Math 113 Homework 1

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Problems 1-7 should be written up (on paper or LaTeX) and handed in via Gradescope on Tuesday, September 10.

The words "map" and "function" mean the same thing.

1. Let S be a set with three elements, and let T be a set with five elements.
   (a) Find the number of functions from S to T.
   (b) Find the number of bijections from S to itself.
2. Let S be a set with two elements, and let T be a set with three elements. As discussed in class, there are 9 functions from S to T. How many of these are injective?
3. Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 + 5x$. Is f injective? Prove your answer.
4. Let A, B be sets, with A nonempty. Show that $f\colon A \to B$ is injective if and only if it has a *left inverse*, i.e., a map $g\colon B \to A$ such that $g \circ f = \mathrm{id}_A$. [Hint: try to define a function g. Explain why the property of injectivity means that g is actually a function!]
5. Find a set S, and function from S to itself that is injective but not surjective.
6. Consider the relation on $\mathbb{R}$ for which $x \sim y$ if and only if $x - y$ is an integer. Show that this is an equivalence relation.
7. Let $S = \{1, 2, 3, 4, 5, 6\}$ and $S_1 = \{1, 2, 3, 4\}$ and $S_2 = \{3, 5\}$. Define a relation on S as follows. For $m, n \in S$, we declare $m \sim n$ if and only if m and n are both in S_1 or are both in S_2 (remember, 'or' is not the same as 'xor,' so it's fine if they are both in both). Does this give an equivalence

relation? If not, which of the properties of an equivalence relation does it satisfy?

In other words, $\sim$ corresponds to the subset $U \subseteq S \times S$ of $(m, n) \in S \times S$ for which $m, n \in S_1$ or $m, n \in S_2$.

Optional Problems

Look at the problems below and make sure you understand how you might do them. You don't need to write them up. But it gives you an idea of the kind of concepts you need to understand in this course. You can also use them later on as exam practice if you like.

8. Let C be the set of numbers appearing on a standard clock face. We define a relation $U \subseteq C \times C$ such that $(a, b) \in U$ if $a = b$, or a is next to b on the clock. Is U an equivalence relation? If not, which properties does it satisfy?

9. (a) Let $f \colon S \to S'$ be a map of sets. Define a relation $U \subseteq S \times S$ where $(x, y) \in U$ iff $f(x) = f(y)$. Show that U is an equivalence relation.

   (b) In the case of $f \colon \mathbb{R}^2 \to \mathbb{R}$ where $f(x, y) = x - y$, describe the equivalence classes geometrically.

10. Let S be a set and $U \subseteq S \times S$ an equivalence relation on S. If $a, b \in S$, and $b \in [a]$, prove that $[b] = [a]$.

11. In the notation of the previous problem that if $[a]$ and $[b]$ have a nonempty intersection, then $[a] = [b]$ (as subsets of S).

12. Let S and T be two sets and $f \colon S \to T$ a function. Under what condition is it true that f is injective if and only if it's surjective?

13. Let S and T be sets. If $U \subseteq S$ and $V \subseteq T$, then $U \times V \subseteq S \times T$. Are all subsets of $S \times T$ of this form? If yes, prove it, but if not, find a counterexample.

14. Consider the function $f \colon \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x^3 + 3x$. Is it injective? Is it surjective? Justify your answer. [Hint: think about its derivative.]

15. Let S be a set with five elements.

    (a) Find the number of functions from S to itself.

    (b) Find the number of bijections from S to itself.