# Math 113 Homework 10

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There are five problems due Saturday, December 7.

For $p$ prime, let $\mathbb{F}_p$ denote $\mathbb{Z}/p\mathbb{Z}$. Remember that it is a field.

1. Let R be a UFD in which the units, together with 0, form a subring $F \subseteq R$. Show that $F$ is in fact a subfield of $R$ and if $F \neq R$, then $R$ contains infinitely many non-associated primes. (Hint: mimic the usual argument for proving there are infinitely many primes in $\mathbb{Z}$.)

2. Determine all irreducible polynomials of degrees 2 and 3 in $R = \mathbb{F}_2[x]$. [Remark: when you take the quotient of $R$ by these polynomials, you get finite fields of sizes 4 and 8, respectively.]

3. For each of the following ideals (written in the notation of HW 8 Question 7(a)), say whether they are prime, maximal (hence also prime), or neither:

   (a) $(x^2 + 1) \subseteq \mathbb{C}[x]$

   (b) $(x^2 + 1) \subseteq \mathbb{R}[x]$

   (c) $(\{4, 2x - 1\}) \subseteq \mathbb{Z}[x]$

   (d) $(2x - 1) \subseteq \mathbb{Z}[x]$

4. Consider the ring $R = \mathbb{Z}[\sqrt{-5}]$.

   (a) Find the group of units in $R$ [Hint: if $z = a + b\sqrt{-5} \in R$, note that $|z|^2$ is an integer, and that $|z_1 z_2| = |z_1||z_2|$.]

   (b) Show that the element $2 \in \mathbb{Z}[\sqrt{-5}]$ is irreducible. [Same hint.]

5. Let $R$ be an integral domain.

(a) Let $a, b \in R$, and suppose that $(a, b)$ is principal. Show that $a$ and $b$ have an HCF/GCD (in the sense of the definition in Section 5.1.3 of the notes). [Hint: let $d$ be such that $dR = (d) = (a, b)$.]

(b) Show that the ideal $(2, 1+\sqrt{-5}) \subseteq \mathbb{Z}[\sqrt{-5}]$ is prime but not principal.