Math 113 Homework 9

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There are six problems due Saturday, November 30.

1. Let R be a commutative ring. Recall that there is a unique homomorphism from $\mathbb{Z}$ to R. For two rings A and B, let $\operatorname{Hom}(A, B)$ denote the set of ring homomorphisms from A to B.

   (a) Give an example of R for which $\operatorname{Hom}(R, \mathbb{Z})$ is empty.

   (b) Give an example of R for which $\operatorname{Hom}(R, \mathbb{Z})$ is infinite.

   (c) Prove that the set $\operatorname{Hom}(\mathbb{Z}[x], R)$ can be naturally put into bijection with the set R [Hint: where does x go?]

2. Is there an integral domain containing exactly 10 elements?

3. Let R be an integral domain of characteristic p. Consider the map $\phi\colon R \to R$ sending x to x^p.

   (a) Show that ϕ is a ring homomorphism.

   (b) Show that ϕ is an automorphism if R is finite.

   (c) Find the image of ϕ when $R = \mathbb{F}_p[x]$.

4. Show that $\mathbb{Q}[\sqrt{2}, \sqrt{3}] = \mathbb{Q}[\sqrt{2}+\sqrt{3}]$. [Hint: show this by showing that if T is a subring of $\mathbb{C}$ containing $\mathbb{Q}$, then T contains $\sqrt{2}$ and $\sqrt{3}$ iff it contains $\sqrt{2} + \sqrt{3}$.]

5. Determine whether the following elements are associate in the given ring:

   (a) $a = 2x - 14$ and $b = 3x - 21$ in $R = \mathbb{Q}[x]$.

   (b) $a = x - 7$ and $b = 3x - 21$ in $R = \mathbb{Z}[x]$.

   (c) $a = 8$ and $b = 9$ in $R = \mathbb{Z}[1/6]$.

(d) $a = 4$ and $b = 9$ in $R = \mathbb{Z}[1/2]$.

(e) $a = x^2 - 2x + 3$ and $b = x^2 + x - 4$ in $R = \mathbb{C}[x]$.

6. Give examples of the following:

   (a) A ring R where 1_R has infinite additive order (i.e., characteristic 0), and R has zero divisors.

   (b) An ideal $I \subseteq \mathbb{C}[X]$ for which there exists $f(X) \in \mathbb{C}[X]$ such that $f(X)^5 \in I$, but $f(X) \notin I$.