MATH 110-1 Linear Algebra
Spring 2000 I. Novik

Midterm I

1. **(18pts)** This part consists of 6 questions. Each question is worth **3**pts. In each question give an example with the required properties or explain why such an example does not exist.

   (a) A vector space over $\mathbf{R}$ of dimension 100.

   (b) Two **isomorphic** vector spaces, one of which has dimension 17 and the second one has dimension 13.

   (c) A linear transformation $T : P_4(\mathbf{R}) \to M_{2\times 2}(\mathbf{R})$ which is **one-to-one.**

   (d) A generating set for $P_2(\mathbf{R})$ which is not a basis.

   (e) An infinite-dimensional vector space.

   (f) A linear transformation $T : \mathbf{R}^{80} \to \mathbf{R}^{170}$ of rank 90.

2. **(16pts)** Suppose that V is a vector space of dimension 9 and that W is a vector space of dimension 11. Let T be a linear transformation

$$T : L(V, W) \to L(V, W).$$

Show that the nullity of T cannot equal the rank of T.

3. **(16pts)** Let V be a vector space over $\mathbf{R}$. Prove that every non-zero linear transformation $T : V \to \mathbf{R}$ is **onto**, and that every non-zero linear transformation $S : \mathbf{R} \to V$ is one-to-one.

4. **(50pts)** Let W_1, W_2 be subspaces of a vector space V. Define the **sum** of W_1 and W_2, denoted $W_1 + W_2$, to be the set $\{x + y \mid x \in W_1 \text{ and } y \in W_2\}$.

   (a) **(10pts)** Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

   (b) **(10pts)** Let B_1 be a basis for W_1, let B_2 be a basis for W_2, Show that $B_1 \cup B_2$ generates $W_1 + W_2$.

   (c) **(20pts)** Suppose that B_1 and B_2 are disjoint. Prove that $B_1 \cup B_2$ is a basis for $W_1 + W_2$ if and only if $W_1 \cap W_2 = \{0\}$.

   (d) **(10pts)** Prove that if V is finite-dimensional, then

   $$\dim(W_1 + W_2) \leq \dim(W_1) + \dim(W_2).$$

   When does equality hold?