Midterm III/math110/fall 2003 LIU
Total Score: 100 pts
Time: 2:10-3:30pm Nov 6

#1. (15%) Please determine if the following statements are correct or not. Please give a brief explanation for each of your answers.

(a). If a linear transformation $T \in \mathcal{L}(V)$ has an eigenvalue $\lambda = 0$, then T cannot be invertible. (3%)

(b). Let $V = P_n(\mathbf{R})$ and let $T \in \mathcal{L}(V)$ be $T = d/dx$. Then V itself is a T-cyclic subspace generated by some $f \in V$. (4%)

(c). Let V be a complex vector space over $\mathbf{C}$ and let $<,>$ be an inner product on V. Then $V^{\perp}$ can never be V itself. (4%)

(d). Let $T \in \mathcal{L}(V)$ and let $W \subset V$ be a $T-$invariant subspace of V. If the characteristic polynomial of T_W splits, then the characteristic polynomial of T also splits. (4%)

#2. (15%) Let $T : \mathbf{R}^4 \mapsto \mathbf{R}^4$ be a linear transformation defined by $(a, b, c, d) \mapsto T(a, b, c, d) = (a+c, a+c, 2a+2c, -a-c)$.

(a). Please determine the characteristic polynomial of T. (hint: you may find the eigenvalues before you find the characteristic polynomial. What is the rank of T? What is its range?) (8%)

(b). Determine if T is diagonalizable or not. If it is diagonalizable, please determine a basis of eigenvectors. If not, explain why not. (7%)

#3. (15%) (a). Let $V = P_n(\mathbf{R})$ and let $D \in \mathcal{L}(V)$ be $D = d/dx$. Give a uniform proof that for all $1 \le i \le n$ the linear operators D^i are not diagonalizable. (5%)

(b). Let V and D still be as in (a). Let $T = I_V + D + D^2 + \cdots + D^n$. Please determine the characteristic polynomial of T and show that it splits. (5%)

(c). Continue the question (b). Find all the eigenspaces of T. Determine if T is diagonalizable. (some knowledge in O.D.E. may help) (5%)

#4. (20%) (a). Let λ_0 be a root of the characteristic equation of $T \in \mathcal{L}(V)$, $dimV < \infty$. Show that the eigenspace associated to λ_0, $E_\lambda \neq \{\mathbf{0}\}$. (7%)

(b). Let $T \in \mathcal{L}(V)$, $dimV = n$. Suppose that T has n distinct eigenvalues. Prove that the eigenvectors associated with these n distinct eigenvalues are linearly independent. (13%)

#5. (20%) Let $V = \mathbf{M}_{n\times n}(\mathbf{R})$ be a vector space over $\mathbf{R}$ with the inner product $< A, B >= tr(AB^t)$. Define $\mathbf{S}_n = \{m | m \in V, m = m^t\}$, $\mathbf{A}_n = \{m | m \in V, m^t = -m\}$, $\mathbf{D}_n$=the subspace of the diagonal $n \times n$ matrices in V, $\mathbf{T}_n$= the subspace of the upper triangular matrices in V.

(a). Please list all the pairs of subspaces (from $\mathbf{S}_n, \mathbf{A}_n, \mathbf{D}_n, \mathbf{T}_n$) such that the direct sums exist. (6%)

(b). Let $T : V \mapsto V$ be $T(m) = m^t + m$. Determine the eigenvalues of T and all its eigenspaces. (8%)

(c). When $n = 3$, find an orthonormal basis of $\mathbf{S}_3$. (6%)

#6. (15%) Please state and prove the Caley-Hamilton theorem for linear transformations $T \in \mathcal{L}(V)$ on a finite dimensional vector space V. (If you use some propositions or theorems proved in the lectures, please state them clearly.)

Extra Credit: (10%) Under the same setup as in #4, prove that $\mathbf{S}_n^{\perp} = \mathbf{A}_n$.