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Spring 2006, Math 104
Solutions to the **Final Examination**

16 May, 2006
5:00-8:00 PM

1. (12 points, 4 points each.) Compute the following. A correct answer will get full credit whether or not work is shown. An incorrect answer may get partial credit if work is shown that uses a basically correct method.

(a) $\lim_{n\to\infty} (n^7+2^n)/(n^6+2^{n+1})$ *Answer:* ½.

(b) The radius of convergence of the power series $\Sigma_{n=1}^{\infty} \frac{3^n+(-4)^n}{5} z^n$. *Answer:* ¼.

(c) $\int_0^1 e^x \, d\alpha(x)$, where $\alpha(x) = \begin{cases} -1 & \text{for } 0 \le x \le ½, \\ 0 & \text{for } ½ < x < ¾, \\ 2 & \text{for } ¾ \le x \le 1. \end{cases}$ *Answer:* $e^{½} + 2e^{¾}$.

2. (24 points, 4 points each.) Complete the following definitions. You may use, without defining them, terms or symbols that Rudin defines before he defines the word or symbol asked for. Your definitions do not have to have exactly the same wording as those in Rudin, but for full credit they should be clear, and mean the same thing as his.

(a) If X is a metric space, E a subset of X, and p a point of X, then p is said to be a *limit point* of E if . . .

Answer: every neighborhood of p in X contains a point $q \neq p$ such that $q \in E$.

(b) If E is a subset of a metric space X, then an *open covering* of E in X means . . .

Answer: a set $\{G_\alpha\}$ of open subsets $G_\alpha \subseteq X$ such that $E \subseteq \bigcup_\alpha G_\alpha$.

(c) If $a_1, a_2, \dots, a_n, \dots$ is a sequence of complex numbers, and s is a complex number, we write $\Sigma_{n=1}^{\infty} a_n = s$ if . . . *Answer:* $\lim_{n\to\infty} \Sigma_{k=1}^{n} a_k = s$.

(d) If X and Y are metric spaces, and $f: X \to Y$ a map, then f is said to be *uniformly continuous* if . . .

Answer: for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $p, q \in X$ with $d(p, q) < \delta$ we have $d(f(p), f(q)) < \varepsilon$.

(e) If X is a metric space, then a sequence (f_n) (in Rudin's notation, $\{f_n\}$) of complex-valued functions on X is said to be *pointwise bounded* if

Answer: for every $x \in X$ there exists a real number c such that for all $n \geq 1$, one has $|f_n(x)| < c$. (Rudin gives two slightly different wordings on p.155; either of these is, of course, also acceptable.)

(f) If X is a metric space and $f \in \mathscr{C}(X)$, then $\|f\|$ denotes . . . *Answer:* $\sup_{x\in X} |f(x)|$.

3. (24 points, 4 points each.) For each of the items listed below, either *give an example* with the properties stated, or give a brief reason why *no such example exists.*

If you give an example, you do *not* have to prove that it has the property stated; however, your examples should be specific; i.e., even if there are many objects of a given sort, you should name a particular one. If you give a reason why no example exists, don't worry about giving reasons for your reasons; a simple statement will suffice.

(a) An unbounded subset E of a compact metric space K.

Answer: Does not exist. A compact set is bounded, and a subset of a bounded set is bounded.

(b) A continuous one-to-one and onto function between metric spaces, $f: X \to Y$, such that the inverse function $f^{-1}: Y \to X$ is not continuous.

Answer: Let $X = [0, 2\pi) \subseteq R$, let $Y = \{(x, y) \in R^2 \mid x^2 + y^2 = 1\}$, and let f be defined by $f(t) = (\cos t, \sin t)$. Or let $X = [0, 1) \cup \{2\}$, let $Y = [0, 1]$, and let f be defined by $f(x) = x$ for $x < 1$, $f(2) = 1$.

(c) Two monotonically increasing functions α and β on the interval $[0,1]$, and a real-valued function f on that set which belongs to $\mathscr{R}(\alpha)$ but not to $\mathscr{R}(\beta)$.

Answer: Many examples. E.g., you could choose for α any continuous monotonically increasing function, and for β the function which has value 0 on $[0, ½)$ and 1 on $[½, 1]$, and let $f = \beta$. Or you could let α be the zero function, $\beta(x) = x$, and f be the function that is 1 on rationals and 0 on irrationals.

(d) A metric space X, a sequence of continuous real-valued functions (f_n) on X, and a continuous real-valued function f on X such that $f_n \to f$ pointwise, but not uniformly.

Answer: Many examples. E.g., let $X = [0,1]$, let f_n be defined by $f_n(x) = nx$ if $x \in [0, 1/(2n)]$, $f_n(x) = 1 - nx$ if $x \in [1/(2n), 1/n]$, and $f_n(x) = 0$ if $x \in [1/n, 1]$, and let f be the zero function.

(e) A metric space X, a sequence of continuous real-valued functions (f_n) on X, and a *discontinuous* function f on X such that $f_n \to f$ uniformly.

Answer: Does not exist: A uniform limit of continuous functions is continuous.

(f) Two distinct algebras $\mathscr{A} \neq \mathscr{B}$ of continuous real-valued functions on $[0,1]$ both of which separate points of $[0,1]$, vanish at no point of $[0,1]$, and are uniformly closed.

Answer: Do not exist. By the Stone-Weierstrass theorem, the first two assumptions on $\mathscr{A}$ and $\mathscr{B}$ imply that each of them has all of $\mathscr{C}([0,1])$ as its uniform closure; but the final condition is equivalent to saying that each of them is its own uniform closure, hence both of them must equal $\mathscr{C}([0,1])$.

4. (8 points) Suppose F is an ordered, field, and S a subset of F which has a least upper bound $\alpha \in F$. Let x be an element of F satisfying $x > 0$. Show that the set $xS = \{xs \mid s \in S\}$ also has a least upper bound in F, namely $x\alpha$. Note: Rudin proves that in any ordered field, the usual laws relating inequalities and the field operations (addition, subtraction, multiplication and division) hold; so you may assume these properties.

Answer: Since α is an upper bound for S, every $s \in S$ satisfies $s \leq \alpha$. Since x is positive, this implies $xs \leq x\alpha$, showing that $x\alpha$ is an upper bound for xS.

To show that $x\alpha$ is the least upper bound *for xS, let us take any $\beta < x\alpha$ in F and show that it is not an upper bound for xS. Multiplying the inequality $\beta < x\alpha$ by x^{-1} we get $x^{-1}\beta < \alpha$; hence as α is the least upper bound of S, $x^{-1}\beta$ is not an upper bound of S, so there is some $s \in S$ satisfying $s > x^{-1}\beta$. Multiplying by x gives $xs > \beta$, showing, as required, that β is not an upper bound for xS.*

5. (10 points) Suppose (f_n) is a sequence of real-valued differentiable functions on R, and that its sequence of derivatives, (f_n') is uniformly bounded. Show that the sequence of functions (f_n) is equicontinuous.

(Recall that for any sequence of real-valued functions (g_n) on a metric space X, the statement that (g_n) is *uniformly bounded* means that there exists a real number M such that for all x and n, $|g_n(x)| < M$, while to say that a sequence of real-valued functions (f_n) on X is *equicontinuous* means that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all n and all $p, q \in X$, we have $d(p, q) < \delta \Rightarrow d(f_n(p), f_n(q)) < \varepsilon$.)

Answer: Let M be as in the above definition of uniform boundedness for (f_n'). Given $\varepsilon > 0$, let $\delta = \varepsilon/M$.

Suppose $p, q \in R$ satisfy $d(p, q) < \delta$ i.e., $|p - q| < \delta$. If $p = q$ then for every n we have $f_n(p) = f_n(q)$; so $d(f_n(p), f_n(q)) = 0 < \varepsilon$. In the contrary case, applying the Mean Value Theorem to f_n on the segment (p, q) if $p < q$, or on the segment (q, p) if $p > q$, we conclude that for some x in that segment we have $f_n(q) - f_n(p) = (q - p)f_n(x)$. Taking absolute values, and substituting in the assumptions $|p - q| < \delta$, $|f_n'(x)| < M$, and $\delta = \varepsilon/M$, we get $|f_n(p) - f_n(q)| < \varepsilon$, i.e., $d(f_n(p), f_n(q)) < \varepsilon$, as required.

6. (8 points) Let $\mathscr{A}$ be an algebra of continuous complex-valued functions on a metric space X. Show that if $\mathscr{A}$ contains the algebra of all continuous real-valued functions on X, then $\mathscr{A}$ is precisely the algebra of all continuous complex-valued functions on X. (This is an argument used by Rudin in proving the complex case of the Stone-Weierstrass theorem; so you will be more or less repeating what he did there.)

Each time you use in your proof one of the conditions defining the statement that $\mathscr{A}$ is an algebra, state that condition.

Answer: By assumption, $\mathscr{A}$ is contained in the algebra of all continuous complex-valued functions on X, so it suffices to show that if f is such a function, then $f \in \mathscr{A}$.

Any continuous complex-valued function f on X can be written $f = f_1 + if_2$, where f_1 and f_2 are continuous real-valued functions, hence, by assumption, are

members of $\mathcal{A}$. An algebra of complex-valued functions is closed under multiplication by complex constants, so since $f_2 \in \mathcal{A}$ we have $if_2 \in \mathcal{A}$. An algebra of complex-valued functions is also closed under addition, so as f_1 and if_2 are in $\mathcal{A}$, so is $f_1 + if_2 = f$, as required.

7. (14 points, 2 points each.) Below, a generalization of a theorem in Rudin is stated and proved. After certain steps of the proof I have inserted parenthetical questions such as "(0 Why?)" Answer each of these questions at the bottom of the page, after the corresponding number. Your answers can be results proved in Rudin (you don't have to specify their statement-numbers!), observations about the given situation, or calculations. You should seldom need as much space as is given for the answers; one key fact or calculation is what is wanted in each case. If you can't justify some step, you may still assume it in justifying later steps.

Note that each question is about the assertion that *immediately* precedes it – not about earlier assertions.

Theorem. *Let α be a monotonically increasing real-valued function on an interval $[a, b]$, and let $f_1, \dots, f_k$ be real-valued functions on $[a, b]$ which each belong to $\mathcal{R}(\alpha)$. We shall write $\mathbf{f}: [a, b] \to R^k$ for the function defined by $\mathbf{f}(x) = (f_1(x), \dots, f_k(x))$. Let K be any compact subset of R^k containing $\mathbf{f}([a, b]) = \{\mathbf{f}(x) \mid x \in [a, b]\}$, and let φ be any continuous real-valued function on K.*

Then the function $\varphi \circ \mathbf{f} : [a, b] \to R$ defined by $(\varphi \circ \mathbf{f})(x) = \varphi(\mathbf{f}(x))$ also belongs to $\mathcal{R}(\alpha)$.

Proof. We shall show that there exist partitions P of $[a, b]$ making the differences

(i) $$U(P, \alpha, \varphi \circ \mathbf{f}) - L(P, \alpha, \varphi \circ \mathbf{f})$$

arbitrarily small. This is equivalent to the desired integrability statement.

Take any $\varepsilon > 0$.

The function φ is uniformly continuous on K (1 Why? *Answer: A continuous function on a compact set is uniformly continuous*), hence we may choose $\delta > 0$ so that for any points p and q of K we have

(ii) $$d(p, q) < \delta \Rightarrow |\varphi(p) - \varphi(q)| < \varepsilon.$$

Let us now choose, for each $j \in \{1, \dots, k\}$, a partition P_j of $[a, b]$ such that

(iii) $$U(P_j, \alpha, f_j) - L(P_j, \alpha, f_j) < \delta\varepsilon.$$

(2 Which of our assumptions implies that such partitions exist? *Answer: $f_j \in \mathcal{R}(\alpha)$ $(j = 1, \dots, k)$.*) Let $P = (x_0, \dots, x_n)$ be a common refinement of these partitions $P_1, \dots, P_k$.

Our plan will be to divide the set of n intervals $[x_{i-1}, x_i]$ $(i = 1, \dots, n)$ into two subsets, such that on each interval in the first subset, the difference between $\sup(\varphi \circ \mathbf{f})$ and $\inf(\varphi \circ \mathbf{f})$ is small, while in the other subset, the sum of the lengths $\Delta\alpha_i$ is small, and show that these properties together make (i) small. To do this, let A be the set of all $i \in \{1, \dots, n\}$ such that

(iv) $$\sup_{x \in [x_{i-1}, x_i]} f_j(x) - \inf_{x \in [x_{i-1}, x_i]} f_j(x) < \delta/\sqrt{k} \quad \text{for } j = 1, \dots, k,$$

and let B consist of the remaining elements of $\{1, \dots, n\}$, that is, those i such that the inequality of (iv) fails for at least one j.

Note that if $i \in A$ and if x, y are points of $[x_{i-1}, x_i]$, then (iv) implies that for $j = 1, \dots, k$ we have $|f_j(x) - f_j(y)| < \delta/\sqrt{k}$. Hence $d(\mathbf{f}(x), \mathbf{f}(y)) < \delta$, by the formula for distance in R^k. (3 What is that formula? *Answer: $d(p, q) = ((p_1 - q_1)^2 + \dots + (p_k - q_k)^2)^{1/2}$.*) So by (ii), for such x and y we have $|(\varphi \circ \mathbf{f})(x) - (\varphi \circ \mathbf{f})(y)| < \varepsilon$; hence

$$\sup_{x \in [x_{i-1}, x_i]} (\varphi \circ \mathbf{f})(x) - \inf_{x \in [x_{i-1}, x_i]} (\varphi \circ \mathbf{f})(x) \leq \varepsilon.$$

Multiplying each of these inequalities by $\Delta\alpha_i$, and summing them over $i \in A$, we conclude that the contribution to (i) of the intervals $[x_{i-1}, x_i]$ with $i \in A$ is

(v) $$\leq (\Sigma_{i \in A} \Delta\alpha_i)\varepsilon \leq (\alpha(b) - \alpha(a))\varepsilon.$$ (4 Explain the second "$\leq$" *Answer: The sum is of a subset of the numbers $\Delta\alpha_i$, hence it is $\leq$ the sum of all $\Delta\alpha_i$, which is $\alpha(b) - \alpha(a)$.*)

We next consider the intervals in our partition P indexed by elements $i \in B$. The fact that these have small total length will be a consequence of the conditions $f_j \in \mathcal{R}(\alpha)$, which we embodied in condition (iii). Let us combine these conditions into one inequality by summing the inequality (iii) over j, getting:

(vi) $$\Sigma_{j=1}^{k} (U(P_j, \alpha, f_j) - L(P_j, \alpha, f_j)) < k\delta\varepsilon.$$

(5 Where did the k on the right hand side come from? *Answer: We have added up k inequalities, so right-hand side is sum of k terms $\delta\varepsilon$.*) Now if we expand the left-hand side of (vi) by using the definition of $U(P_j, \alpha, f_j)$ and of $L(P_j, \alpha, f_j)$ as sums of terms, one for each interval $[x_{i-1}, x_i]$, then for each $i \in \{1, \ldots, n\}$, the terms corresponding to the interval $[x_{i-1}, x_i]$ add up to

(vii) $$\Sigma_{j=1}^{k} (\sup_{x \in [x_{i-1}, x_i]} f_j(x) - \inf_{x \in [x_{i-1}, x_i]} f_j(x)) \Delta\alpha_i.$$

Now for those i belonging to our set B, (vii) is $\geq (\delta/\sqrt{k})\Delta\alpha_i$. (6 Why? *Answer: Because $i \in B$ means that the inequality of* (iv) *fails for at least one j.*) Hence summing over i, we see that the contribution of intervals with $i \in B$ to the left-hand side of (vi) is $\geq (\delta/\sqrt{k}) \Sigma_{i \in B} \Delta\alpha_i$, and substituting this into (vi) we get

$$(\delta/\sqrt{k}) \Sigma_{i \in B} \Delta\alpha_i < k\delta\varepsilon.$$

Dividing both sides by $\delta/\sqrt{k}$ we get our desired result that the $\Delta\alpha_i$ with $i \in B$ have small sum:

(viii) $$\Sigma_{i \in B} \Delta\alpha_i < k^{3/2}\varepsilon.$$

Now the contribution to (i) of the intervals indexed by terms $i \in B$ is at most

$$\Sigma_{i \in B} (\sup_{p \in K} \varphi(p) - \inf_{p \in K} \varphi(p)) \Delta\alpha_i$$

(7 What general result implies that the above sup and inf are finite? *Answer: A continuous function on a compact set is bounded.*), and by (viii) the above sum is

(ix) $$\leq (\sup_{p \in K} \varphi(p) - \inf_{p \in K} \varphi(p)) k^{3/2}\varepsilon.$$

So (v) and (ix) give bounds on the contributions to (i) of the intervals $[x_{i-1}, x_i]$ with i in A and B respectively. Adding these, we conclude that (i) is at most

$$((\sup_{p \in K} \varphi(p) - \inf_{p \in K} \varphi(p)) k^{3/2} + \alpha(b) - \alpha(a))\varepsilon.$$

By taking ε sufficiently small, this can be made arbitrarily small, as required.

(*Remark: The theorem in Rudin generalized above is Theorem 6.11, p.127, which is the $k=1$ case of the above result.*)

I HOPE YOU DID WELL!

Good luck on your remaining exams
and
HAVE A GOOD SUMMER!