Math 53
Summer 2009
D. Kaspar

Final

August 14

Full name: ______________________________

- This exam begins at 2:10 and finishes at 4:00.
- Outside scratch paper, formula sheets, and calculators are not allowed.
- Write your answers in the spaces following each question. If you need more room, use the reverse side of the page and check the box indicating I should look at the back.
- Circle or box important parts of your work and cross out portions you do not want evaluated.
- You may ask me questions, but unless there is an error on the exam I will probably refuse to answer.
- Simplify your answers if you have time, but this is less important than the calculus part of the problem.
- Unless told otherwise, **show all your work**.

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	10	10	10	10	10	10	10	10	10	10	100
Score:											

Work on reverse ☐

1. (10 points) Indicate whether the statement is true (**T**) or false (**F**). All have equal weight and you shouldn't justify your answers. Assume all functions and vector fields are smooth. **Read the statements carefully.**

   (a) $\frac{d}{dt}[f(\mathbf{r}(t))] = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$.

   (a) ____________

   (b) If f defined on $x^2 + y^2 < 1$ is continuous, then f has an absolute minimum and an absolute maximum.

   (b) ____________

   (c) $\int_0^{2\pi} \int_1^2 (1)\, dr\, d\theta$ is the area of the region $1 \leq x^2 + y^2 \leq 4$.

   (c) ____________

   (d) $\lim_{a \to 0^+} \iiint_{B_a} (x^2 + y^2 + z^2)^{-1}\, dV$ is finite, where $B_a = \{(x, y, z) : a^2 \leq x^2 + y^2 + z^2 \leq 1\}$.

   (d) ____________

   (e) If $\mathbf{F}(x, y)$ is defined on the region $1 < x^2 + y^2 < 4$ and $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path independent, then $\mathbf{F}$ is conservative.

   (e) ____________

   (f) $\int_{-C} f\, ds = -\int_C f\, ds$.

   (f) ____________

   (g) $\operatorname{curl}(\operatorname{div} \mathbf{F}) = \mathbf{0}$.

   (g) ____________

   (h) $\iint_M \mathbf{F} \cdot d\mathbf{S} = 0$, where M is the Möbius strip.

   (h) ____________

   (i) $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$, where S is the unit sphere $x^2 + y^2 + z^2 = 1$.

   (i) ____________

   (j) $\operatorname{div}(f\mathbf{F}) = \nabla f \cdot \mathbf{F} + f \operatorname{div} \mathbf{F}$.

   (j) ____________

Work on reverse ☐

2. (a) (7 points) Show that $u(x, y, t)$ defined by

$$u(x, y, t) = \frac{e^{-(x^2+y^2)/(2t)}}{2\pi t}$$

is a solution to the partial differential equation $u_t - \frac{1}{2}(u_{xx} + u_{yy}) = 0$. You may use your calculation for u_{xx} to guess u_{yy} without penalty, provided you do so correctly.

(b) (3 points) For any fixed (x, y), determine (with justification) $\lim_{t \to +\infty} u(x, y, t)$.

3. Let $f(x, y) = x^4 - 4xy + y^4$.

   (a) (5 points) Find the local minima, local maxima, and saddle points of f.

   (b) (5 points) Find the tangent plane to the graph of $z = f(x, y)$ at $(0, 0, 0)$. Near $(0, 0, 0)$ but not at $(0, 0, 0)$, is the graph of $z = f(x, y)$ strictly above the tangent plane, strictly below, or neither? Explain how you came to your conclusion.

Work on reverse ☐

4. (10 points) Explain why $f(x, y) = e^{-x^2-y^2}(x^2 + 2y^2)$ has an absolute minimum and an absolute maximum on the region $x^2 + y^2 \leq 4$, and then find all absolute minima and absolute maxima. To compare values at the end, remember that $e > 2$.

Work on reverse ☐

5. (10 points) Find the mass and center of mass for the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$, $z = 3 - x$, with constant density k. You may use symmetry to skip certain calculations, but indicate this in your solution. The identity $\cos^2 \alpha = \frac{1}{2}(1 + \cos 2\alpha)$ may be useful.

Work on reverse □

6. (10 points) Evaluate $\iint_R \cos(9x^2 + 4y^2)\, dA$, where R is the region in the first quadrant bounded by the ellipse $9x^2 + 4y^2 = \frac{\pi}{2}$.

Work on reverse □

7. (10 points) Evaluate $\iiint_E e^{(x^2+y^2+z^2)^{3/2}}\,dV$, E enclosed by sphere $x^2+y^2+z^2=4$ in the first octant.

Work on reverse □

8. (10 points) Evaluate $\int_C y^3\,dx - x^3\,dy$, where C is the circle $x^2+y^2=4$, oriented *clockwise*.

Work on reverse ☐

9. Do **not** use the theorems of Green, Stokes, or Gauss in parts (a) or (b).

   (a) (4 points) Evaluate $\int_C (y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}) \cdot d\mathbf{r}$, where C is the circle $x^2 + y^2 = 1$, $z = 0$, oriented counterclockwise as viewed from above.

   (b) (4 points) Evaluate $\iint_S (-\mathbf{i} - \mathbf{j} - \mathbf{k}) \cdot d\mathbf{S}$ where S is the hemisphere $x^2 + y^2 + z^2 = 1$, where $z \geq 0$, oriented upward.

   (c) (2 points) Explain the similarity of the results above.

Work on reverse ☐

10. (10 points) Suppose that $u(x, y, z)$ is a smooth function and solves the partial differential equation

$$\nabla^2 u = 1.$$

If a simple solid region E has outward-oriented boundary surface S, show $\iint_S \nabla u \cdot d\mathbf{S}$ is the volume of E.

Extra sheet