Math 113 Homework 11

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There are six problems due Saturday, May 4.

For p prime, let $\mathbb{F}_p$ denote $\mathbb{Z}/p\mathbb{Z}$. Remember that it is a field.

1. Let R be a UFD in which the units, together with 0, form a subring $F \subseteq R$. Show that F is in fact a subfield of R and if $F \neq R$, then R contains infinitely many non-associated primes. (Hint: mimic the usual argument for proving there are infinitely many primes in $\mathbb{Z}$.)

2. Determine all irreducible polynomials of degrees 2 and 3 in $\mathbb{F}_2[x]$.

3. For each of the following ideals (written in the notation of HW 9 Question 4(a)), say whether they are prime, maximal (hence also prime), or neither:

   (a) $(x^2 + 1) \subseteq \mathbb{C}[x]$

   (b) $(x^2 + 1) \subseteq \mathbb{R}[x]$

   (c) $(4, 2x - 1) \subseteq \mathbb{Z}[x]$

4. Consider the ring $R = \mathbb{Z}[\sqrt{-5}]$.

   (a) Find the group of units in R.

   (b) Show that the element $2 \in \mathbb{Z}[\sqrt{-5}]$ is irreducible. [Hint: think about the absolute value of a general element $a + b\sqrt{-5}$ for $a, b \in \mathbb{Z}$, and remember that $|xy| = |x||y|$ for $x, y \in \mathbb{C}$.]

5. Show that every ideal in $\mathbb{Z}$ is principal. [Hint: let I be an ideal, assume $I \neq \{0\}$, and let m be the smallest positive element of I. For every $n \in I$, use the Remainder Theorem to show that m divides n.]

6. Let R be an integral domain.

(a) Let $a, b \in R$, and suppose that (a, b) is principal. Show that a and b have an HCF (in the sense of the definition on p.64 of the notes). [Hint: let d be such that $dR = (d) = (a, b)$.]

(b) Show that the ideal $(2, 1+\sqrt{-5}) \subseteq \mathbb{Z}[\sqrt{-5}]$ is prime but not principal.

Remark: 6(b) shows that in a PID, any two elements have an HCF. This is one step toward proving that PID's are UFD's.