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Fall 2002, Math 113, Sec. 5
Final Examination

13 Dec., 2002
12:30-3:30

1. (12 points, 4 points each.) Complete the following definitions.

(a) If X is a set, then a *permutation* of X means ...

(b) If R and S are commutative rings, then a map $f: R \to S$ is called a *homomorphism* of commutative rings if ...

(c) A proper ideal I of a commutative ring R is said to be a *maximal* ideal if ...

2. (36 points; 4 points each.) For each of the items listed below, either *give an example,* or give a brief reason why *no example exists.* (If you give an example, you do *not* have to prove that it has the property stated. Examples should be specific for full credit; i.e., even if there are many objects of a given sort, you should name one.)

(a) An element $\sigma \in S_6$ such that $\sigma\,(1\ 2\ 3)\,\sigma^{-1} = (2\ 4\ 6)$.

(b) An injective (i.e., one-to-one) homomorphism $f: \mathbb{Z} \to \mathbb{R}^\times$. (Recall that $\mathbb{R}^\times$ denotes the group of nonzero real numbers under multiplication.)

(c) A factorization of the polynomial $3x^3 + 29x^2 - 4x - 2$ as a product of two polynomials of lower degree in $\mathbb{Q}[x]$.

(d) A ring R, an ideal $I \subseteq R$, and elements $a \neq b$ of R such that $a + I = b + I$.

(e) A polynomial $f(x) \in \mathbb{Q}[x]$ which has no root in $\mathbb{Q}$, but which is reducible in $\mathbb{Q}[x]$.

(f) A field with exactly 100 elements.

(g) An ideal $I \subseteq \mathbb{Z}[x]$ such that $\mathbb{Z}[x]/I$ is isomorphic to $\mathbb{Z}[i]$, the ring of Gaussian integers.

(h) Two elements of $\mathbb{Z}_5[x]$ that are associates, but are not equal.

(i) A unique factorization domain which is not a principal ideal domain.

3. Short proofs. (22 points = 6 + 8 + 8.)

(a) If $f: X \to Y$ and $g: Y \to Z$ are set maps such that the composite map $g \circ f: X \to Z$ is injective, show that f is injective.

(b) Suppose a group G acts on a set X, and let $S = \{\sigma \in G \mid \forall\, x \in X,\ \sigma x = x\}$. Show that S is a normal subgroup of G. (You must show both that it is a subgroup and that it is normal.)

(c) Suppose $P_1 \geq P_2 \geq \ldots \geq P_n \geq P_{n+1} \geq \ldots$ is a decreasing sequence of prime ideals. Show that the ideal $I = \bigcap_{n \geq 1} P_n$ is prime. (Here you are to take for granted that I is an ideal, in contrast to the homework problem this is taken from, where you had to prove both that it was an ideal and that it was prime.)