Math 113 Homework 6

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There are six problems due Monday, March 18.

Let G be a group and $H \subseteq G$ a subgroup. Given $g \in G$, let

$$gHg^{-1} := \{ghg^{-1} \mid h \in H\}.$$

Notice that gHg^{-1} is a subgroup of G. Letting $Sub(G)$ denote the set of all subgroups of G, this construction defines an action of G on $Sub(G)$.

1. Let Sym_3 act on $Sub(Sym_3)$ as described above. Describe all orbits of this action. Calculate the stabilizer subgroups of the subgroup generated by (12) and of the subgroup generated by (123). [Hint: remember that you determined the set $Sub(Sym_3)$ in the previous homework.]

2. (a) Prove that $Alt_n \subseteq Sym_n$ is a normal subgroup.

   (b) There are seven conjugacy classes in Sym_5. How many of them are contained in Alt_5?

3. Let G be a group and $H \subseteq G$ a subgroup. Define the *right cosets* of H in G as subsets of the form

   $$Hg = \{h * g \mid h \in H\}.$$

   Prove that H is normal if and only if $Hg = gH$ for all $g \in G$. Using this or otherwise, prove that any subgroup of index two must be normal. (Hint: distinct left cosets are disjoint, and index 2 means there are exactly two left cosets. Same for right cosets.)

4. Let $n \in \mathbb{N}$ and $n > 2$. Determine the number of conjugacy classes of the dihedral group D_n.

5. Determine all subgroups of D_4. Which ones are normal?

6. Let B denote the set of upper triangular real invertible 2×2 matrices, which is a subgroup of $GL_2(\mathbb{R})$. Consider the following subgroups of B:

$$T = \left\{ \begin{matrix} a & 0 \\ 0 & b \end{matrix} \mid a, b \in \mathbb{R}^\times \right\},$$

$$U = \left\{ \begin{matrix} 1 & a \\ 0 & 1 \end{matrix} \mid a \in \mathbb{R} \right\}.$$

   (a) Prove that U is normal in B.
   (b) Prove that T is not normal in B.
   (c) Prove that $f \colon T \to B/U$ sending $t \in T$ to the coset tU is an isomorphism.
   (d) Explain why B is not isomorphic to $T \times U$.

1 Extra Practice Problems

7. For $x \in \mathbb{R}^2$, let $Rot(x) \subseteq Isom(\mathbb{R}^2)$ denote the set of all rotations about x. Let $X \subseteq \mathbb{R}^2$ have the property that $Rot(x) \subseteq Sym(X)$ (in other words, X is symmetric around x). Prove that $Sym(X)$ contains a reflection. Is it true that $Sym(\{x\}) = Sym(X)$?

8. Let $\mathbb{Q}/\mathbb{Z}$ denote the quotient of $(\mathbb{Q}, +)$ by its subgroup $\mathbb{Z}$. Find a subset $X \subseteq \mathbb{R}^2$ such that $Sym(X) \cong \mathbb{Q}/\mathbb{Z}$. (Hint: $\mathbb{R}/\mathbb{Z}$ is isomorphic to the group of rotational symmetries of the circle.)

9. Prove that if every subgroup of G is normal, then elements of coprime order commute. (Hint: $x, y \in G$ commute if and only if $x^{-1}y^{-1}xy = e$.)

10. Determine all possible orders of elements in Sym_5.

11. Prove that the intersection of two normal subgroups is normal.