Comments, corrections, and related references welcomed, as always!

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HOMOMORPHIC IMAGES OF PRO-NILPOTENT ALGEBRAS

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ABSTRACT. It is shown that any finite-dimensional homomorphic image of an inverse limit of nilpotent not-necessarily-associative algebras over a field is nilpotent. More generally, this is true of algebras over a general commutative ring k, with "finite-dimensional" replaced by "of finite length as a k-module".

These results are obtained by considering the multiplication algebra $M(A)$ of an algebra A (the associative algebra of k-linear maps $A \to A$ generated by left and right multiplications by elements of A), and its behavior with respect to nilpotence, inverse limits, and homomorphic images.

As a corollary, it is shown that a finite-dimensional homomorphic image of an inverse limit of finite-dimensional *solvable* Lie algebras over a field of characteristic 0 is solvable.

It is also shown by example that *infinite*-dimensional homomorphic images of pro-nilpotent algebras can have properties far from those of nilpotent algebras; in particular, properties that imply that they are not residually nilpotent.

Several open questions and directions for further investigation are noted.

1. General definitions.

Throughout this note, k will be a commutative associative unital ring, and an "algebra" will mean a k-algebra; i.e., a k-module A given with a k-bilinear multiplication $A \times A \to A$, not necessarily associative or unital.

A right, left, or 2-sided ideal of a k-algebra A will mean a k-submodule closed under left, respectively right multiplication by elements of A, respectively both.

Recall that if A is a *nonunital associative* algebra contained in a *unital* associative algebra A', then the identity

(1) $\qquad (1+x)(1+y) \ = \ 1+(x+y+xy) \quad (x, y \in A)$

which holds in A' motivates one to define, on A, the operation of *quasimultiplication*,

(2) $\qquad x * y \ = \ x+y+xy.$

This is again associative, and has 0 as identity element; an element $x \in A$ is called *quasiinvertible* if there exists $y \in A$ such that $x * y = y * x = 0$; equivalently, if $1+x$ is invertible in the submonoid $\{1+u \mid u \in A\}$ of A' with respect to ordinary multiplication. In particular, every nilpotent element $x \in A$ is quasiinvertible, with quasiinverse $-x+x^2-\dots+(-x)^n+\dots$. The Jacobson radical of A is the largest ideal consisting of quasiinvertible elements; so an associative algebra is Jacobson radical if and only if every element is quasiinvertible. We shall write "Jacobson radical" and "radical" interchangeably in this note, using the former mainly in statements of results. We shall only use these terms in reference to associative algebras.

If A is a not-necessarily-associative algebra, let us write $\mathrm{Endo}(A)$ for the associative unital k-algebra of all endomorphisms of A as a k-module. (Since $\mathrm{End}(A)$ generally denotes the monoid of algebra endomorphisms of A, we use this slightly different symbol for its algebra of module endomorphisms.) For every $x \in A$, we define the left and right multiplication maps $l_x,\ r_x \in \mathrm{Endo}(A)$ by

(3) $\qquad l_x(y) \ = \ x\,y, \quad r_x(y) \ = \ y\,x,$

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and denote by $M(A)$ the generally *nonunital* subalgebra of $\mathrm{Endo}(A)$ generated by these maps, as x runs over A; this is called the *multiplication algebra* of A [20, p.14].

An algebra A is called *nilpotent* if for some $n > 0$, all length-n products of elements of A, no matter how bracketed, are zero. We shall see that $M(A)$ is nilpotent if and only if A is nilpotent (not hard to prove, but not quite trivial).

2. Preview of the proof of our main result, and of a counterexample.

If A is a nilpotent algebra, then the associative algebra $M(A)$, being nilpotent, will in particular be radical. Now though the property of being nilpotent is not preserved by inverse limits, that of being radical is, and is likewise preserved under surjective homomorphisms. To use these facts, we have to know how $M(A)$ behaves with respect to homomorphisms and inverse limits.

In general, a homomorphism of algebras $h : A \to B$ does not induce a homomorphism $M(h) : M(A) \to M(B)$; but we shall see that it does if h is surjective, and that $M(h)$ is then also surjective. The need for h to be surjective will not be a problem for us, because if an algebra A is an inverse limit of nilpotent algebras A_i, then by replacing the A_i with appropriate subalgebras, we can get a new system having the same inverse limit A, and such that the new projection maps $A \to A_i$ and connecting maps $A_i \to A_j$ are surjective. Once these conditions hold, we shall find that

$$(4)\qquad M(\varprojlim A_i) \ \subseteq\ \varprojlim M(A_i) \ \subseteq\ \mathrm{Endo}(\varprojlim A_i).$$

Hence, if the A_i are all nilpotent, the elements of $M(\varprojlim A_i)$ will all have quasiinverses in the radical algebra $\varprojlim M(A_i)$, and hence in $\mathrm{Endo}(\varprojlim A_i)$. From this we shall be able to deduce that if B is a homomorphic image of $A = \varprojlim A_i$, then for all $u \in M(B)$, the linear map $1 + u \in \mathrm{Endo}(B)$ is surjective.

If, moreover, B has finite length as a k-module, this surjectivity makes these maps $1 + u$ $(u \in M(B))$ invertible; i.e., it makes the elements u quasiinvertible in $\mathrm{Endo}(B)$. If we could say that they were quasi-invertible in $M(B)$, this would make $M(B)$ radical. We can't initially say that; but we shall find that the quasiinvertibility of these images in $\mathrm{Endo}(B)$ allows us to extend the domain of our map $M(A) \to \mathrm{Endo}(B)$ to a radical subalgebra of $\mathrm{Endo}(A)$ containing $M(A)$. Since a homomorphic image of a radical algebra is radical, we get a radical subalgebra of $\mathrm{Endo}(B)$ containing $M(B)$. Using once more the finite length assumption on B, we will conclude that that subalgebra of $\mathrm{Endo}(B)$ is nilpotent, hence so is $M(B)$; hence so is B, yielding our main result (first paragraph of abstract).

It is curious that in the above development, before assuming that B had finite length, we could conclude that the maps $1+u$ $(u \in M(B))$ were surjective, but not that they were injective. Let me sketch a concrete example (to be given in detail in §8) showing how injectivity can fail, and why surjectivity must nonetheless hold.

Suppose one takes an inverse limit A of nilpotent associative algebras A_i, and divides out by the two-sided ideal (w) generated by an element of the form

$$(5)\qquad w \ =\ y \ -\ x\,y\,z \ =\ (1 - l_x r_z)(y),$$

where $x, y, z \in A$. In the resulting algebra $A/(w)$, let us, by abuse of notation, use the same symbols x, y, z for the images of the corresponding elements of A. Thus, in that algebra we have $y = x\,y\,z$; equivalently, y is annihilated by the operator $1 - l_x r_z$. Hence the latter operator will not be injective if $y \neq 0$ in $A/(w)$, in other words, if $y \notin (w)$ in A.

Can y in fact fail to lie in (w)? Note that we can formally solve (5) for y, getting

$$(6)\qquad y \ =\ w + xwz + x^2wz^2 + \dots\,.$$

In each of the nilpotent algebras A_i of which A is the inverse limit, (6) is literally true, since the images of x and z are nilpotent; so in each of those algebras, the image of y does lie in the image of (w). But as we pass to larger and larger algebras A_i, the number of nonzero terms of (6) can grow without bound, so there is no evident way to express $y \in A$ as a member of the ideal (w); and indeed, we shall see in §8 that for appropriate choice of these algebras and elements, y does not belong to that ideal, so that on $A/(w)$, $1 - l_x r_z$ is non-injective. By the above considerations, this makes $A/(w)$ non-residually-nilpotent; a quicker way to see this is to note that the equations $y = x\,y\,x = x^2 y\,x^2 = \dots$ show that $y \in \bigcap_n (A/(w))^n$, whence the image of y in any nilpotent homomorphic image of $A/(w)$ must be zero.

On the other hand, I claim that whenever x and z are elements of a homomorphic image A/U of an inverse limit A of nilpotent associative algebras A_i, the operator $1-l_x r_z$ will be *surjective.* Given an element $w\in A/U$ which we want to show is in the range of this operator, let us lift $x,\,w,z$ to elements of A, which we will denote by the same symbols. Seeking an element $y\in A$ mapped by $1-l_x r_z$ to w, we get the same formal expression as before, $y=w+xwz+x^2wz^2+\dots$. Again, this sum cannot be evaluated using the algebra operations of A; but it can in each of the A_i, and we find that the resulting elements of the A_i yield, in the inverse limit A, an element y satisfying $w=y-xyz$, as desired.

3. Acknowledgements, and some related literature.

I am indebted to Nazih Nahlus for conjecturing the main result of this note in the case where the A_i are finite-dimensional nilpotent Lie algebras over an algebraically closed field of characteristic 0. I am also grateful to Georgia Benkart, Christian Jensen, Karl H. Hofmann, Greg Marks, Nazih Nahlus, and the referee of this note for helpful comments on earlier drafts.

For related results, by Nahlus and the present author, on homomorphic images of *direct products* of algebras, see [5], [6].

Inverse limits of finite-dimensional Lie groups and Lie algebras are also studied in [13], though with a different emphasis from this note, focusing on homomorphisms continuous in the inverse limit topology.

4. Basic properties of nilpotence.

The condition of nilpotence on a nonassociative algebra A can be characterized in several ways.

In what follows, whenever B and C are k-submodules of A, we understand BC to mean the k-submodule of A spanned by all products bc $(b\in B,\,c\in C)$. Let us define recursively k-submodules $A_{[n]}$ and $A_{(n)}$ $(n=1,2,\dots)$ of any algebra A by

(7) $$A_{[1]} \;=\; A, \qquad A_{[n+1]} \;=\; A\,A_{[n]}+A_{[n]}\,A,$$

(8) $$A_{(1)} \;=\; A, \qquad A_{(n+1)} \;=\; \textstyle\sum_{0<m<n+1} A_{(m)}\,A_{(n+1-m)}.$$

It is easy to see by induction that these yield descending chains of submodules:

(9) $$\text{for } n>0, \quad A_{[n]} \;\supseteq\; A_{[n+1]} \quad\text{and}\quad A_{(n)} \;\supseteq\; A_{(n+1)},$$

and also that

(10) $$\text{for all } n, \quad A_{[n]} \;\subseteq\; A_{(n)}.$$

If A is associative, then $A_{[n]}$ and $A_{(n)}$ clearly coincide, their common value being the submodule of A spanned by all n-fold products, which we shall write A^n. In the next lemma, for an arbitrary algebra A, we apply the latter notation to the associative algebra $M(A)\subseteq \mathrm{Endo}(A)$, defined in §1.

Lemma 1. *If A is an algebra, then the following conditions are equivalent:*

(i) *There exists a positive integer n_1 such that $A_{[n_1]}=\{0\}$.*

(ii) *There exists a positive integer n_2 such that $A_{(n_2)}=\{0\}$.*

(iii) *There exists a positive integer n_3 such that $M(A)^{n_3}=\{0\}$.*

Moreover, if the above equivalent conditions hold, then letting $N_1,\ N_2,\ N_3$ be the least $n_1,\ n_2,\ n_3$ as in those statements, we have

(11) $$N_3 \;=\; \max(1,\,N_1-1), \qquad N_1 \;\le\; N_2 \;\le\; 2^{N_1-2}+1.$$

Proof. We will first establish the stated relations between conditions (i) and (iii), and between N_1 and N_3. Let $l_A\subseteq M(A)$ denote the k-submodule of all left-multiplication operators l_x $(x\in A)$ and r_A the k-submodule of all right-multiplication operators r_x. We claim that

(12) $$\text{for all } n>0, \quad M(A)^{n+1} \;=\; (l_A+r_A)\,M(A)^n.$$

Here "$\supseteq$" is clear. To see "$\subseteq$", note that $M(A)$ consists of all sums of products of one or more elements of l_A+r_A, hence $M(A)^{n+1}$ consists of all sums of products of $n+1$ or more such elements. If such a product has more than $n+1$ such factors, we can, by associativity, group these into $n+1$ subproducts, of which the first is a single factor. Moreover, the assumption $n>0$ assures us that the first of $n+1$ factors is not the only one. So written, our product clearly belongs to $(l_A+r_A)M(A)^n$, giving "$\subseteq$".

Now the recursive step of (7) says that $A_{[n+1]} = (l_A + r_A)A_{[n]}$, so using (12), and induction from the case $n = 1$, one concludes that

(13) for all $n > 0$, $\quad A_{[n+1]} \;=\; M(A)^n(A)$

This gives the equivalence of (i) and (iii) on the one hand, and the initial equality of (11) on the other.

Turning to the submodules $A_{(n)}$, the inclusion (10) yields the implication (ii) $\implies$ (i) and the first inequality of (11). To get the reverse implication and the final inequality of (11), we first note that both hold trivially if $A = \{0\}$, in which case $N_1 = N_2 = 1$. To prove them for nonzero A, in which case any n_1 as in (i), or n_2 as in (ii), must be ≥ 2, it suffices to show that

(14) for $n \geq 2$, $\quad A_{(2^{n-2}+1)} \;\subseteq\; A_{[n]}$.

For $n = 2$, we have equality. Assuming we know (14) for some $n \geq 2$, we look at the definition of $A_{(2^{n-1}+1)}$ as in (8), and note that in each of the summands $A_{(m)}A_{((2^{n-1}+1)-m)}$, one of the indices m or $(2^{n-1}+1) - m$ will be $\geq 2^{n-2}+1$; hence the summand will be contained in $A_{(2^{n-2}+1)}A + A\,A_{(2^{n-2}+1)}$. By inductive hypothesis, this is $\subseteq A_{[n]}A + A\,A_{[n]} = A_{[n+1]}$, as required. $\square$

(If we think of an arbitrarily parenthesized nonassociative product as representing a binary tree of multiplications, the last part of the above proof is essentially a calculation showing that a binary tree with $2^{n-2}+1$ leaves $(n \geq 2)$ must contain a chain with n nodes.)

Definition 2. *An algebra A satisfying the equivalent conditions* (i)-(iii) *of Lemma 1 will be called* nilpotent*.*

Lemma 1 now gives

Corollary 3. *For any algebra A, $M(A)$ is nilpotent if and only if A is nilpotent.* $\square$

(In the sketch in the preceding section, we defined nilpotence in terms of condition (ii) of Lemma 1, as is often done. The verification that this is equivalent to (iii) required the "$2^{n-2}+1$" part of the proof of that lemma, which is why we described it as not quite trivial.)

We end this section with some further observations on nilpotence that will not be needed for our main results.

In the inequality $N_1 \leq N_2$ of (11), we have equality whenever A is associative, by the sentence following (10). For examples where the upper bound $N_2 \leq 2^{N_1-2}+1$ is achieved, take any positive integer n, and consider the (nonassociative) k-algebra A such that

(15) A is free as a k-module on a basis $\{x_1, \dots, x_{n-1}\}$, with multiplication given by $x_m\, x_m \;=\; x_{m+1}$ for $1 \leq m \leq n-2$, and all other products of basis elements equal to zero (including $x_{n-1}\, x_{n-1}$).

It is easy to verify by induction that for every $i \leq n$, $A_{[i]}$ is the submodule spanned by $\{x_i, \dots, x_{n-1}\}$. In particular, $A_{[i]}$ becomes $\{0\}$ starting with $i = n$, so the N_1 of Lemma 1 is n for this algebra. Less obvious, but no harder to verify, is the statement that

(16) for every $i \leq n$, and j with $2^{i-2} < j \leq 2^{i-1}$, $A_{(j)}$ is the submodule spanned by $\{x_i, \dots, x_{n-1}\}$.

Indeed, note that if $i > 1$, and j lies in the above range, then j can be written as the sum of two integers $\leq 2^{i-2}$, but not as the sum of two integers $\leq 2^{i-3}$. Using this fact, and the definitions (8) and (15), one gets (16) by induction on i. So for this algebra, $N_2 = 2^{n-2}+1 = 2^{N_1-2}+1$.

The next lemma shows that Lie algebras behave like associative algebras in this respect.

Lemma 4. (i) *If A is an associative or Lie algebra, then for all positive integers p and q, $A_{[p]}A_{[q]} \subseteq A_{[p+q]}$.*

(ii) *If A is any algebra for which the conclusion of* (i) *holds, then for every positive integer n, $A_{[n]} = A_{(n)}$.*

Proof. For associative algebras, (i) is a weakening of the familiar identity $A^pA^q = A^{p+q}$.

For Lie algebras, let us switch to bracket notation, and note that by anticommutativity, the recursive step of our definition (7) can be written $A_{[n+1]} = [A,\, A_{[n]}]$. This immediately gives (i) for $p = 1$ and arbitrary q. So let $p > 1$, and assume inductively that the result is true for all smaller p. Using the Jacobi identity at the second step below, and our inductive assumption at the third and fourth steps, we compute

(17)
$$\begin{aligned}[A_{[p]},\, A_{[q]}] \;&=\; [\,[A,\, A_{[p-1]}],\, A_{[q]}] \;\subseteq\; [A,\, [A_{[p-1]},\, A_{[q]}]\,] \;+\; [A_{[p-1]},\, [A,\, A_{[q]}]\,]\\ &\subseteq\; [A,\, A_{[p+q-1]}] \;+\; [A_{[p-1]},\, A_{[q+1]}] \;\subseteq\; A_{[p+q]} \;+\; A_{[p+q]} \;=\; A_{[p+q]}.\end{aligned}$$

To get (ii), recall from (10) that $A_{[n]} \subseteq A_{(n)}$ for arbitrary algebras, and note that by definition we have equality when $n = 1$. Thus, it suffices to prove the inclusion $A_{[n]} \supseteq A_{(n)}$ when $n > 1$, inductively assuming this inclusion for smaller n. The inclusion we are to prove is clearly equivalent to the statement that each summand $A_{(m)}A_{(n-m)}$ in the definition of $A_{(n)}$ is contained in $A_{[n]}$. By our inductive hypothesis, $A_{(m)}A_{(n-m)}$ is contained in $A_{[m]}A_{[n-m]}$, and since we are assuming the conclusion of (i), this is contained in $A_{[n]}$. $\square$

5. Properties of $M(A)$.

As recalled above, the multiplication algebra $M(A)$ of any algebra A is the (generally nonunital) subalgebra of the associative algebra $\mathrm{Endo}(A)$ generated by the left and right multiplication operators l_x and r_x, as x ranges over A.

For a general homomorphism of algebras $h : A \to B$, there is no natural way to map $M(A)$ to $M(B)$. For instance, if h is the inclusion of a subalgebra A in an algebra B, such that a central element $x \in A$ becomes noncentral in B, then $l_x = r_x$ in $M(A)$, but the corresponding members of $M(B)$ are distinct. For surjective homomorphisms, however, this problem goes away:

Lemma 5. *If $h : A \to B$ is a surjective homomorphism of algebras, then there exists a unique homomorphism $M(h) : M(A) \to M(B)$ such that*

(18) *for all $x \in A$,* $\quad M(h)(l_x) \ = \ l_{h(x)}$ *and* $\quad M(h)(r_x) \ = \ r_{h(x)}$,

equivalently, such that

(19) *for all $u \in M(A)$ and $a \in A$,* $\quad (M(h)(u))(h(a)) \ = \ h(u(a))$.

Moreover, $M(h)$ is surjective.

Proof. $\mathrm{Ker}(h)$ is an ideal of A, hence it is carried into itself by every map l_x and every map r_x, and thus by every element u of the algebra $M(A)$ generated by such maps. Hence if two elements $a,\ a' \in A$ differ by an element of $\mathrm{Ker}(h)$, so do $u(a)$ and $u(a')$; that is, if $h(a) = h(a')$, then $h(u(a)) = h(u(a'))$; so as $B = h(A)$, we get a well-defined linear map $M(h)(u) : B \to B$ satisfying (19).

It is immediate that $M(h)$ is an algebra homomorphism, and acts by (18) on elements l_x and r_x. It is surjective because it carries the generating set $\{l_x,\, r_x \mid x \in A\}$ of $M(A)$ to the corresponding generating set of $M(B)$. $\square$

It is also immediate that for a composable pair of surjective algebra homomorphisms $h,\ g$, we have $M(hg) = M(h)M(g)$; and that if we write id_A for the identity homomorphism $A \to A$, then $M(\mathrm{id}_A) = \mathrm{id}_{M(A)}$. Thus, M is a functor from the category whose objects are k-algebras, and whose morphisms are *surjective* algebra homomorphisms, to the category of associative k-algebras.

Now suppose we are given an inverse system of k-algebras; i.e., that for some inversely directed partially ordered set I, we are given a family of algebras $(A_i)_{i\in I}$ and algebra homomorphisms $f_{ji} : A_i \to A_j$ $(i \leq j)$, such that $f_{ii} = \mathrm{id}_{A_i}$ for $i \in I$, and $f_{kj}f_{ji} = f_{ki}$ for $i \leq j \leq k$. Recall that the *inverse limit* $\varprojlim_I A_i$ of this system can be constructed (or alternatively, may be defined) as the subalgebra $A \subseteq \prod_I A_i$ consisting of those elements $(a_i)_{i\in I}$ such that $f_{ji}(a_i) = a_j$ for all $i \leq j$. Thus, the projection maps $p_j : A \to A_j$ carrying $(a_i)_{i\in I}$ to $a_j \in A_j$ satisfy

(20) $\quad f_{ji}\, p_i \ = \ p_j \ \ (i \leq j)$.

The algebra A, with these maps, is universal for (20) (see [2, §§7.4-7.5] for motivation and details).

For a general inverse system of algebras A_i, we cannot talk of applying M to the f_{ji} and p_i, since these may not be surjections. (Even if all the f_{ji} are surjective, the resulting p_i may fail to be: [10] [11] [24].) However, given any inverse system of algebras $(A_i)_{i\in I}$, and writing A for its inverse limit, if we replace each A_i with its subalgebra $p_i(A)$, the result will be an inverse system still having inverse limit A, but where the restricted maps f_{ji} and p_i are all surjective. (Actually, surjectivity of the p_i implies surjectivity of the f_{ji}, in view of (20).) Also, of course, if the original algebras A_i were nilpotent, the subalgebras with which we have replaced them will still be. Hence in what follows, we shall often restrict attention to inverse systems where all f_{ji} and p_i are surjective.

Lemma 6. *Let* $(A_i, f_{ji})_{i,j\in I}$ *be an inverse system of* k*-algebras, and* $A = \varprojlim A_i$ *its inverse limit, with projection maps* $p_i : A \to A_i$; *and suppose the* p_i (*and hence the* f_{ji}) *are all surjective.*

Then $\varprojlim_I M(A_i)$ *may be identified with a subalgebra of* $\mathrm{Endo}(A)$ *containing* $M(A)$, *by letting each* $(u_i)_{i\in I} \in \varprojlim_I M(A_i)$ *act on* A *by sending* $(a_i)_{i\in I} \in A$ *to* $(u_i(a_i))_{i\in I} \in A$.

Proof. The condition for $(a_i)_{i\in I}$ to belong to $A = \varprojlim_I A_i$ says that each f_{ji} takes a_i to a_j, and the condition for $(u_i)_{i\in I}$ to belong to $\varprojlim_I M(A_i)$ says that each $M(f_{ji})$ takes u_i to u_j. By (19), with f_{ji} for h, the latter condition tells us that $u_j(f_{ji}(a_i)) = f_{ji}(u_i(a_i))$, and by the former, the left-hand side of this relation equals $u_j(a_j)$. This shows that the I-tuple $(u_i(a_i))_{i\in I}$ again belongs to $A = \varprojlim_I A_i$. Thus, each $u \in \varprojlim_I M(A_i)$ induces a map $A \to A$ acting as described in the last phrase of the lemma.

It is routine to verify that these maps $A \to A$ are module endomorphisms, that this action of $\varprojlim_I M(A_i)$ on A respects the ring operations of $\varprojlim_I M(A_i)$, and that it is faithful; so we get an identification of $\varprojlim_I M(A_i)$ with a subalgebra of $\mathrm{Endo}(A)$. Finally, for any $x = (x_i)_{i\in I} \in A$, one easily verifies that $(l_{x_i})_{i\in I}$ is an element of $\varprojlim_I M(A_i)$ that acts on A as l_x; so as a subalgebra of $\mathrm{Endo}(A)$, $\varprojlim_I M(A_i)$ contains each operator l_x. It similarly contains each r_x, hence it contains $M(A)$.

In fact, one easily checks that each $u \in M(A)$ agrees with the element $(M(p_i)(u))_{i\in I} \in \varprojlim_I M(A_i)$. $\square$

In general, $\varprojlim_I M(A_i)$ will be larger than $M(A)$. Indeed, as noted in §2, if all A_i are nilpotent, then the algebras $M(A_i)$ are nilpotent, hence are radical, hence $\varprojlim_I M(A_i)$ is radical. But in the example we sketched there (to be given in detail in §8), $M(A)$ was not radical (since the image of $1 - l_x\, r_z$ under the map $M(A) \to M(B)$ was not invertible, so that element could not have been invertible in $M(A)$). Thus, in such an example, $M(A)$ cannot coincide with $\varprojlim_I M(A_i)$, and, indeed, must fail to be closed therein under quasiinverses.

6. Hopfian modules, and modules of finite length.

As also noted in §2, the operator $1 - l_x\, r_z$ of the example referred to above will nevertheless be surjective on any homomorphic image B of A. A key to the proof of our main result will be to restrict attention to image algebras B whose k-module structure is such that every surjective module endomorphism is injective. In getting our main conclusion, we will need the stronger assumption that B has finite length as a k-module; but let us take a look at the weaker condition just stated, under which we will be able to carry our proof part of the way.

An algebraic structure is said to be *Hopfian* if it has no surjective endomorphisms other than automorphisms [12] [23]. Here are some quick examples of Hopfian modules: A vector space is Hopfian if and only if it is finite-dimensional. A Noetherian module M over any ring is Hopfian, for if $h : M \to M$ were surjective but not injective, then the chain

(21) $\quad \{0\} \subsetneq h^{-1}(\{0\}) \subsetneq h^{-1}(h^{-1}(\{0\})) \subsetneq \dots$

would contradict the Noetherian condition [1, Prop. IV.5.3(i)] [12, Prop. 6(i)] [16, Prop. 1.14]. In particular, any module of finite length is Hopfian. Over a commutative ring, every finitely generated module is Hopfian [1, Prop. IV.5.3(ii)], and over a commutative integral domain k with field of fractions F, any k-submodule of a finite-dimensional F-vector-space is Hopfian (cf. [12, Prop. 11]). So, for instance, $\mathbb{Q}$ is a Hopfian $\mathbb{Z}$-module – though its homomorphic image $\mathbb{Q}/\mathbb{Z}$ is an example of a non-Hopfian module. (The classes of Hopfian modules listed above are all closed under finite direct sums; however, examples are known of non-Hopfian finite direct sums of Hopfian modules; indeed, of a Hopfian abelian group A such that $A \oplus A$ is not Hopfian [8].)

The next result only assumes A and B are modules over a ring, and does not require that ring to be commutative. In view of our convention that k denotes a commutative ring, we shall call the ring there K. (In our application of the result, however, K will be our commutative ring k.)

Proposition 7. *Suppose* A *and* B *are right modules over an associative unital ring* K, *let* $h : A \to B$ *be a surjective module homomorphism, and let* $\mathrm{Endo}(A; \ker(h))$ *be the subring of the endomorphism ring* $\mathrm{Endo}(A)$ *consisting of the endomorphisms that carry* $\ker(h)$ *into itself* (*and hence induce endomorphisms of* B).

Suppose R is a radical subring of $\mathrm{Endo}(A)$, *and B is Hopfian as a K-module. Then $R \cap \mathrm{Endo}(A; \ker(h))$ is also a radical ring; hence its image in* $\mathrm{Endo}(B)$ *is a radical subring of* $\mathrm{Endo}(B)$.

Proof. To show that the ring $R \cap \mathrm{Endo}(A; \ker(h))$ is radical, it suffices to verify that it is closed under quasiinverses in R. Let $r \in R \cap \mathrm{Endo}(A; \ker(h))$, and s be its quasiinverse in R. Thus, $1+r$ and $1+s$ are mutually inverse elements of $\mathrm{Endo}(A)$.

Since $1+r$ is invertible as an endomorphism of A, it is in particular surjective, from which it is easy to see that the endomorphism of B it induces is surjective. Since B is Hopfian, that endomorphism is also injective, and this says that back in $\mathrm{Endo}(A)$, $1+r$ carries no element from outside $\ker(h)$ into $\ker(h)$. Thus, the inverse map $1+s \in \mathrm{Endo}(A)$ carries no element of $\ker(h)$ out of $\ker(h)$, i.e., $1+s \in \mathrm{Endo}(A; \ker(h))$; hence $s \in \mathrm{Endo}(A; \ker(h))$, proving the latter ring radical. The final assertion follows immediately. $\square$

We shall use the above result in conjunction with part (iii) of the next lemma. Note that in that lemma, we return to the general hypothesis of a commutative base field k; and that parts (i) and (ii), but not part (iii), assume B an algebra. (Even in part (iii), it will be an algebra in our application.)

Lemma 8. (i) *In a radical associative algebra B, a finite set of elements $X \subseteq B$ which are not all zero cannot satisfy $X \subseteq BX$.*

(ii) *A radical associative algebra B cannot have a nonzero subalgebra which is both idempotent $(S = S^2)$ and finitely generated as an algebra.*

(iii) *If B is a k-module of finite length, then any radical subalgebra $R \subseteq \mathrm{Endo}(B)$ is nilpotent.*

Proof. (i): Writing $U = (k+B)X$ for the left ideal of B generated by X, the condition $X \subseteq BX$ implies that $U = BU$; so by Nakayama's Lemma [15, Lemma 4.22(2)] [17, Exercise XVII.7.4, p.661], $U = \{0\}$, hence $X \subseteq \{0\}$, contradicting our hypothesis. (The references cited state Nakayama's Lemma for unital rings. In our present context, we can apply that version of the lemma to the left module U over the unital ring $k+B$, using the fact that B is an ideal contained in the radical thereof.)

(ii): Suppose S were an idempotent subalgebra of B generated as a k-algebra by a finite set X. The fact that S is generated by X implies that $S \subseteq (k+S)X$, giving the third inclusion of

$$X \subseteq S = S^2 \subseteq B\,S \subseteq B(k+S)X \subseteq B\,X, \tag{22}$$

which contradicts (i).

(iii): Since B has finite length as a k-module, the chain of submodules $B \supseteq RB \supseteq R^2B \supseteq \dots$ stabilizes; say $R^{n+1}B = R^nB$. Again using finite length of B, we see that R^nB is finitely generated as a k-module, hence as a $(k+R)$-module; hence, since it is carried to itself by the radical ideal R of $k+R$, Nakayama's lemma shows that it is zero. Hence $R^n = \{0\}$. $\square$

7. The main theorem.

Definition 9. *A k-algebra A which can be written as an inverse limit of nilpotent k-algebras is called* pro-nilpotent.

Part (iii) of the next result is what we have been aiming at. The first two parts note what can be said under weaker assumptions.

Theorem 10. *Let $B = h(A)$ be a surjective homomorphic image of a pro-nilpotent k-algebra A. Then*

(i) *For every $r \in M(B)$, the operator $1+r \in \mathrm{Endo}(B)$ is surjective.* (*More generally, for every $n > 0$ and $r \in \mathrm{Mat}_n(M(B))$, $1+r$ acts surjectively on the direct sum of n copies of B.*)

(ii) *If B is Hopfian as a k-module, $M(B)$ is contained in a Jacobson radical subalgebra of* $\mathrm{Endo}(B)$.

(iii) *If B is of finite length as a k-module, then it is nilpotent as an algebra.*

Proof. Let $A = \varprojlim_I A_i$, where $(A_i, f_{ji})_{i,j \in I}$ is an inverse system of nilpotent k-algebras.

As noted earlier, if we replace each A_i by the image $p_i(A)$ therein, and restrict the f_{ji} to these subalgebras, we get a new inverse system having the same inverse limit A, and such that the restricted maps p_i and f_{ji} are surjective; moreover, the new A_i, being subalgebras of the given algebras, are still nilpotent. Hence without loss of generality, let us assume all the p_i and f_{ji} surjective.

By Corollary 3, the multiplication algebras $M(A_i)$ are nilpotent, hence are radical, and an inverse limit of radical rings is radical; so under the identification of Lemma 6, $\varprojlim_I M(A_i)$ is a radical subalgebra of $\mathrm{Endo}(A)$ containing $M(A) \subseteq \mathrm{Endo}(A; \ker(h))$.

With no additional assumptions, we see that the radicality of $\varprojlim_I M(A_i)$ implies that for every $u \in M(A)$, the operator $1+u$ is invertible on A, hence in particular, acts surjectively, hence that its image in $M(B)$ acts surjectively on B, giving the first statement of (i). Since a full matrix algebra $\mathrm{Mat}_n(R)$ over a radical algebra R is radical, this argument applies, more generally, to $\mathrm{Mat}_n(M(A))$ and $\mathrm{Mat}_n(M(B))$, acting on a direct sum of copies of A, respectively B, yielding the parenthetical generalization.

If B is Hopfian as a k-module, then by Proposition 7, $(\varprojlim_I M(A_i)) \cap \mathrm{Endo}(A; \ker(h))$ is a radical k-algebra. Since $\ker(h)$ is an ideal of A, it is carried into itself by the operators l_x and r_x $(x \in A)$, so $M(A) \subseteq (\varprojlim_I M(A_i)) \cap \mathrm{Endo}(A; \ker(h))$, hence its image $M(B) = M(h)(M(A)) \subseteq \mathrm{Endo}(B)$ is contained in the radical subalgebra $M(h)((\varprojlim_I M(A_i)) \cap \mathrm{Endo}(A; \ker(h)))$, giving (ii).

Finally, if B has finite length, Lemma 8(iii) shows that the above radical subalgebra of $\mathrm{Endo}(B)$ is nilpotent, hence its subalgebra $M(B)$ is nilpotent, hence by Corollary 3 again, B is nilpotent. $\square$

In the next section we will give counterexamples to the conclusions of Theorem 10 and Lemma 8 in the absence of some of the hypotheses; in particular, the finite length hypothesis of Theorem 10(iii). On the other hand, in §10 (after some general observations in §9), we will get a few additional positive results. In §11 we note a consequence of our main theorem for solvable Lie algebras, and in the final §§12-13, some questions and topics for further study.

8. Counterexamples.

The first example below is the promised case of a homomorphic image B of a pro-nilpotent algebra containing elements x, z such that the map $1 - l_x r_z \in M(B)$ is not one-to-one.

In constructing that example and the next, we shall make use of unital free associative algebras $k\langle X\rangle$ on finite sets X of noncommuting indeterminates (e.g., $X = \{x, y, z\}$) over a field k, their completions, which are noncommuting formal power series algebras $k\langle\langle X\rangle\rangle$, and the nonunital versions of these two constructions (their "augmentation ideals", i.e., the kernels of the unital homomorphisms to k sending the indeterminates to zero), which we will denote $[k]\langle X\rangle$, respectively $[k]\langle\langle X\rangle\rangle$.

Within these algebras, we shall write $(a, b, \dots)$ for the 2-sided ideal generated by elements $a, b, \dots$. In the completed algebras, we shall also write $((a, b, \dots))$ for the *closure* of such an ideal in the inverse limit topology.

These examples will start by taking a set T of monomials in the given free generators, which does not contain the monomial 1, and forming the factor algebra $k\langle X\rangle/(T)$. Note that this has a k-basis consisting of all monomials not containing any subword belonging to T. We shall then form the completion $k\langle\langle X\rangle\rangle/((T))$ and take for our A the subalgebra $[k]\langle\langle X\rangle\rangle/((T))$. It is not hard to see that $k\langle\langle X\rangle\rangle/((T))$ is the inverse limit of the factor-algebras $k\langle X\rangle/(T \cup X^i)$ where X^i denotes the set of monomials of length i in the given generators, so that $[k]\langle\langle X\rangle\rangle/((T))$ is the inverse limit of the nilpotent algebras $A_i = [k]\langle X\rangle/(T \cup X^i)$. This inverse limit consists of all formal infinite k-linear combinations of monomials having no subword in T.

By abuse of notation, we use the same symbols $x, \dots$ for our original generators and for their images in our various factor-algebras.

Example 11. *For any field k there exists a pro-nilpotent associative k-algebra A having elements x, y, z such that $y \notin (y - xyz)$.*

Thus, in the algebra $B = A/(y - xyz)$, the operator $1 - l_x r_z$ annihilates the nonzero element y. In particular, $0 \neq y \in B\,y\,B$, so B cannot be residually nilpotent.

Hence also, though the algebras A and B are Jacobson radical, their multiplication algebras $M(A)$ and $M(B)$ are not: in each, the element $-l_x r_z$ is not quasiinvertible (*though it is the product of the quasiinvertible elements $-l_x$ and r_z*).

Construction and proof. Since it is easier to study the ideal of an algebra $[k]\langle\langle X\rangle\rangle/((T))$ generated by one of the indeterminates than the ideal generated by a more complicated expression, we shall take for A an algebra of the form $[k]\langle\langle x, w, z\rangle\rangle/((T))$, find a $y \in A$ such that $w = y - xyz$, and then obtain B by dividing A by the ideal generated by the indeterminate w.

Let the set of monomials T be chosen so that the only nonzero monomials in $[k]\langle x, w, z\rangle/(T)$ are the words

(23) $\quad x^i w\, z^j \quad (i, j \geq 0), \quad$ and subwords of such words.

Thus, we take

(24) $T \ = \ \{xz,\, wx,\, ww,\, zw,\, zx\}.$

and let

(25) $A \ = \ [k]\langle\langle x, w, z\rangle\rangle/((T)).$

For convenient calculation with ideals, we also introduce the notation

(26) $k+A \ = \ k\,\langle\langle x, w, z\rangle\rangle/((T)).$

On A, which by our preceding discussion is pro-nilpotent, consider the operator $-l_x r_z \in M(A) \subseteq \varprojlim M(A_i)$. Since the latter algebra is Jacobson radical, $-l_x r_z$ is quasiinvertible in $\mathrm{Endo}(A)$; so let $y = (1-l_x r_z)^{-1}(w)$. Clearly,

(27) $y \ = \ w + xwz + x^2wz^2 + \dots + x^nwz^n + \dots\ .$

(Indeed, one can see immediately that this series satisfies $w = y - xyz$.)

We claim that $y \notin (w)$. To see this, note that every element of (w) is a *finite* sum

(28) $\sum_{i=1}^n a_i\, w\, b_i \quad (a_1,\dots,a_n,\, b_1,\dots,b_n \in k+A).$

Now an element $a\in A$ such that every monomial occurring in a contains a factor w or z will annihilate w on the left, and elements in which all monomials occurring contain factors w or x likewise annihilate w on the right (see (23), (24)); so let us write each a_i in (28) as $a'_i + a''_i$, where $a'_i \in k[[x]]$, while the monomials occurring in a''_i all have factors w or z, and each b_i as $b'_i + b''_i$, where $b'_i \in k[[z]]$ while the monomials occurring in b''_i all have factors w or x. Then (28) becomes

(29) $\sum_{i=1}^n a'_i\, w\, b'_i \quad (a'_1,\dots,a'_n \in k[[x]],\ b'_1,\dots,b'_n \in k[[z]]).$

We now see that if in (29) we take the right coefficient, in $k[[z]]$, of x^jw for any $j\geq 0$, this will be a k-linear combination of $b'_1,\dots,b'_n$. In particular,

(30) the k-vector-subspace of $k[[z]]$ spanned by the right coefficients in that algebra of the words x^jw $(j=0,1,\dots)$ in any element (29) of (w) is finite-dimensional over k.

However, by (27), the right coefficient of x^jw in y is z^j. The elements z^j span an infinite-dimensional subspace of $k[[z]]$, so $y\notin (w) = (y-xyz)$, proving our first assertion about this example. The second assertion is a restatement thereof.

Since $y = xyz$ in B, we have $y\in ByB \subseteq B(ByB)B \subseteq \dots$, hence y maps to 0 in every nilpotent homomorphic image of B, so B is not residually nilpotent.

A is radical because it is an inverse limit of radical algebras, and B is radical because it is a homomorphic image of A. We have shown that $l_x r_z \in M(B)$ is not quasiinvertible, hence the same is true of the element of $M(A)$ (denoted by the same symbol) which maps to it; so neither $M(A)$ nor $M(B)$ is radical. Finally, the maps $A\to M(A)$ given by $a\mapsto l_a$ and $a\mapsto r_a$ are a homomorphism and an antihomomorphism, so the quasiinvertibility of x and z in A implies quasiinvertibility of l_x and r_z in $M(A)$, hence also in $M(B)$. $\square$

We remark that if we write $A' = [k]\langle\langle x,w,z\rangle\rangle$, so that $A = A'/((T))$, and define $y = (1-l_x l_z)^{-1}(w)$ in the pro-nilpotent algebra A', then the fact that $y\notin (w)$ in A implies that the same holds in A'; so the conclusions proved above for A and B also hold for A' and $B' = A'/(w)$. Dividing out by $((T))$ just made it easier to see what we were doing.

In a different direction, suppose that instead of dividing a pro-nilpotent algebra A by the ideal generated by an element of the form $r = y - xyz = (1-l_x r_z)(y)$, we had divided such an algebra by the ideal generated by an element of the form $s = y - xy = (1-l_x)(y)$. Using the fact that $-x$, and hence l_{-x}, are quasiinvertible, we find that in this case, y *does* belong to the ideal (s), so it goes to zero in our factor-ring. Thus, this simpler construction does not give an example of non-injectivity. The same, of course, happens if we divide out by an element of the form $t = y - yz = (1-r_z)(y)$. Thus, the two-sided nature of the operator $l_x r_z$ was needed to make Example 11 work.

However, the fact we just called on, that $a\mapsto l_a$ is a homomorphism $A\to M(A)$, holds only for associative algebras A. In the next example, we shall see that on an inverse limit of nilpotent Lie algebras, an operator of the form $1-l_x$ can fail to be quasiinvertible. (Our example will in fact be "one-sided" from the Lie point of view, but "two-sided" from the associative point of view.) The construction will be formally a little simpler than the preceding example, but the verification will be a bit more complicated.

Example 12. *There exists a pro-nilpotent associative algebra A over any field k having elements x, y such that $y \notin (y - xy + yx)$. Thus, under commutator brackets, A is a pro-nilpotent* Lie algebra *with elements x, y such that $y \notin (y - [x,y])_{\mathrm{Lie}}$ (where $(\ )_{\mathrm{Lie}}$ denotes "Lie ideal generated by").*

Hence, in the Lie algebra $B = A/(y - [x,y])_{\mathrm{Lie}}$, $1 - \mathrm{ad}_x$ annihilates the nonzero element y. In particular, $0 \neq y \in [B, y]$, so B is not residually nilpotent.

As in the previous example, the associative algebras A and $A/(y - [x,y])$ are Jacobson radical, while their multiplier algebras are not: $r_x - l_x$ is not quasiinvertible (though r_x and l_x are).

Construction and proof. This time, let us start with the associative algebra $[k]\langle\langle x, w\rangle\rangle/((T))$, with T chosen so that the only nonzero monomials are the words

(31) $\quad x^i w\, x^j \ \ (i, j \geq 0),\quad$ and their subwords.

Thus, we take

(32) $\quad T \ = \ \{w x^i w \mid i \geq 0\},$

and let

(33) $\quad A \ = \ [k]\langle\langle x, w\rangle\rangle/((T)).$

We now define

(34) $\quad y \ = \ (1 - l_x + r_x)^{-1}(w) \ \in \ A.$

Though the obvious way to begin the calculation of this element would be to write $(1 - l_x + r_x)^{-1} = \sum_{i=0}^{\infty} (l_x - r_x)^i$, we can get the right coefficient of $x^i w$ in (34) more quickly if we instead use the formula

(35) $\quad (1 - l_x + r_x)^{-1} \ = \ \sum_{i=0}^{\infty} \ l_x^i\, (1 + r_x)^{-1-i},$

which is valid because l_x and $1 + r_x$ commute. This gives

(36) $\quad y \ = \ \sum_{i=0}^{\infty} \ x^i\, w\, (1 + x)^{-1-i}$

in the formal power series algebra $A = [k]\langle\langle x, w\rangle\rangle/((T))$.

Again, if this lay in (w), it would follow that the right factors $(1+x)^{-1-i}$ $(i = 0, 1, \dots)$ of the monomials $x^i\, w$ would lie in a finite-dimensional k-subspace of $[k][[x]]$. But they do not: the positive and negative powers of $1 + x$ are k-linearly independent in the field $k(x)$, so they are k-linearly independent in the larger formal Laurent series field $k((x))$, hence in the smaller formal power series algebra $k[[x]] \subseteq A$.

Hence $y \notin (w) = ((1 - l_x + r_x)(y)) = (y - xy + yx)$, and since the Lie ideal generated by $y - xy + yx = y - [x, y]$ is contained in the associative ideal generated by that element, we likewise have $y \notin (y - [x,y])_{\mathrm{Lie}}$.

The other assertions follow as before. $\square$

The above example may seem suspicious: In B, $y = [x, y]$, so x and y span a 2-dimensional sub-Lie-algebra $B^* \subseteq B$. Suppose we let A^* be the inverse image of this algebra in A, and replace each member of the family of algebras A_i of which A is the inverse limit by the image A_i^* of A^* therein. Won't the resulting inverse system have A^* as inverse limit, giving a description of the finite-dimensional non-nilpotent Lie algebra B^* as a homomorphic image of an inverse limit A^* of nilpotent Lie algebras?

What is wrong with this argument is the assumption that the inverse limit of the A_i^* will be A^*. Rather, one finds that that inverse limit will be the closure of A^* in the inverse limit topology on A. Since the map $A \to B$ is not continuous in that topology on A and the discrete topology on B (since its kernel is $(y - [x,y])_{\mathrm{Lie}}$, not $((y - [x,y]))_{\mathrm{Lie}}$), the closure of A^* may have a much larger image than B^*.

Another thought: Looking intuitively at Examples 11 and 12, we can say that in each, we took a pro-nilpotent algebra A, and were able to arrange for an element $y \in A$ to "survive" under a homomorphism $A \to B$ that made it fall together with a member of AyA or $Ay + yA$. In these cases, y survived "with the help of" other elements (x, and possibly z) which did not themselves fall together with higher-degree expressions. We may ask whether a family X of elements can all "help one another" to survive under a homomorphism that makes each fall together with a linear combination of higher degree monomials in it and the others. One way of posing this question is: Can a homomorphic image B of a pro-nilpotent algebra A contain a nonzero subalgebra S that is idempotent, i.e., satisfies $S = S^2$?

If our algebras are associative, and the set X generating S is finite, the answer is no. Indeed, A, and hence B, will be Jacobson radical, and Lemma 8(ii) says that such an algebra cannot have a nonzero finitely generated idempotent subalgebra. Lemma 8(i) describes a more general restriction.

However, these restrictions fail for nonassociative algebras. Indeed, in Example 12 we had $y \in [B, y]$, contradicting the analog of Lemma 8(i). We record next a much simpler (though non-Lie) example with the same property (which we will want to call on, for a different property, later), then an example with a finite-dimensional simple subalgebra.

Example 13. *Another pro-nilpotent algebra A over any field k having elements x, y such that $y \notin (y - xy)$, hence such that on $B = A/(y - xy)$, $1 - l_x$ annihilates the nonzero element y; hence such that $0 \neq y \in B\,y$ (in contrast to Lemma 8(*i*)).*

Construction and proof. A natural approach, paralleling our earlier constructions, would be to start with a nonassociative k-algebra on generators x, w, in which all monomials are set to zero except for x, w, xw, $x(xw)$, $x(x(xw)), \dots$. But rather than dealing with a free nonassociative algebra, and the resulting proliferation of parentheses, let us simply name the resulting basis of our algebra, and say how the multiplication acts. (The "free associative algebra modulo monomials" approach of our previous examples insured that the algebra described was associative; but no such condition is needed here.)

So let us start with an algebra having a basis $\{x, w_0, w_1, w_2, \dots\}$, and multiplication given by

(37) $\quad x\,w_i \ = \ w_{i+1} \quad (i = 0, 1, \dots),$ and all other products of basis elements zero.

Clearly, for each $i \geq 0$, this algebra has a homomorphic image A_i in which all w_j with $j \geq i$ are set to zero, and these images form an inverse system of nilpotent algebras, whose inverse limit A consists of all formal infinite sums $\alpha\,x + \sum_{i=0}^{\infty} \beta_i\,w_i \ \ (\alpha,\, \beta_i \in k)$.

The ideal (w_0) of A is easily shown to consist of the finite sums $\beta_0\,w_0 + \dots + \beta_n\,w_n$. In particular, it does not contain the element $y = \sum_{i=0}^{\infty} w_i = (1 - l_x)^{-1} w_0$, which satisfies $y - xy = w_0$. This gives the first assertion; the remaining assertions follow immediately. $\square$

Still more striking is

Example 14. *There exists a pro-nilpotent algebra A over any field k having an element y such that $y \notin (y - y^2)$.*

*Hence in $B = A/(y - y^2)$, the element y spans an idempotent 1-dimensional (associative!) subalgebra, in contrast to Lemma 8(*ii*).*

Construction and proof. This time, we start with an algebra having basis $\{w_0, w_1, \dots\}$, and multiplication given by

(38) $\quad w_i\,w_i \ = \ w_{i+1} \quad (i = 0, 1, \dots),$ and all other products of basis elements zero.

We again get nilpotent homomorphic images A_i on setting w_j equal to zero for all $j \geq i$. The inverse limit A of these algebras consists of all formal infinite sums

(39) $\quad \sum_{i=0}^{n} \alpha_i\,w_i \quad (\alpha_i \in k).$

Again, it is not hard to see that

(40) the ideal (w_0) of A consists of all finite sums $\alpha_0\,w_0 + \dots + \alpha_n\,w_n$.

Again let $y = \sum_{i=0}^{\infty} w_i$. We find that $y - y^2 = w_0$, so $(y - y^2)$ is (w_0), described in (40), which clearly does not contain y. This proves the main assertion; the final statement again follows. $\square$

Even for associative algebras, Lemma 8(ii) only excludes nonzero *finitely generated* idempotent subalgebras. An easy example of a radical associative algebra B with a non-finitely-generated idempotent subalgebra S is gotten by taking for both B and S the maximal ideal of any nondiscrete valuation ring. It is harder to get an example with B a homomorphic image of a pro-nilpotent algebra, but the following celebrated construction of Sąsiada and Cohn [19] has that property. (For parallelism with the other examples of this section, I have interchanged below the use of the symbols x and y in [19].) Note that the ideal (y) in the statement, though generated by a single element as a 2-sided ideal of B, must, by Lemma 8(i)-(ii), require infinitely many elements to generate it as a left ideal or as a subalgebra of B.

Example 15 (Sąsiada and Cohn [19]). *If k is a field, then in the pro-nilpotent associative algebra $A = [k]\langle\langle x, y\rangle\rangle$, one has $y \notin (y - x\,y^2\,x)$.*

Thus, in $B = A/(y - x\,y^2\,x)$, we have $0 \neq y = xy^2x$, so the ideal (y) is a nonzero left ideal satisfying $B(y) = (y)$ and a nonzero subalgebra satisfying $(y)^2 = (y)$. Moreover, if we take any ideal U of B maximal for the property of not containing y, then in $B' = B/U$, the subalgebra (y) is simple. Thus, $(y) \subseteq B'$ is a simple Jacobson radical algebra.

Sketch of proof. The proof that $y \notin (y - x\,y^2\,x)$ occupies most of the five pages of [19], and I will not discuss it here.

The relation $y = xy \cdot yx$ in B yields the asserted equalities, $B(y) = (y)$ (which one sees also holds in Example 11), and $(y)^2 = (y)$ (which does not hold in our previous associative examples).

Let us now verify the simplicity as a ring of the ideal (y) of $B' = B/U$ (though this is also done in [19]). By maximality of U,

(41) $\quad (y)$ contains no proper nonzero B'-ideal.

Suppose, however, by way of contradiction, that it contained a proper nonzero (y)-ideal V. If $(y)V = \{0\}$, then the right annihilator of (y) in (y) is a nonzero ideal of B' contained in (y), hence, by (41), is all of (y). But this implies that in B', $y\,y = 0$, hence $0 = xyyx = y$, a contradiction. So $(y)V \neq \{0\}$. Now from this it follows in turn that if $(y)V(y) = \{0\}$, the left annihilator of (y) in (y) gives the same contradiction. So $(y)V(y) \neq \{0\}$. But this is an ideal of B' contained in V, hence properly contained in (y), a final contradiction that completes the proof.

In the final assertion of the lemma (the goal of [19]), radicality holds because any ideal of a radical ring is radical. $\square$

Each of Examples 11-15 above shows, among other things, that the conclusion of Theorem 10(ii) can fail if one deletes the condition that the underlying k-module of B be Hopfian. We end this section with a much easier example showing that even if that module is Hopfian, this is not enough to give the full assertion of part (iii).

Example 16. *There exist a commutative ring k and a pro-nilpotent commutative associative k-algebra A which is Hopfian as a k-module, but not nilpotent as an algebra.*

Construction and proof. Let k be a complete discrete valuation ring, with maximal ideal (p), and consider the inverse system of nilpotent algebras $A_i = (p)/(p^i)$ $(i \geq 1)$ with the obvious surjective connecting homomorphisms. Because k is complete, the inverse limit A of this system is isomorphic as a k-algebra to the ideal $(p) \subseteq k$, which is free of rank 1 as a k-module, hence is Hopfian, but is not nilpotent as an algebra. $\square$

(If we had left out the assumption that k was complete, our A would have been the maximal ideal $p\,\hat{k}$ of the completion $\hat{k}$ of k. In that situation, $\hat{k}$, and hence that ideal, are again Hopfian, not only as $\hat{k}$-modules but as k-modules, though for less obvious reasons.)

9. A chain of conditions.

The proof of Theorem 10 involves a chain of conditions on an algebra A :

(42) $\quad A$ is nilpotent; equivalently, $M(A)$ is nilpotent.

$$\Downarrow$$

(43) $\quad M(A)$ is Jacobson radical; equivalently, for every $u \in M(A)$, $1 + u$ is invertible in $1 + M(A)$.

$$\Downarrow$$

(44) $\quad M(A)$ is contained in a Jacobson radical subalgebra of $\mathrm{Endo}(A)$.

$$\Downarrow$$

(45) $\quad$ For every $n > 0$ and $u \in \mathrm{Mat}_n(M(A))$, $1 + u$ is surjective as a map on the direct sum of n copies of A.

$$\Downarrow$$

(46) For every $u \in M(A)$, $1+u$ is surjective as a map $A \to A$.

We note some quick examples showing that the first four of these conditions are distinct:

In any commutative local integral domain R which is not a field, the maximal ideal A is a radical subalgebra, and satisfies $M(A) \cong A$, hence A satisfies (43), but not (42).

Amplifying the first sentence of the paragraph preceding Example 16, we note that for the algebras A of Examples 11-15 the inclusion $M(A) \subseteq \varprojlim_I M(A_i)$ yields (44), but that these algebras cannot satisfy (43), since they have homomorphic images B on which certain operators $1+u$ $(u \in M(B))$ are non-invertible.

The algebras B of those same examples satisfy (45) by Theorem 10(i), but they have elements $u \in M(B)$ such that $1+u$ is not injective, hence not invertible, so they do not satisfy (44).

I do not know whether there are algebras satisfying (46) but not (45).

Conditions (42), (43), (45) and (46) clearly carry over to homomorphic images; but Examples 11-15 show that (44) does not; though Proposition 7 shows that it does when the image algebra is Hopfian as a k-module.

Condition (42) carries over to subalgebras, but none of the others do. E.g., in a discrete valuation ring, such as the localization $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ at a prime p (notation unrelated to the $A_{(n)}$ of §4!), or a formal power series algebra $k[[t]]$ over a field k, the maximal ideal (in these cases, $p\,\mathbb{Z}_{(p)}$, respectively $[k][[t]]$), regarded as an algebra, satisfies (43), and hence (44)-(46); but in these two examples, the $\mathbb{Z}$-subalgebra $p\,\mathbb{Z} \subseteq p\,\mathbb{Z}_{(p)}$, respectively the $k[t]$-subalgebra $[k][t] \subseteq [k][[t]]$, fail to satisfy (46), hence likewise (43)-(45).

What about inverse limits; say with respect to inverse systems where the p_i are surjective? Examples 11-15 show that (42) and (43) fail to carry over to these. Probably (45) and (46) do not carry over either – those conditions make the maps $1+u$ $(u \in M(A))$ surjective, but do not make inverse images of elements under those maps unique, and this leads to no way of lifting such inverse images to the inverse limit algebra. However, (44) does carry over. Indeed, for each i let $N(A_i)$ denote the least radical subalgebra of $\mathrm{Endo}(A_i)$ containing $M(A_i)$, i.e., the closure of $M(A_i)$ under quasiinverses and the algebra operations. It is not hard to verify that the f_{ji} induce (surjective) homomorphisms $N(A_i) \to N(A_j)$. The inverse limit of this system of homomorphisms will be a radical subalgebra of $\mathrm{Endo}(A)$ containing $M(A)$.

We remark that variants of (45) and (46) in which "surjective" is replaced by "injective" or by "bijective" might also be of interest.

10. A Nakayama-like property.

In the proof of Theorem 10, we obtained statement (iii) from statement (ii) essentially by showing that for an algebra of finite length as a module, condition (44) implies (42). On the other hand, we did not obtain (ii) directly from (i) – I do not know whether for algebras that are Hopfian as modules, (45) or (46) implies (44). (If the implication requires the stronger statement (45), then it probably needs not only the hypothesis that A is Hopfian, but that all finite direct sums $\bigoplus_n A$ are Hopfian.) If A is Hopfian and satisfies (46), the maps $1+u$ $(u \in M(A))$ are invertible, hence all $u \in M(A)$ are quasiinvertible in $\mathrm{Endo}(A)$; the difficulty is that if try to close $M(A)$ within $\mathrm{Endo}(A)$ under quasiinverses and the algebra operations, some of the quasiinverses we introduce may have sums or products that fail to be quasiinvertible (like the $l_x r_z$ and $l_x + r_x$ of Examples 11 and 12), leading to the failure of (44).

However, whether or not we can deduce (44), if (45) holds and the modules $\bigoplus_n A$ are Hopfian, then $M(A)$ will behave somewhat like a radical algebra, in that it will satisfy a version of Nakayama's lemma. Let us formulate this result with $M(A)$ and A generalized to an arbitrary ring and module.

Lemma 17. *Let R be a nonunital associative ring, A a left R-module, and n a positive integer. Suppose that for every $u \in \mathrm{Mat}_n(R)$, the element $1+u$ acts in a one-to-one fashion on $\bigoplus_n A$.*

Then for any n-generator R-submodule C of A (or more generally, for any n-generator R-submodule C of a direct product of copies of A), one has $RC = C \implies C = \{0\}$.

Proof. The case where C is a submodule of a direct product reduces immediately to the case $C \subseteq A$ by projecting onto any coordinate where some member of C has nonzero component. So assume C is a submodule of A, generated by $x_1, \ldots, x_n$.

The condition $RC = R$ means that each x_m can be written as an R-linear combination of itself and the others; which says that if we let $x = (x_1, \dots, x_n) \in \bigoplus_n A$, then for some $u \in \mathrm{Mat}_n(R)$, we have $x = ux$. But this says that $1 - u$ annihilates x, contradicting our hypothesis. $\square$

We can now get part (iii) of Theorem 10 directly from part (i), via the following consequence of the above lemma.

Corollary 18. *For A an algebra of finite length as a k-module,* (45) *implies* (42). (*So for such A,* (42)-(45) *are equivalent.*)

Proof. If (42) fails, then the decreasing chain of submodules $M(A)^d(A)$ of A $(d = 0, 1, \dots)$ never becomes zero; but by the finite length assumption, it must stabilize. Thus, say $\{0\} \neq C = M(A)^d(A)$ satisfies $C = M(A)(C)$. By our finite length hypothesis, C is finitely generated, say by n elements.

Since a module of finite length is Hopfian, condition (45) says that all elements $1 + u$ $(u \in \mathrm{Mat}_n(M(A)))$ act invertibly on $\bigoplus_n A$, hence in a one-to-one fashion; so Lemma 17 says that $C = \{0\}$, a contradiction. $\square$

The next corollary to Lemma 17 shows that Theorem 10(ii) has concrete consequences for the structure of Hopfian homomorphic images of pro-nilpotent algebras. (Note that the hypothesis of finite generation as an ideal is weaker than finite generation as a one-sided ideal, which is weaker in turn than finite generation as an algebra.)

Corollary 19. *A nonzero k-algebra A satisfying* (45) (*in particular, a nonzero homomorphic image of a pro-nilpotent algebra*), *which has the property that for all n the k-module $\bigoplus_n A$ is Hopfian, cannot be both idempotent as a k-algebra and finitely generated as an ideal.*

Proof. For any algebra A, the ideals of A are its $M(A)$-submodules, and we see that the conditions of idempotence as an algebra and finite generation as an ideal say that $M(A)A = A$ and A is finitely generated as an $M(A)$-module. However, the Hopfian condition together with (45) yield the hypothesis of Lemma 17, implying that $A = \{0\}$. $\square$

11. Solvable Lie algebras.

The *derived series* of an algebra A is the sequence of subalgebras $A^{(n)}$ $(n = 0, 1, \dots)$ defined by

(47) $$A^{(0)} = A, \qquad A^{(n+1)} = A^{(n)}\, A^{(n)}.$$

This concept is standard in the theory of Lie algebras (where the $A^{(n)}$ are in fact ideals). It is less so for general nonassociative algebras, but is introduced in that context in [20, p.17].

An algebra A is called *solvable* if $A^{(n)} = \{0\}$ for some $n \geq 0$. It is easy to see that $A^{(n)} \subseteq A_{(2^n)}$ $(n = 0, 1, \dots)$, so every nilpotent algebra is solvable; but the converse is not true, as shown by the 2-dimensional Lie algebra with basis $\{x, y\}$ and multiplication $[x, y] = y$.

There is, however, in the classical context, a characterization of solvable Lie algebras in terms of nilpotent Lie algebras:

(48) [14, Corollary 1 to Theorem 13, p.51.] If A is a finite-dimensional Lie algebra over a field of characteristic 0, then A is solvable if and only if its commutator ideal $A^{(1)} = [A, A]$ is nilpotent.

Nazih Nahlus has pointed out to me that using this fact, one gets as a consequence of Theorem 10 the following result, which he conjectured some years ago.

Corollary 20 (to Theorem 10(iii). N. Nahlus (personal communication))**.** *Let A be an inverse limit of finite-dimensional solvable Lie algebras A_i over a field k of characteristic* 0. *Then any finite-dimensional homomorphic image B of A is solvable.*

Proof. By (48), the commutator ideals of the A_i form an inverse system of nilpotent algebras. The inverse limit $A^* \subseteq A$ of this system contains all brackets of elements of A; so when we map A homomorphically onto the finite-dimensional algebra B, the image of A^* contains all brackets of elements of B. Theorem 10(iii) tells us that that image is nilpotent, so B is solvable. $\square$

On the other hand, there are both infinite-dimensional Lie algebras A in characteristic 0, and finite-dimensional Lie algebras A in positive characteristic, which are solvable, but for which $A^{(1)}$ is not nilpotent.

An example of the former is given by the vector space A of operators on $\mathbb{R}[x]$ spanned by the operators X^n of multiplication by x^n $(n = 0, 1, \dots)$, the operator $D = d/dx$, and the composite operator $XD = x\,d/dx$. Indeed, one verifies that A is closed under commutator brackets, hence forms a Lie algebra (the semidirect product of the 2-dimensional Lie algebra L spanned by $\{D,\, XD\}$, and the L-module $\mathbb{R}[x]$). One finds that the subalgebra $A^{(1)} = [A, A]$ is spanned by all the above operators except XD. (In particular, $[D,\, XD] = D$ does appear.) This subalgebra is not nilpotent, since $[D, X^n] = nX^{n-1}$, so that there are elements which can be bracketed with D arbitrarily many times before going to zero. At the next step, however, $A^{(2)} = [A^{(1)}, A^{(1)}]$ is spanned by the operators X^n only, and hence has zero bracket operation, so $A^{(3)} = \{0\}$, showing that A is solvable, even though $A^{(1)}$ is not nilpotent.

To get a finite-dimensional example in positive characteristic, let us first note a variant of the above characteristic 0 example. Consider the ring of functions $\mathbb{R}[x, e^x]$, and the space of operators on that ring spanned by D and XD as above, together with (rather than the operators X^n) the operators $X^n Y$ $(n \geq 0)$, where Y is the operator of multiplication by e^x. Again, one verifies that this is closed under commutator brackets, and so gives a Lie algebra A (the semidirect product of L as above and the L-module $\mathbb{R}[x]\, e^x$). Where in the preceding example, the infinite-dimensionality of $\{X^0,\, X^1,\, X^2, \dots\}$ was involved in establishing the non-nilpotence of $A^{(1)}$, here non-nilpotence follows from the single relation

$$[D,\, X^0\, Y] = X^0\, Y. \tag{49}$$

This does not allow us to pass to a finite-dimensional subalgebra with the desired properties, because the iterated action of XD on $X^0\, Y$ brings in all the $X^n\, Y$. However, one finds that the structure constants of this Lie algebra with respect to our basis are integers, and that when one reduces them modulo a prime p, then the span of $\{X^p\, Y,\, X^{p+1} Y, \dots\}$ becomes an ideal. (Key calculation: in the original algebra, $[D,\, X^p Y] = pX^{p-1} Y + X^p Y$, and modulo p, the first term of that expression vanishes.) The factor-algebra by that ideal is thus a $(p+2)$-dimensional Lie algebra B, and the relation (49) shows that $B^{(1)}$ is still not nilpotent. However, we find that $B^{(1)}$ again loses the operator XD, and $B^{(2)}$ likewise loses D, hence has zero bracket operation, so that again $B^{(3)} = \{0\}$ and B is solvable. Further examples in prime characteristic may be found in [7].

So if Corollary 20 is to be extended to positive characteristic, or to inverse limits of not necessarily finite-dimensional Lie algebras (or, indeed, to non-Lie algebras), a very different proof will be needed.

One can, of course, generalize that corollary and its present proof by strengthening the hypothesis to *assume* A is an inverse limit of Lie algebras for which $A^{(1)}$ is nilpotent. Indeed, one can generalize the resulting statement to arbitrary algebras, replacing solvability by any condition specifying that the values of a given family of algebra terms should generate a nilpotent subalgebra.

12. Possible variants of our main theorem.

Let us look at a few ways Theorem 10 can, or might, be generalized.

We start with one which as a generalization proves disappointing – but does so by showing that our present Theorem 10 is stronger than we realized.

12.1. **General limits.** Recall that the concept of the inverse limit of an inversely directed system of algebraic structures is a case of the more general category-theoretic notion of the "limit" of a functor [2, §§7.6] [18, §III.4], other important examples of which are the fixed-point algebra of a group acting on an algebra, and the equalizer of a pair of algebra homomorphisms. If one examines the proof of Theorem 10, one sees no reason why it should not work for limits in this general sense. It does – but that extension gives nothing new:

Lemma 21. *For a k-algebra A, the following conditions are equivalent.*

(i) *A can be written as the limit of a system of nilpotent k-algebras indexed by a small category.*

(ii) *A is pro-nilpotent, i.e., can be written as an inverse limit of an inversely directed system of nilpotent k-algebras.*

Sketch of proof. Clearly, (ii) $\implies$ (i).

Conversely, suppose $F : \mathbf{C} \to \mathbf{Alg}_k$ is a functor from a small category $\mathbf{C}$ to the category $\mathbf{Alg}_k$ of not-necessarily-associative k-algebras, such that for all $X \in \mathrm{Ob}(\mathbf{C})$, $F(X)$ is nilpotent.

Let I be the partially ordered set of finite subsets of $\mathrm{Ob}(\mathbf{C})$, ordered by reverse inclusion; clearly, I is inversely directed. For each $i \in I$, let $\mathbf{C}_i$ be the full subcategory of $\mathbf{C}$ with object-set i, and let $A_i = \varprojlim (F \,|\, \mathbf{C}_i)$, where $F \,|\, \mathbf{C}_i$ denotes the restriction of F to $\mathbf{C}_i$.

Each A_i is a subalgebra of the finite product $\prod_{X \in i} F(X)$, and the class of nilpotent algebras is closed under finite products and subalgebras, hence each A_i is nilpotent. Given $i \leq j \in I$, which by our ordering of I means $i \supseteq j$, the inclusion $\mathbf{C}_j \subseteq \mathbf{C}_i$ induces a restriction homomorphism $A_i \to A_j$. It is straightforward to verify that $\varprojlim F = \varprojlim_I A_i$, yielding (ii). □

12.2. **Variant sorts of nilpotence.** Within the multiplier algebra $M(A)$ of an algebra A, we may look at the subalgebra $M_l(A)$ generated by the left multiplication operators l_x, and the subalgebra $M_r(A)$ generated by the right multiplication operators r_x.

If A is associative, these give nothing very new: $M_l(A)$ is isomorphic to the factor-algebra of A by its left annihilator ideal $\{x \in A \mid xA = \{0\}\}$, and $M_r(A)$ is antiisomorphic to the factor-algebra of A by the analogous right annihilator ideal; in particular, each is nilpotent if and only if A is. If, rather, A is anticommutative (e.g., is a Lie algebra) or is commutative (e.g., is a Jordan algebra), then $M_l(A)$ and $M_r(A)$ coincide with $M(A)$.

But for a general nonassociative algebra A, these two subalgebras of $M(A)$ can look very different. For instance, for the algebra with multiplication (37), it is easy to see that $(AA)A = \{0\}$, so that $M_r(A)^2 = \{0\}$, but that $M_l(A)^n \neq \{0\}$ for all n.

The conditions $(\exists\, n)\, M_l(A)^n = \{0\}$ and $(\exists\, n)\, M_r(A)^n = \{0\}$ are known as *left* nilpotence and *right* nilpotence [22]. An algebra can be both left and right nilpotent without being nilpotent, as shown by the algebra with basis $x, w_0, w_1, \dots$, and multiplication

$$(50) \qquad x\, w_{2i} \;=\; w_{2i+1}, \quad w_{2i+1}\, x \;=\; w_{2i+2}, \quad \text{all other products of basis elements being zero.}$$

The development of Theorem 10 goes over, with no change, if the condition of nilpotence is replaced by that of left nilpotence or of right nilpotence! It is not clear to me what the most useful common generalization of these various sorts of nilpotence is, so I leave it to the experts in nonassociative rings to develop that observation further.

Let us record a few other versions of nilpotence, corresponding to still other subalgebras of $M(A)$.

Given any $\alpha, \beta \in k$, one can define a new multiplication on any k-algebra A by

$$(51) \qquad x * y \;=\; \alpha\, x\, y + \beta\, y\, x,$$

(from which the original multiplication is recoverable by a transformation of the same form if $\alpha^2 - \beta^2$ is invertible in k). Left nilpotence of this operation is a property of A that is not, in general, equivalent to either nilpotence, left nilpotence, or right nilpotence of the original operation; but any results on left nilpotence of a general algebra will necessarily apply to left nilpotence of this operation.

Recall next that for any algebra A one can define the family of *associator* operations,

$$(52) \qquad a_{x,z}(y) \;=\; x(yz) - (xy)z \qquad (x, y, z \in A).$$

Hence we may consider the subalgebra $M_a(A) \subseteq M(A)$ generated by all these maps, and study algebras A for which $M_a(A)$ is nilpotent. Does the fact that the generating set of maps $\{a_{x,z} \mid x, z \in A\}$ is not a linear image of A but a bilinear image of $A \times A$ affect the usefulness of this construction? I don't know.

Finally, note that to every finite binary tree with n leaves, one can associate a way of bracketing n symbols, and hence a way of associating to every algebra A a derived n-ary operation. Various nilpotence-like conditions can be expressed conveniently in terms of this formalism. Thus, an algebra A is left nilpotent if and only if for some n, the $n+1$-ary operation induced by the length-n right-branching chain is zero on A; right nilpotent, likewise, if and only if for some n, the operation induced by the length-n left-branching chain is zero. (Here we call a tree a "chain" if after pruning all leaves, it has the form usually called a chain.) An algebra A is nilpotent if and only if for some n, the operations induced by all length-n chains are zero; equivalently, if and only if for some n' the operations induced by all trees with n' leaves are zero. An algebra is solvable if and only if for some n the operation induced by the full depth-n binary tree (with $2^{n+1} - 1$ nodes) is zero. One might develop a general framework for studying such conditions, associating a nilpotence-like condition on algebras to every filter of subsets of the set of finite binary trees.

12.3. **What about restricted Lie algebras?** Over a field k of characteristic $p > 0$, a more useful concept than that of a Lie algebra is that of a *restricted* Lie algebra or *p-Lie algebra*: a Lie algebra given with an additional operation, $x \mapsto x^{(p)}$, satisfying certain identities which, in associative k-algebras, relate the p-th power map with the k-module structure and commutator brackets. Though p-Lie algebras are not algebras in the sense of this note, it would, of course, be of interest to know whether versions of our results hold for these objects.

12.4. **What about nilpotent groups?** The relation between nilpotence of Lie algebras over $\mathbb{R}$ and $\mathbb{C}$, and nilpotence, in the sense of group theory, of the corresponding Lie groups makes it natural to ask whether the methods and results of this note have analogs for groups G (not necessarily Lie).

In view of the way the brackets of a Lie algebra are related to the group operation, the natural analogs of the maps r_x and l_x in the above development would seem to be the commutator maps $c_x(y) = x^{-1}y^{-1}x\,y$. The analog of "quasiinvertibility" for a map $u: G \to G$ might be invertibility of the set-map $g \mapsto g\,u(g)$ of G to itself. But it is not clear under what operations it would be natural to close the set of commutator maps to form the analog of $M(A)$, and whether this (or any method) will lead to an analog of Theorem 10.

12.5. **A different connection with groups.** Yiftach Barnea has pointed out a lemma in group theory which is similar to our main theorem in a different way. It says that a finite homomorphic image of a pro-p group is a p-group [9, Lemma 1.18]. The proof can, in fact, be put in a form strikingly similar to that of our Theorem 10: Note that in a p-group, for every integer q relatively prime to p, the operation $(\ )^q$ of exponentiation by q is invertible. Hence this remains true in an inverse limit of such groups. Hence in any homomorphic image G of such an inverse limit, $(\ )^q$ is surjective. In that situation, $(\ )^q$ need not be injective; but if G is finite, surjectivity implies injectivity, so all the maps $(\ )^q$ are invertible, forcing G to be a p-group.

That lemma is a key step in Serre's proof that any homomorphism from a finitely generated (in the topological sense) pro-p group to a finite group is continuous in the natural topology [9, Theorem 1.17], [21, §I.4.2, Exercise 5-6, p.32]. Following the parallism noted above, and using Theorem 10 of this note in place of [9, Lemma 1.18], we obtain in [3] an analog of that result for pro-nilpotent algebras in a large class of varieties of algebras over a field k, including those of associative, Lie and Jordan algebras.

13. Questions.

Several topics for further investigation were noted above. Here are some more specific questions.

Regarding the chain of conditions in §9, we ask

Question 22. (i) *For A an algebra, is the implication* (45) $\Longrightarrow$ (46) *reversible? More generally, if A is a module over an associative nonunital ring R such that for each $r \in R$, the operator $1 + r$ is surjective on A, does the action of* $\mathrm{Mat}_n(R)$ *on the direct sum of n copies of A have the same property?*

(ii) *For A an algebra which is Hopfian as a k-module, is any of the the implications* (43) $\Longrightarrow$ (44) $\Longrightarrow$ (45) $\Longrightarrow$ (46) *reversible?*

Examples 14 and 15 show that a homomorphic image of a pro-nilpotent algebra over a field can contain a simple subalgebra, and so, in particular, an idempotent subalgebra. This leaves open the question

Question 23. *Can a nonzero homomorphic image B of a pro-nilpotent algebra A over a field k itself be idempotent? Simple? If so, can this happen when our algebras are associative?*

Of course, by Theorem 10(iii), such a B cannot be finite-dimensional and by Lemma 8(i)-(ii), if our algebras are associative, B cannot be finitely generated. For the case where k is not, as assumed above, a field, Corollary 19 gives a weaker restriction of the same sort.

We have seen ways in which Lie algebras behave like associative algebras (Lemma 4), and ways in which they differ (the contrast between Lemma 8(i) and Example 12). The next question concerns some cases where it isn't clear on which side of the fence Lie algebras will fall.

Question 24. *Can a homomorphic image B of a pro-nilpotent Lie algebra have a nonzero finitely generated idempotent subalgebra?*

If so, can it have a nonzero finitely generated simple subalgebra?

If so, can such a subalgebra be finite-dimensional?

(A curious difference between the behaviors of Lie and associative algebras is noted in [4, Example 25.49], where it is observed that a topological Lie algebra (over a field) with a linearly compact topology need not be an inverse limit of finite-dimensional Lie algebras. That example is the Lie algebra spanned by the operator denoted D in the discussion following Corollary 20, together with formal power series (rather than just polynomials) in the operator there denoted X. Under the duality between vector spaces and linearly compact vector spaces, this shows that the "Fundamental theorem on coalgebras", a result on coassociative coalgebras, is not valid for co-Lie-algebras.)

In §11, where we considered solvable Lie algebras, we raised

Question 25. *In Corollary 20, is it possible to remove or weaken* (i) *the condition that the* A_i *be finite-dimensional, or* (ii) *the condition of characteristic* 0, *or* (iii) *the condition that the algebras be Lie?*

In [5] and [6], N. Nahlus and the present author study homomorphic images of *direct product* algebras $\prod_I A_i$. The form of the results obtained there suggests some possible strengthenings of Theorem 10(iii):

Question 26. *In Theorem 10*(iii), *if* k *is a field* (*or perhaps, more restrictively, an infinite field*), *can the hypothesis that* B *is finite-dimensional* (*the form that the finite-length hypothesis takes for vector spaces*) *be weakened to* countable-*dimensional?* (*Cf.* [5, Theorem 11], [6, Theorem 8].)

For any algebra B, *let us write* $Z(B)$ *for the ideal* $\{b \in B \mid b\,B = B\,b = \{0\}\}$. *Then if* k *is infinite and* $\mathrm{card}(I)$ *is less than any uncountable measurable cardinal, can the conclusion of Theorem 10*(iii) *be strengthened to say that the composite map* $A \to B \to B/Z(B)$ *factors through one of the projections* $p_i : A \to A_i$ (*equivalently, is continuous in the pro-discrete topology*)*? Without those cardinality hypotheses, can we say that* $B/Z(B)$ *is a homomorphic image of one of the* A_i *?* (*Cf.* [5, Proposition 16].)

The concept of measurable cardinal is reviewed in [5, §15]. The need, in the second paragraph of the above question, for the cardinality conditions and for the denominator "$Z(B)$" arises from the need for these same restrictions in [5] and [6]. Indeed, an infinite direct product of algebras is an inverse limit of finite subproducts, so counterexamples to statements for infinite products are also counterexamples for inverse limits.

On the other hand, in [3] we obtain positive results without the cardinality restriction on I, or passing to $B/Z(B)$, under the assumption that A has a finitely generated subalgebra dense in the inverse limit topology, and satisfies identities with certain properties (which hold, inter alia, for associative, Lie, and Jordan algebras).

Thinking about the counterexamples in §8, and the differences between the kinds of examples that can exist for associative and for nonassociative algebras, suggests

Question 27. *If an associative algebra* B *can be written as a homomorphic image of a pro-nilpotent algebra, can it be written as a homomorphic image of an associative pro-nilpotent algebra?*

Same question, with associativity replaced by an arbitrary family of identities.

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