 10.

1. Let R be a commutative ring.
   (a) Let $a \in R$, and set $aR = \{ar \mid r \in R\}$. Prove that aR is an ideal. (This is known as a *principal ideal.*)
   (b) If I and J are ideals of R, show that $I \cap J$ is an ideal.
   (c) In the case that $R = \mathbb{Z}$, $I = n\mathbb{Z}$, and $J = m\mathbb{Z}$, what is $I \cap J$?
2. Let R be a non-trivial commutative ring.
   (a) Prove that if $I \subseteq R$ is an ideal, then $1_R \in I$ if and only if $I = R$.
   (b) Prove that R is a field if and only if its only ideals are $\{0\}$ and R.
3. Let $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$.
   (a) Prove that $\mathbb{Q}[\sqrt{2}]$ is a subring of $\mathbb{C}$,
   (b) Prove that it is in fact a field.
4. Let R be a ring. We say that $r \in R$ is *idempotent* if $r^2 = r$. Show that if R is a ring in which every element is idempotent, then R is commutative, and $r + r = 0_R$ for all $r \in R$.
5. Let R be a commutative ring. We say that $r \in R$ is *nilpotent* if there is $n \in \mathbb{N}$ such that $r^n = 0_R$.
   (a) Show that the set of nilpotent elements of R forms an ideal.
   (b) Find the set of nilpotent elements of $R = \mathbb{Z}/2700\mathbb{Z}$.