Math 113 Homework 3

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There are five problems, due Tuesday, September 24.

1. Let $(G, *)$ be a group. Define the center $Z(G)$ to be the subset

$$Z(G) = \{g \in G \mid g * h = h * g \, \forall h \in G\}$$

   (a) Prove that $Z(G)$ is a subgroup of G.
   (b) What is the center of $(M_n(\mathbb{R}), +)$?
   (c) What is the center of $(\mathrm{GL}_2(\mathbb{R}), \times)$?

2. Let

$$\frac{1}{3}\mathbb{Z} := \{r \in \mathbb{Q} \mid 3r \in \mathbb{Z}\}.$$

   (a) Show that $\frac{1}{3}\mathbb{Z}$ is a subgroup of $(\mathbb{Q}, +)$.
   (b) What are the left cosets of $H = \mathbb{Z}$ in $G = (\frac{1}{3}\mathbb{Z}, +)$? I.e., give a representative for each left coset, and state how many left cosets there are.

3. Let H, K be two subgroups of G such that $H \cup K$ is also a subgroup of G. Show that either $H \subseteq K$ or $K \subseteq H$. [Hint: Suppose that $H \not\subseteq K$, so that there exists $a \in H \setminus K$. Show that for any $b \in K$, we have $a * b \in H$. Use that to show that $K \subseteq H$.]

4. Let $a \in \mathbb{Z}$ and $n \in \mathbb{N}$.

   (a) Let $a\mathbb{Z}/n\mathbb{Z}$ denote the set of equivalence classes modulo n that contain a multiple of a. Show that $a\mathbb{Z}/n\mathbb{Z}$ is a subgroup of $\mathbb{Z}/n\mathbb{Z}$.
   (b) When is this subgroup the whole group?

5. Let $\phi \colon G \to H$ be a homomorphism, and suppose $g \in G$ has order k. [This means that $g^k = e$, and $g^m \neq e$ for any positive integer m less than k.]

   (a) Show that the order of $\phi(g)$ divides k.

   (b) Show that if ϕ is injective, then the order of $\phi(g)$ is k.