George M. Bergman
247 Cory Hall

Spring 2006, Math 104
Final Examination

16 May, 2006
5:00-8:00 PM

1. (12 points, 4 points each.) Compute the following. A correct answer will get full credit whether or not work is shown. An incorrect answer may get partial credit if work is shown that uses a basically correct method.

(a) $\lim_{n\to\infty} (n^7+2^n)/(n^6+2^{n+1})$

(b) The radius of convergence of the power series $\Sigma_{n=1}^{\infty} \dfrac{3^n+(-4)^n}{5} z^n$.

(c) $\int_0^1 e^x \, d\alpha(x)$, where $\alpha(x) = \begin{cases} -1 & \text{for } 0 \le x \le 1/2, \\ 0 & \text{for } 1/2 < x < 3/4, \\ 2 & \text{for } 3/4 \le x \le 1. \end{cases}$

2. (24 points, 4 points each.) Complete the following definitions. You may use, without defining them, terms or symbols that Rudin defines before he defines the word or symbol asked for. Your definitions do not have to have exactly the same wording as those in Rudin, but for full credit they should be clear, and mean the same thing as his.

(a) If X is a metric space, E a subset of X, and p a point of X, then p is said to be a *limit point* of E if . . .

(b) If E is a subset of a metric space X, then an *open covering* of E in X means . . .

(c) If $a_1, a_2, \ldots, a_n, \ldots$ is a sequence of complex numbers, and s is a complex number, we write $\Sigma_{n=1}^{\infty} a_n = s$ if . . .

(d) If X and Y are metric spaces, and $f: X \to Y$ a map, then f is said to be *uniformly continuous* if . . .

(e) If X is a metric space, then a sequence (f_n) (in Rudin's notation, $\{f_n\}$) of complex-valued functions on X is said to be *pointwise bounded* if

(f) If X is a metric space and $f \in \mathscr{C}(X)$, then $\|f\|$ denotes . . .

3. (24 points, 4 points each.) For each of the items listed below, either *give an example* with the properties stated, or give a brief reason why *no such example exists.*

If you give an example, you do *not* have to prove that it has the property stated; however, your examples should be specific; i.e., even if there are many objects of a given sort, you should name a particular one. If you give a reason why no example exists, don't worry about giving reasons for your reasons; a simple statement will suffice.

(a) An unbounded subset E of a compact metric space K.

(b) A continuous one-to-one and onto function between metric spaces, $f: X \to Y$, such that the inverse function $f^{-1}: Y \to X$ is not continuous.

(c) Two monotonically increasing functions α and β on the interval $[0,1]$, and a real-valued function f on that set which belongs to $\mathscr{R}(\alpha)$ but not to $\mathscr{R}(\beta)$.

(d) A metric space X, a sequence of continuous real-valued functions (f_n) on X, and a continuous real-valued function f on X such that $f_n \to f$ pointwise, but not uniformly.

(e) A metric space X, a sequence of continuous real-valued functions (f_n) on X, and a *discontinuous* function f on X such that $f_n \to f$ uniformly.

(f) Two distinct algebras $\mathscr{A} \neq \mathscr{B}$ of continuous real-valued functions on $[0,1]$ both of which separate points of $[0,1]$, vanish at no point of $[0,1]$, and are uniformly closed.

4. (8 points) Suppose F is an ordered, field, and S a subset of F which has a least upper bound $\alpha \in F$. Let x be an element of F satisfying $x > 0$. Show that the set $xS = \{xs \mid s \in S\}$ also has a least upper bound in F, namely $x\alpha$. Note: Rudin proves that in any ordered field, the usual laws relating inequalities and the field operations (addition, subtraction, multiplication and division) hold; so you may assume these properties.

5. (10 points) Suppose (f_n) is a sequence of real-valued differentiable functions on R, and that its sequence of derivatives, (f_n') is uniformly bounded. Show that the sequence of functions (f_n) is equicontinuous.

(Recall that for any sequence of real-valued functions (g_n) on a metric space X, the statement that (g_n) is *uniformly bounded* means that there exists a real number M such that for all x and n, $|g_n(x)| < M$, while to say that a sequence of real-valued functions (f_n) on X is *equicontinuous* means that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all n and all $p, q \in X$, we have $d(p,q) < \delta \Rightarrow d(f_n(p), f_n(q)) < \varepsilon$.)

6. (8 points) Let $\mathscr{A}$ be an algebra of continuous complex-valued functions on a metric space X. Show that if $\mathscr{A}$ contains the algebra of all continuous real-valued functions on X, then $\mathscr{A}$ is precisely the algebra of all continuous complex-valued functions on X. (This is an argument used by Rudin in proving the complex case of the Stone-Weierstrass theorem; so you will be more or less repeating what he did there.)

Each time you use in your proof one of the conditions defining the statement that $\mathscr{A}$ is an algebra, state that condition.

7. (14 points, 2 points each.) Below, a generalization of a theorem in Rudin is stated and proved. After certain steps of the proof I have inserted parenthetical questions such as "([0] Why?)". Answer each of these questions at the bottom of the page, after the corresponding number. Your answers can be results proved in Rudin (you don't have to specify their statement-numbers!), observations about the given situation, or calculations. You should seldom need as much space as is given for the answers; one key fact or calculation is what is wanted in each case. If you can't justify some step, you may still assume it in justifying later steps.

Note that each question is about the assertion that *immediately* precedes it – not about earlier assertions.

Theorem. *Let* α *be a monotonically increasing real-valued function on an interval* $[a, b]$, *and let* $f_1, \ldots, f_k$ *be real-valued functions on* $[a, b]$ *which each belong to* $\mathscr{R}(\alpha)$. *We shall write* $\mathbf{f}: [a, b] \to R^k$ *for the function defined by* $\mathbf{f}(x) = (f_1(x), \ldots, f_k(x))$. *Let* K *be any compact subset of* R^k *containing* $\mathbf{f}([a, b]) = \{\mathbf{f}(x) \mid x \in [a, b]\}$, *and let* φ *be any continuous real-valued function on* K.

Then the function $\varphi \circ \mathbf{f} : [a, b] \to R$ *defined by* $(\varphi \circ \mathbf{f})(x) = \varphi(\mathbf{f}(x))$ *also belongs to* $\mathscr{R}(\alpha)$.

Proof. We shall show that there exist partitions P of $[a, b]$ making the differences

(i) $$U(P, \alpha, \varphi \circ \mathbf{f}) - L(P, \alpha, \varphi \circ \mathbf{f})$$

arbitrarily small. This is equivalent to the desired integrability statement.

Take any $\varepsilon > 0$.

The function φ is uniformly continuous on K ([1] Why?), hence we may choose $\delta > 0$ so that for any points p and q of K we have

(ii) $$d(p, q) < \delta \Rightarrow |\varphi(p) - \varphi(q)| < \varepsilon.$$

Let us now choose, for each $j \in \{1, \ldots, k\}$, a partition P_j of $[a, b]$ such that

(iii) $$U(P_j, \alpha, f_j) - L(P_j, \alpha, f_j) < \delta\varepsilon.$$

([2] Which of our assumptions implies that such partitions exist?) Let $P = (x_0, \ldots, x_n)$ be a common refinement of these partitions $P_1, \ldots, P_k$.

Our plan will be to divide the set of n intervals $[x_{i-1}, x_i]$ $(i=1, \ldots, n)$ into two subsets, such that on each interval in the first subset, the difference between $\sup(\varphi \circ \mathbf{f})$ and $\inf(\varphi \circ \mathbf{f})$ is small, while in the other subset, the sum of the lengths $\Delta\alpha_i$ is small, and show that these properties together make (i) small. To do this, let A be the set of all $i \in \{1, \ldots, n\}$ such that

(iv) $$\sup_{x \in [x_{i-1}, x_i]} f_j(x) - \inf_{x \in [x_{i-1}, x_i]} f_j(x) < \delta/\surd k \quad \text{for } j = 1, \ldots, k,$$

and let B consist of the remaining elements of $\{1, \ldots, n\}$, that is, those i such that the inequality of (iv) fails for at least one j.

Note that if $i \in A$ and if x, y are points of $[x_{i-1}, x_i]$, then (iv) implies that for $j=1, \ldots, k$ we have $|f_j(x) - f_j(y)| < \delta/\surd k$. Hence $d(\mathbf{f}(x), \mathbf{f}(y)) < \delta$, by the formula for distance in R^k. ([3] What is that formula?) So by (ii), for such x and y we have $|(\varphi \circ \mathbf{f})(x) - (\varphi \circ \mathbf{f})(y)| < \varepsilon$; hence

$$\sup_{x \in [x_{i-1}, x_i]} (\varphi \circ \mathbf{f})(x) - \inf_{x \in [x_{i-1}, x_i]} (\varphi \circ \mathbf{f})(x) \leq \varepsilon.$$

Multiplying each of these inequalities by $\Delta\alpha_i$, and summing them over $i \in A$, we conclude that the contribution to (i) of the intervals $[x_{i-1}, x_i]$ with $i \in A$ is

(v) $\leq (\Sigma_{i\in A} \Delta\alpha_i)\,\varepsilon \leq (\alpha(b)-\alpha(a))\,\varepsilon$. (4 Explain the second "$\leq$")

We next consider the intervals in our partition P indexed by elements $i \in B$. The fact that these have small total length will be a consequence of the conditions $f_j \in \mathscr{R}(\alpha)$, which we embodied in condition (iii). Let us combine these conditions into one inequality by summing the inequality (iii) over j, getting:

(vi) $\Sigma_{j=1}^{k} (U(P_j, \alpha, f_j) - L(P_j, \alpha, f_j)) < k\,\delta\,\varepsilon.$

(5 Where did the k on the right hand side come from?) Now if we expand the left-hand side of (vi) by using the definition of $U(P_j, \alpha, f_j)$ and of $L(P_j, \alpha, f_j)$ as sums of terms, one for each interval $[x_{i-1}, x_i]$, then for each $i \in \{1, \dots, n\}$, the terms corresponding to the interval $[x_{i-1}, x_i]$ add up to

(vii) $\Sigma_{j=1}^{k} (\sup_{x\in[x_{i-1},x_i]} f_j(x) - \inf_{x\in[x_{i-1},x_i]} f_j(x))\,\Delta\alpha_i.$

Now for those i belonging to our set B, (vii) is $\geq (\delta/\surd k)\,\Delta\alpha_i$. (6 Why?) Hence summing over i, we see that the contribution of intervals with $i \in B$ to the left-hand side of (vi) is $\geq (\delta/\surd k)\,\Sigma_{i\in B}\Delta\alpha_i$, and substituting this into (vi) we get

$$(\delta/\surd k)\,\Sigma_{i\in B}\Delta\alpha_i < k\,\delta\,\varepsilon.$$

Dividing both sides by $\delta/\surd k$ we get our desired result that the $\Delta\alpha_i$ with $i \in B$ have small sum:

(viii) $\Sigma_{i\in B}\Delta\alpha_i < k^{3/2}\,\varepsilon.$

Now the contribution to (i) of the intervals indexed by terms $i \in B$ is at most

$$\Sigma_{i\in B} (\sup_{p\in K} \varphi(p) - \inf_{p\in K} \varphi(p))\,\Delta\alpha_i$$

(7 What general result implies that the above sup and inf are finite?), and by (viii) the above sum is

(ix) $\leq (\sup_{p\in K} \varphi(p) - \inf_{p\in K} \varphi(p))\,k^{3/2}\,\varepsilon.$

So (v) and (ix) give bounds on the contributions to (i) of the intervals $[x_{i-1}, x_i]$ with i in A and B respectively. Adding these, we conclude that (i) is at most

$$((\sup_{p\in K} \varphi(p) - \inf_{p\in K} \varphi(p))\,k^{3/2} + \alpha(b) - \alpha(a))\,\varepsilon.$$

By taking ε sufficiently small, this can be made arbitrarily small, as required.