

Math 113 Homework 6

David Corwin

October 20, 2019

There are six problems, due Tuesday, October 29.

Given $g \in G$, let

$$gHg^{-1} := \{ghg^{-1} \mid h \in H\}.$$

Show that this is a subgroup of G isomorphic to H . Notice that gHg^{-1} is a subgroup of G . Letting $\text{Sub}(G)$ denote the set of all subgroups of G , this construction defines an action of G on $\text{Sub}(G)$ (you don't have to prove this).

1. Let Q be the quaternion group, and let Q act on $\text{Sub}(Q)$ as described above.
 - (a) Describe all orbits of this action. [Hint: isomorphic groups have the same size, so this question is not too difficult.]
 - (b) Calculate the stabilizer subgroups of the subgroup generated by i and of the subgroup generated by $-k$. [Hint: remember that you determined the set $\text{Sub}(Q)$ in a previous homework.]
2.
 - (a) Prove that the stabiliser of $\text{Alt}_n \in \text{Sub}(\text{Sym}_n)$ is Sym_n . [Hint: show that it is a normal subgroup.]
 - (b) There are seven conjugacy classes in Sym_5 [see the HW 5 solutions for a description of them]. How many of them are contained in Alt_5 ?
3. Let $n \in \mathbb{N}$ and $n > 2$. Determine the number of conjugacy classes of the dihedral group D_n . [Hint: this works differently depending on whether n is even or odd.]
4. Let G be a group and $H \subseteq G$ a subgroup. Define the *right cosets* of H in G as subsets of the form

$$Hg = \{h * g \mid h \in H\}.$$

- (a) Prove that H is normal if and only if $Hg = gH$ for all $g \in G$.
 - (b) Using this or otherwise, prove that any subgroup of index two must be normal. (Hint: distinct left cosets are disjoint, and index 2 means there are exactly two left cosets. Same for right cosets.)
5. Determine all subgroups of D_4 . Which ones are normal?
6. Let B denote the set of upper triangular real invertible 2×2 matrices, which is a subgroup of $GL_2(\mathbb{R})$. Consider the following subgroups of B :

$$T = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{R}^\times \right\},$$

$$U = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{R} \right\}.$$

- (a) Prove that U is normal in B .
- (b) Prove that T is not normal in B .
- (c) Prove that $f: T \rightarrow B/U$ sending $t \in T$ to the coset tU is an isomorphism.
- (d) Explain why B is not isomorphic to $T \times U$.

1 Extra Practice Problems

- 7. For $x \in \mathbb{R}^2$, let $Rot(x) \subseteq Isom(\mathbb{R}^2)$ denote the set of all rotations about x . Let $X \subseteq \mathbb{R}^2$ have the property that $Rot(x) \subseteq Sym(X)$ (in other words, X is symmetric around x). Prove that $Sym(X)$ contains a reflection. Is it true that $Sym(\{x\}) = Sym(X)$?
- 8. Let \mathbb{Q}/\mathbb{Z} denote the quotient of $(\mathbb{Q}, +)$ by its subgroup \mathbb{Z} . Find a subset $X \subseteq \mathbb{R}^2$ such that $Sym(X) \cong \mathbb{Q}/\mathbb{Z}$. (Hint: \mathbb{R}/\mathbb{Z} is isomorphic to the group of rotational symmetries of the circle.)
- 9. Prove that if every subgroup of G is normal, then elements of coprime order commute. (Hint: $x, y \in G$ commute if and only if $x^{-1}y^{-1}xy = e$.)
- 10. Determine all possible orders of elements in Sym_5 .
- 11. Prove that the intersection of two normal subgroups is normal.