Math 113 Homework 2

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There are five problems, due Wednesday, February 20.

1. Let $(G, *)$ be a group. Define the center $Z(G)$ to be the subset

$$Z(G) = \{g \in G \mid g * h = h * g \, \forall h \in G\}$$

   Prove that $Z(G)$ is a subgroup of G.

2. Let H, K be two subgroups of G such that $H \cup K$ is also a subgroup of G. Show that either $H \subseteq K$ or $K \subseteq H$.

3. Let $a \in \mathbb{Z}/n\mathbb{Z}$. Show that a generates $\mathbb{Z}/n\mathbb{Z}$ (under addition) if and only if a is coprime to n.

4. Let $\phi \colon G \to H$ be a homomorphism, and suppose $g \in G$ has order k. Show that the order of $\phi(g)$ divides k.

5. Let $(G, *)$ and $(H, \circ)$ be groups, and suppose that G is generated by the set $\{x_1, \cdots, x_n\}$. Let ϕ and ψ be two group homomorphisms from $(G, *)$ to $(H, \circ)$. Show that if $\phi(x_i) = \psi(x_i)$ for $i = 1, \cdots, n$, then ϕ and ψ are the same homomorphism (i.e., $\phi(g) = \psi(g)$ for all $g \in G$).

   Remark. This proves the important fact that a homomorphism is completely deter-mined by what it does to a generating set. [CAUTION: it does not follow, as in the case of linear algebra, that we can define a homomorphism simply byspecifying where to send the generators; one has to be careful about possiblerelations the generators may satisfy]