Math 104, Final, 20081213 1230 – 15:30. O. Hald.

Solve 4 of the 5 problems.

① Let $f(x)$ be a continuous, non-negative function. Show that $f(x) = 0$ for $a \leq x \leq b$ if $\int_a^b f(x)\,dx = 0$.

② Let $u(x) = \int_0^x \frac{\sin(x-y)}{1+y^2}\,dy$ and $v(x) = \int_0^x \sin(x-y)\sin(y)\,dy$

(a) Is $u(x)$ uniformly continuous on $[0,\infty)$? (b) Is $v(x)$?

Hint: $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$. $\cos(2a) = 1 - 2\sin^2(a)$.

③ Assume that $f(x)$ and $g(x)$ are Riemann integrable on $[a,b]$.

(a) Show that $|f(x) - f(y)| \leq \sup_J f - \inf_J f$ for all $x, y \in J \subset [a,b]$

(b) Show that $\sup_J (f \cdot g) - \inf_J (f \cdot g) \leq$

$$(\sup_J f - \inf_J f) \sup_J |g| + (\sup_J g - \inf_J g) \sup_J |f|.$$

(c) Show that $f \cdot g$ is Riemann integrable on $[a,b]$.

④ Let $f_n(x) = \sum_{i=0}^{n-1} \sin^2(x) \cos^{2i}(x)$ for $-\pi \leq x \leq \pi$

(a) Find $f(x) = \lim_{n\to\infty} f_n(x)$

(b) Does $f_n(x) \to f(x)$ uniformly on $[-\pi/4, \pi/4]$?

(c) Does $f_n(x) \to f(x)$ uniformly on $[\pi/4, \frac{3\pi}{4}]$?

⑤ Let $f(x) = \begin{cases} \cos(1/x) & \text{if } x > 0 \\ 1 & \text{if } x \leq 0, \end{cases}$ and set $Z = \{x : \max(0, f(x)) = 0\}$

(a) Is $f(x)$ continuous? (b) Is Z closed?