Department of Mathematics, University of California, Berkeley

Mathematics 140

Metric Differential Geometry

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Midterm Exam, October 11, 2001

Instructions. Do all of the problems below. You may refer to the page of notes which you brought to the exam with you. Please hand in these notes along with your blue book.

1. Let C be a plane curve defined by an equation of the form $y = f(x)$, where $f(x)$ is a smooth function of x. Supposed that it is parametrized in the "left to right" direction.
A. Find a formula for the signed curvature of C at $(0, f(0))$ in terms of derivatives of f at $x = 0$. You may use the formula

$$\kappa = \frac{||\ddot{\gamma} \times \dot{\gamma}||}{||\dot{\gamma}||^3}, \qquad (1)$$

for the curvature of a curve in $\mathbb{R}^3$.
B. For **extra credit**, *without* using Equation (1) or the result of part A, show that, if $f'(0) = 0$, then the signed curvature of C at the point $(0, f(0))$ is equal to $f''(0)$.

2.
A. Show that the circular cylinder in (x, y, z) space defined by the equation $x^2 + y^2 = 1$ can be covered by a single surface patch whose domain is the "punctured disc" defined by $0 < u^2 + v^2 < \pi^2$ in the (u, v) plane.
B. Is this surface patch an isometry? Is it conformal? Is it equiareal? Justify your answers. You should be able to answer the questions by geometric arguments, without computing the first fundamental form of the patch. In fact, you might even be able to answer them without having solved part A.
C. For **extra credit**, compute the first fundamental form of your surface patch.

3.
Let ϕ be the mapping from $\mathbb{R}^3$ to itself defined by $\phi(\mathbf{x}) = a\mathbf{x}$, where a is some positive real number. Let C be a space curve whose curvature is nowhere zero, and let D be the curve $\phi(C)$. State clearly the relation between the torsion of C at a point $\mathbf{x}$ and the torsion of D at the corresponding point $\phi(\mathbf{x})$, and prove this relation. [*Hint*: choose a parametrization of C, and then use ϕ to get a parametrization of D. Compare the tangent, normal, and binormal vectors of the two curves.]