MATH 113: INTRODUCTION TO ABSTRACT ALGEBRA (Section 4)

Midterm 1

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Be sure to justify your answers. You may use any preceding parts to answer each question. Good luck!

1. Consider the following two elements of the symmetric group S_8:

$$\sigma - (13458)(1245)(678) \quad \text{and} \quad \tau - (156).$$

   (a) (*5 points*) Express σ as a product of disjoint cycles.

   (b) (*5 points*) Find the orders of σ and τ.

   (c) (*5 points*) Express $\sigma\tau$ as a product of disjoint cycles.

   (d) (*5 points*) What is the order of the cyclic subgroup generated by $\sigma\tau$?

   (e) (*5 points*) Express $\sigma\tau\sigma^{-1}$ as a product of disjoint cycles.

2. (a) (*8 points*) Recall that the dihedral group D_n is generated by two elements x and y satisfying the relations $x^n = e$, $y^2 = e$ and $xy = yx^{-1}$. Find the order of the group D_n.

   (b) (*8 points*) Find the order of each element in D_n.

   (c) (*9 points*) Find all cyclic subgroups of D_3. Given the fact that all proper subgroups of D_3 are cyclic, draw the lattice diagram of subgroups for D_3.

3. (a) (*8 points*) Find k where $0 \le k < 15$, such that $k \equiv 2^{103} (\text{mod } 15)$. (**Hint**: $103 = 4(25) + 3$.)

   (b) (*8 points*) Find the order of the element $42 \in \mathbb{Z}_{180}$.

   (c) (*9 points*) Let $GL(n, \mathbb{R}) = \{n \times n \text{ real matrices } M \mid \det M \neq 0\}$ be the general linear group under the matrix multiplication. Prove that $SL(n, \mathbb{R}) = \{M \in GL(n.\mathbb{R}) \mid \det M = 1\}$ is a subgroup of $GL(n, \mathbb{R})$.

4. (a) (*8 points*) Prove that the relation on $\mathbb{R}$ defined by $x \sim y$ iff $x - y \in \mathbb{Z}$ is an equivalence relation.

   (b) (*8 points*) Let $[x]$ denote the equivalence class containing $x \in \mathbb{R}$ in part (a). Show that the set of equivalence classes in part (a) is a group under the operation $[x] + [y] = [x + y]$. Be sure to check that the operation is well defined.

   (c) (*9 points*) Show that the group in part (b) is isomorphic to the multiplicative group U of complex numbers on the unit circle, i.e. to $U = \{e^{i\theta} \mid 0 \le \theta < 2\pi, \theta \in \mathbb{R}\}$, where $i = \sqrt{-1}$. (**Hint**: Consider an exponential map, which you should check to be well defined.)