Math 113 Homework 4

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There are five problems, due Wednesday, February 27.

1. Define an action of Sym_3 on $\mathbb{R}^3$ as follows. For $\sigma \in Sym_3$ and $v = (x_1, x_2, x_3) \in \mathbb{R}^3$, let

$$\sigma(v) = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}).$$

   (a) Show that the subspace $V = \{(x_1, x_2, x_3 \mid x_1 + x_2 + x_3 = 0\}$ has the property that if $v \in V$, then $\sigma(v) \in V$.

   (b) Can you find a one-dimensional vector subspace of $\mathbb{R}^3$ that has this same property?

2. Let G be a finite group with 20 elements. Let S be a set with 15 elements. Does there exist a transitive action of G on S?

3. For a positive integer n, let $\phi(n)$ denote the number of positive integers from 1 to n that are relatively prime to n. For example, $\phi(1) = \phi(2) = 1$, $\phi(3) = \phi(4) = 2$, $\phi(5) = 4$, etc.

   (a) Compute $\phi(10)$.

   (b) Let $(\mathbb{Z}/n\mathbb{Z})^\times$ denote the subset of elements of $\mathbb{Z}/n\mathbb{Z}$ that have a multiplicative inverse, where $n > 1$. Show that $((\mathbb{Z}/n\mathbb{Z})^\times, \times)$ is a group with $\phi(n)$ elements.

   (c) Using the previous part, prove that if a is relatively prime to n, then $a^{\phi(n)} - 1$ is divisible by n.

4. Let $n > 1$. Find a faithful action of the group $(\mathbb{Z}/n\mathbb{Z}^\times, \times)$ on the group $(\mathbb{Z}/n\mathbb{Z}, +)$.

5. Is there a group homomorphism from $(\mathbb{Z}/20\mathbb{Z}, +)$ to $(\mathbb{Z}/21\mathbb{Z}, +)$? If so, how many are there?