Math 113 Homework 7

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There are five problems due Saturday, November 9.

1. Show that if m and n are relatively prime, then $\mathbb{Z}/(mn)\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ (as groups under addition). [Hint: for which $a \in \mathbb{Z}$ is $([a],[a]) \in \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ the identity? You may use the fact that $LCM(m,n) = mn$.]

2. Let G be an abelian group, and let G_p be the set of elements of order a power of p. Show that G_p is a subgroup of G.

3. Up to isomorphism, how many abelian groups of size 32 are there? [Hint: use Corollary 3.95 of the notes]

4. Let R be a commutative ring.
   (a) If I and J are ideals of R, show that $I \cap J$ is an ideal.
   (b) In the case that $R = \mathbb{Z}$, $I = n\mathbb{Z}$, and $J = m\mathbb{Z}$, what is $I \cap J$?

5. Let R be a commutative ring. We say that $r \in R$ is *nilpotent* if there is $n \in \mathbb{N}$ such that $r^n = 0_R$.
   (a) Show that the set of nilpotent elements of R forms an ideal [Hint: product is easy. For sums, use the binomial theorem.]
   (b) Find the set of nilpotent elements of $R = \mathbb{Z}/2700\mathbb{Z}$.

Extra Practice Problems

6. Show that $0_R \times a = 0_R$ for any $a \in R$.

7. Show that $-1_R \times a = -a$, where $-a$ denotes the additive inverse of a.

8. Show that if $(R, +, \times)$ satisfies all the axioms for a ring except possibly for the axiom of commutativity of addition, then addition is commutative. [Hint: Note that the last two problems don't require addition to be commutative, and show that the additive inverse of $a + b$ is $-a - b$.]

9. Let R be a ring. We say that $r \in R$ is *idempotent* if $r^2 = r$. Show that if R is a ring in which every element is idempotent, then R is commutative, and $r + r = 0_R$ for all $r \in R$.