Math 113 Homework 5

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There are four problems, due Thursday, October 10.

1. (a) Let $\sigma = (1234)$ and $\tau = (126)(354)$, both in Sym_6. Find $\tau\sigma\tau^{-1}$.
   (b) Show that if τ is any element of Sym_6, and $\sigma = (1234)$, then $\tau\sigma\tau^{-1}$ is a 4-cycle.
   (c) Explain why, if S is any set, σ is a k-cycle in $\Sigma(S)$, and τ is any element of $\Sigma(S)$, then $\tau\sigma\tau^{-1}$ is a k-cycle. [Hint: suppose that $\sigma = (a_1 a_2 \cdots a_k)$. Then show that $\tau\sigma\tau^{-1} = (\tau(a_1)\tau(a_2)\cdots\tau(a_k))$.]

2. (a) How many elements of Sym_5 have order 3?
   (b) Determine all possible orders of elements in Sym_5.

3. Let $G = Sym_3$ and $S = \{1, 2, 3, 4, 5, 6\}$. Define an action of G on S as follows. The action corresponds to a homomorphism $\varphi\colon G \to \Sigma(S)$ sending $(12) \in G$ to $(12)(45) \in Sym_6 = \Sigma(S)$ and $(123) \in G$ to $(123) \in \Sigma(S)$. (Note that by Question 5 on HW 4, this uniquely specifies the action.)
   (a) Find $\varphi((23))$ and $\varphi((132))$.
   (b) For each element of S, find its stabilizer as a subgroup of G.

4. Define an action of $G = Sym_3$ on $S = \mathbb{R}^3$ as follows. For $\sigma \in Sym_3$ and $v = (x_1, x_2, x_3) \in \mathbb{R}^3$, set
   $$\phi(\sigma, v) = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}).$$
   (a) Show that the subspace $V = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 0\}$ has the following property: if $v \in V$, then $\phi(\sigma, v)$ is also in V.
   (b) Explain why the previous fact shows that Sym_3 is isomorphic to a subgroup of $\mathrm{GL}_2(\mathbb{R})$.
   (c) Can you find a line (one-dimensional subspace) that has the same property?