MATH 113 — FALL 2003 FINAL EXAM
INTRODUCTION TO ABSTRACT ALGEBRA

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA, BERKELEY
INSTRUCTOR: ALEXANDER YONG

NAME: ____

ID NUMBER: ____________________

(1) Do not open this exam until you are told to begin.
(2) This exam has 10 pages including this cover. There are 9 questions total. You have 3 hours.
(3) No aids such as calculators, notes or books are permitted.
(4) You will be handed scrap paper for use.
(5) Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to the proctor when you turn in the exam.
(6) You may use any results proved in class. However, you must include complete hypotheses and conclusions to ensure credit.
(7) Please turn off all cell phones.

PROBLEM	POINTS	SCORE
1	12	
2	12	
3	10	
4	12	
5	10	
6	12	
7	10	
8	12	
9	10	
TOTAL	100	

1. Consider the following problems about the S_n, the symmetric group on $\{1, 2, \ldots, n\}$: [12 marks]
   (a) Let $n = 6$ and let $\pi = (1\ 2\ 3\ 4) \in S_6$ be a cycle (i.e., $1 \mapsto 2,\ 2 \mapsto 3,\ 3 \mapsto 4,\ 4 \mapsto 1$). Compute π^2, π^3, π^4. [3 marks]
   (b) Prove that S_n contains as a subgroup, an isomorphic copy of each cyclic group $C_i \cong \mathbb{Z}_i$ for $i = 1, 2, \ldots n$. [5 marks]
   (c) Can S_{100} contain an isomorphic copy of C_{101} as a subgroup? Why or why not? [4 marks]

2. Consider the quotient ring $R = \mathbb{R}[x]/\langle x^2 + x + 1 \rangle$. [12 marks]
   (a) Prove that R is a field. [5 marks]
   (b) Give a description (with proof) of the cosets of R. [5 marks]
   (c) This ring is isomorphic to a well known ring S. What is S? You do not need to prove your assertion. [2 marks]

3. Consider the ring $S = \mathbb{Q}(2^{\frac{1}{2}}, 2^{\frac{1}{3}}, 2^{\frac{1}{4}}, \ldots)$, i.e., the smallest field containing $\mathbb{Q}$ and all the (real) n^{th} roots of 2, for $n = 2, 3, 4, \ldots$. [10 marks]

(a) Prove that S is an algebraic extension over $\mathbb{Q}$. [5 marks]

(b) Prove that S is *not* a finite extension over $\mathbb{Q}$. [5 marks]

4. Let $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ be the ring of Gaussian integers. Determine whether the ideal $\langle 3 + i \rangle$ is a prime ideal in $\mathbb{Z}[i]$. [12 marks]

5. Let G act on itself by conjugation: $g \star h = ghg^{-1}$ where $g, h \in G$; recall this splits G into conjugacy classes. Prove that every normal subgroup H of a group G is a union of conjugacy classes of G, one of which is $\{1\}$. [10 marks]

6. Let G be a finite group with $| G |= p^a m$ where p is prime and $m < p$. [12 marks]

(a) State Sylow's first theorem. [4 marks]

(b) Prove that all of G's Sylow p-subgroups are normal. [8 marks]

6. Prove that no group is the union of two proper subgroups (HINT: you don't need any fancy theorems to prove this one). [10 marks]

8. Let p and q be two distinct prime integers. [12 marks]
   (a) Show that $\mathbb{Q}(\sqrt{p}+\sqrt{q}) = \mathbb{Q}(\sqrt{p}, \sqrt{q})$. [6 marks]
   (b) Find the minimal polynomial of $\sqrt{p}+\sqrt{q}$. [6 marks]

9. Let H be a normal subgroup of a group G, and let $m = [G : H]$. Show that $a^m \in H$ for every $a \in G$. [10 marks]