Math 113 Homework 8

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There are seven problems due Thursday, November 21.

1. Let I be an ideal in $\mathbb{Z}$.
   (a) Suppose I has positive elements, and let a be the smallest positive element of I. Show that $I = a\mathbb{Z}$ using the Remainder Theorem.
   (b) Using the previous part, show that every ideal of $\mathbb{Z}$ is principal. [Hint: Note that if I is the zero ideal, then it is principal, so assume that I is not the zero ideal. Then note that I has positive elements, so let a be the smallest positive element of I.]
2. Let R be a non-trivial commutative ring.
   (a) Prove that if $I \subseteq R$ is an ideal, then $1_R \in I$ if and only if $I = R$.
   (b) Prove that R is a field if and only if its only ideals are $\{0\}$ and R.
   (c) Let I be an ideal of $\mathbb{R}[x]$ that contains both $x+1$ and $x-1$. Show that $I = \mathbb{R}[x]$.
3. Let $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$.
   (a) Prove that $\mathbb{Q}[\sqrt{2}]$ is a subring of $\mathbb{C}$.
   (b) Prove that it is in fact a field.
4. Let R be a ring. We say that $r \in R$ is *idempotent* if $r^2 = r$. Show that if R is a ring in which every element is idempotent, then R is commutative, and $r + r = 0_R$ for all $r \in R$. [Hint: this is the only problem on this list that's a little tricky.]
5. (a) Find a proper subring of $\mathbb{Q}$ other than $\mathbb{Z}$.
   (b) Show that if $\mathbb{R} \subseteq R$ and $R \subseteq \mathbb{C}$, then R is equal to either $\mathbb{R}$ or $\mathbb{C}$.

6. Which of the following sets are ideals in the given ring?

   (a) $\{p(x,y) \mid p(x,x) = 0\} \subseteq \mathbb{C}[x,y]$
   (b) $\{p(x,y) \mid p(x,y) = p(y,x)\} \subseteq \mathbb{C}[x,y]$
   (c) $\{p(x) \mid p \text{ has no real roots}\} \subseteq \mathbb{C}[x]$

7. Let R be a commutative ring with unity.

   (a) Let $X \subseteq R$ be an arbitrary subset. Prove that there exists an ideal $I \subseteq R$ containing X with the following property: if J is an ideal and $X \subseteq J$, then $I \subseteq J$. (We call I the *ideal generated by* X, and denote it $(X) \subseteq R$.) [Hint: define I to be the set of all finite linear combinations of elements of X with coefficients in R. Then it shouldn't be hard to show that I is an ideal, and show that any such J contains I.]
   (b) If $m, n \in \mathbb{Z}$, when is the ideal generated by $\{m, n\}$ equal to all of $\mathbb{Z}$?