DYNAMICAL ZETA FUNCTIONS FOR ANOSOV FLOWS VIA MICROLOCAL ANALYSIS

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Abstract. The purpose of this paper is to give a short microlocal proof of the meromorphic continuation of the Ruelle zeta function for C^∞ Anosov flows. More general results have been recently proved by Giulietti–Liverani–Pollicott [GiLiPo] but our approach is different and is based on the study of the generator of the flow as a semiclassical differential operator.

The purpose of this article is to provide a short microlocal proof of the meromorphic continuation of the Ruelle zeta function for C^∞ Anosov flows on compact manifolds:

Theorem. *Suppose X is a compact manifold and $\varphi_t : X \to X$ is a C^∞ Anosov flow with orientable stable and unstable bundles. Let $\{\gamma^\sharp\}$ denote the set of primitive orbits of φ_t, with $T_\gamma^\sharp$ their periods. Then the Ruelle zeta function,*

$$\zeta_{\rm R}(\lambda) = \prod_{\gamma^\sharp} (1 - e^{i\lambda T_\gamma^\sharp}),$$

which converges for $\operatorname{Im}\lambda \gg 0$ has a meromorphic continuation to $\mathbb{C}$.

In fact the proof applies to any Anosov flow for which linearized Poincaré maps $\mathcal{P}_\gamma$ for closed orbits γ satisfy

$$|\det(I - \mathcal{P}_\gamma)| = (-1)^q \det(I - \mathcal{P}_\gamma), \quad \text{with } q \text{ independent of } \gamma. \tag{1.1}$$

A class of examples is provided by $X = S^*M$ where M is a compact orientable negatively curved manifold with φ_t the geodesic flow – see [GiLiPo, Lemma B.1]. For methods which can be used to eliminate the orientability assumptions see [GiLiPo, Appendix B].

The meromorphic continuation of $\zeta_{\rm R}$ was conjectured by Smale [Sm] and in greater generality it was proved very recently by Giulietti, Liverani, and Pollicott [GiLiPo]. Another recent perspective on dynamical zeta functions in the contact case has been provided by Faure and Tsujii [FaTs1, FaTs2]. Our motivation and proof are however different from those of [GiLiPo]: we were investigating trace formulæ for Pollicott–Ruelle resonances [Po, Ru86] which give some lower bounds on their counting function. Sharp upper bounds were given recently in [DDZ, FaSj].

To explain the trace formula for resonances suppose first that $X = S^*\Gamma\backslash\mathbb{H}^2$ is a compact Riemann surface. Then the Selberg trace formula combined with the Guillemin trace formula [Gu] gives

$$\sum_{\mu \in \mathrm{Res}(P)} e^{-i\mu t} = \sum_\gamma \frac{T_\gamma^\# \delta(t - T_\gamma)}{|\det(I - \mathcal{P}_\gamma)|}, \quad t > 0, \tag{1.2}$$

see [Le] for an accessible presentation in the physics literature and [DFG] for the case of higher dimensions. On the left hand side $\mathrm{Res}(P)$ is the set of resonances of $P = -iV$ where V is the generator of the flow,

$$\mathrm{Res}(P) = \left\{ \mu_{j,k} = \lambda_j - i(k + \tfrac{1}{2}),\ j, k \in \mathbb{N} \right\},$$

where λ_j's are the zeros of the Selberg zeta function included according to their multiplicities. On the right hand side γ's are periodic orbits, $\mathcal{P}_\gamma$ is the linearized Poincaré map, T_γ is the period of γ, and $T_\gamma^\#$ is the primitive period.

The point of view of Faure–Sjöstrand [FaSj] stresses the analogy between analysis of the propagator $\varphi_{-t}^* = e^{-itP}$ with scattering theory for elliptic operators on non-compact manifolds: for flows, the fiber infinity of T^*X is the analogue of spatial infinity for scattering on non-compact manifolds. Melrose's Poisson formula for resonances valid for Euclidean infinities [Me82, SjZw, Zw96] and some hyperbolic infinities [GuZw97] suggests that (1.2) should be valid for general Anosov flows but that seems to be unknown.

In general, the validity of (1.2) follows from the finite order (as an entire function) of the analytic continuation of

$$\zeta_1(\lambda) := \exp\left(- \sum_\gamma \frac{T_\gamma^\# e^{i\lambda T_\gamma}}{T_\gamma |\det(I - \mathcal{P}_\gamma)|} \right). \tag{1.3}$$

The μ's appearing on the left hand side of (1.2) are the zeros of ζ_1 – see [GuZw97, §5] or [Zw96] for an indication of this simple fact. Under certain analyticity assumptions on X and φ_t, Rugh [Ru] and Fried [Fr] showed that the Ruelle zeta function $\zeta_R(\lambda)$ is a meromorphic function of finite order but neither [GiLiPo] nor our paper suggest the validity of such a statement in general.

One reason to be interested in (1.2) in the general case is the following consequence based on [GuZw99, §4]: the counting function for the Pollicott–Ruelle resonances in wide strips cannot be sublinear. More precisely, there exists a constant C_0 such that for each $\varepsilon \in (0, 1)$,

$$\#\{\mu \in \mathrm{Res}(P) \ :\ \mathrm{Im}\,\mu > -C_0/\varepsilon,\ |\mu| \leq r\} \not\leq r^{1-\varepsilon}, \quad r \geq C(\varepsilon). \tag{1.4}$$

We arrived at the proof of main Theorem while attempting to demonstrate (1.2) for C^∞ Anosov flows. We now indicate the idea of that proof in the case of analytic continuation of $\zeta_1(\lambda)$ given by (1.3). It converges for $\mathrm{Im}\,\lambda \gg 0$ – see Lemma 2.2

for convergence and (2.5) below for the connection to the Ruelle zeta function. The starting point is Guillemin's formula,

$$\operatorname{tr}^\flat e^{-itP} = \sum_\gamma \frac{T_\gamma^\# \delta(t - T_\gamma)}{|\det(I - \mathcal{P}_\gamma)|}, \quad t > 0 \tag{1.5}$$

where the trace is defined using distributional operations of pullback by $\iota(t,x) = (t,x,x)$ and pushforward by $\pi : (t,x) \to t$: $\operatorname{tr}^\flat e^{-itP} := \pi_* \iota^* K_{e^{-itP}}$, where $K_\bullet$ denotes the distributional kernel of an operator. The pullback is well-defined in the sense of distributions [HöI–II, §8.2] because the wave front set of $K_{e^{-itP}}$ satisfies

$$\operatorname{WF}(K_{e^{-itP}}) \cap N^*(\mathbb{R}_t \times \Delta(X)) = \emptyset, \quad t > 0, \tag{1.6}$$

where $\Delta(X) \subset X \times X$ is the diagonal and $N^*(\mathbb{R}_t \times \Delta(X)) \subset T^*(\mathbb{R}_t \times X \times X)$ is the conormal bundle. See Appendix B and [Gu, §II] for details.

Since

$$\frac{d}{d\lambda} \log \zeta_1(\lambda) = \frac{1}{i} \sum_\gamma \frac{T_\gamma^\# e^{i\lambda T_\gamma}}{|\det(I - \mathcal{P}_\gamma)|} = \frac{1}{i} \int_0^\infty e^{it\lambda} \operatorname{tr}^\flat e^{-itP} dt,$$

it is enough to show that the right hand side has a meromorphic continuation to $\mathbb{C}$ with simple poles and residues which are non-negative integers. For that it is enough to take $t_0 > 0$ smaller than T_γ for all γ (note that $\operatorname{tr}^\flat e^{-itP} = 0$ on $(0, t_0)$) and consider a continuation of

$$\frac{1}{i} \int_{t_0}^\infty e^{it\lambda} \operatorname{tr}^\flat e^{-itP} dt = \frac{1}{i} e^{it_0\lambda} \int_0^\infty e^{it\lambda} \operatorname{tr}^\flat \varphi_{-t_0}^* e^{-itP} dt.$$

We now note that

$$i \int_0^\infty e^{it\lambda} \varphi_{-t_0}^* e^{-itP} dt = \varphi_{-t_0}^* (P - \lambda)^{-1} \quad \text{for } \operatorname{Im} \lambda \gg 0. \tag{1.7}$$

With a justification provided by a simple approximation argument (see the proof of [HöIII–IV, Theorem 19.4.1] for a similar construction) it is then sufficient to continue

$$\operatorname{tr}^\flat \left(\varphi_{-t_0}^* (P - \lambda)^{-1} \right), \quad \operatorname{Im} \lambda \gg 0, \tag{1.8}$$

meromorphically. As recalled in §3.2, $(P - \lambda)^{-1} : C^\infty(X) \to \mathcal{D}'(X)$ continues meromorphically so to check the meromorphy of (1.8) we only need to check the analogue of the wave front set relation (1.6) for the distributional kernel of $\varphi_{-t_0}^*(P - \lambda)^{-1}$, namely that this wave front set does not intersect $N^*(\Delta(X))$. But that follows from an adaptation of propagation results of Duistermaat–Hörmander [HöIII–IV, §26.1], Melrose [Me94], and Vasy [Va]. The Faure–Sjöstrand spaces [FaSj] provide the a priori regularity which allows an application of these techniques. In fact, we use somewhat simpler anisotropic Sobolev spaces in our argument and provide an alternative approach to the meromorphic continuation of the resolvent – see §§3.1, 3.2.

Remarks. (i) If the coefficients of the generator of the flow are merely C^k for large enough k, then microlocal methods presented in this paper show that the Ruelle zeta function can still be continued meromorphically to a strip $\{\operatorname{Im}\lambda \geq -k/C\}$, where C is a constant independent of k. That follows immediately from the fact that wavefront set statements in H^s regularity depend only on a finite number of derivatives of the symbols involved. In [GiLiPo] a more precise estimate on the width of the strip was provided.

(ii) One conceptual difference between [GiLiPo] and the present paper is the following. In [GiLiPo, (2.11), (2.12)], the resolvent $(P-\lambda)^{-1}$ is decomposed into two pieces, one of which corresponds to resonances in a large disk and the other one to the rest of the resonances; using an auxiliary determinant [GiLiPo, (2.7)], it is shown that it is enough to study mapping properties of large iterates of $(P-\lambda)^{-1}$, which implies that resonances outside the disk can be ignored in a certain asymptotic regime. In our work, however, we show directly that $(P-\lambda)^{-1}$ lies in a class where one can take the flat trace. In terms of the expression (1.7), this requires uniform control of the wavefront set of φ_{-t}^* as $t\to+\infty$. Such a statement does not follow from the analysis for bounded times and this is where the matters are considerably simplified by using radial source/sink estimates originating in scattering theory.

(iii) In this paper we only provide analysis at bounded frequencies, but do not discuss the behavior of $\zeta_R(\lambda)$ as λ goes to infinity. However, a high frequency analysis of the zeta function is possible using the methods of semiclassical analysis, which recover the structure of $(P-\lambda)^{-1}$ modulo $\mathcal{O}(|\lambda|^{-\infty})$, rather than just compact, errors. An example is provided by the bounds on the number of Pollicott–Ruelle resonances in [FaSj, DDZ].

Organization of the paper. In §2 we list the preliminaries from dynamical systems and microlocal analysis. Precise definitions, references and proofs of the statements in §2 are given in the appendices. They are all standard and reasonably well known but as the paper is interdisciplinary in spirit we provide detailed arguments. Except for references to texts [HöI–II, HöIII–IV, Zw], the paper is self-contained.

In §3 we simultaneously prove the meromorphic continuation and describe the wave front set of the Schwartz kernel of $(\mathbf{P}-\lambda)^{-1}$. This is based on results about propagation of singularities. The vector field H_p has *radial-like sets*, that is invariant conic closed sets which are sources/sinks for the flow – they correspond to stable/unstable directions in the Anosov decomposition. Away from those sets the results are classical and due to Duistermaat–Hörmander – see for instance [HöIII–IV, §26.1]. At the radial points we use the more recent propagation results of Melrose [Me94] and Vasy [Va]. The a priori regularity needed there is provided by the properties of the spaces H_{sG}. Finally, in §4 we give our proof of the main theorem which is a straightforward application of the results in §3 and the more standard results recalled in §2.

Notation. We use the following notation: $f = \mathcal{O}_\ell(g)_H$ means that $\|f\|_H \le C_\ell g$ where the norm (or any seminorm) is in the space H, and the constant C_ℓ depends on ℓ. When either ℓ or H are absent then the constant is universal or the estimate is scalar, respectively. When $G = \mathcal{O}_\ell(g)_{H_1 \to H_2}$ then the operator $G : H_1 \to H_2$ has its norm bounded by $C_\ell g$.

2. Preliminaries

2.1. **Dynamical systems.** Let X be a compact manifold and $\varphi_t : X \to X$ be a C^∞ flow, $\varphi_t = \exp tV$, $V \in C^\infty(X;TX)$. The flow is *Anosov* if the tangent space to X has a continuous decomposition $T_xX = E_0(x) \oplus E_s(x) \oplus E_u(x)$ which is invariant, $d\varphi_t(x)E_\bullet(x) = E_\bullet(\varphi_t(x))$, $E_0(x) = \mathbb{R}V(x)$, and for some C and $\theta > 0$ fixed

$$\begin{aligned} |d\varphi_t(x)v|_{\varphi_t(x)} &\le Ce^{-\theta|t|}|v|_x, \quad v \in E_u(x), \quad t < 0, \\ |d\varphi_t(x)v|_{\varphi_t(x)} &\le Ce^{-\theta|t|}|v|_x, \quad v \in E_s(x), \quad t > 0. \end{aligned} \tag{2.1}$$

where $|\bullet|_y$ is given by a smooth Riemannian metric on X. Fix a smooth volume form μ on X. We present here some basic results: an upper bound on the number of closed trajectories of φ_t (Lemma 2.2) and on the volume of the set of trajectories that return to a small neighbourhood of their originating point after a given time (Lemma 2.1). These bounds are used in the proof of Lemma 4.1. See Appendix A for the proofs. The constant L is defined in (A.3).

Lemma 2.1. *Define the following measure on $X \times \mathbb{R}$: $\tilde\mu = \mu \times dt$ and fix $t_e > 0$. Then there exists C such that for each $\varepsilon > 0, T > t_e$, and $n = \dim X$,*

$$\tilde\mu\big(\{(x,t) \mid t_e \le t \le T,\ d(x,\varphi_t(x)) \le \varepsilon\}\big) \le C\varepsilon^n e^{nLT}. \tag{2.2}$$

In particular, by letting $\varepsilon \to 0$, we get a bound on the number of closed trajectories:

Lemma 2.2. *Let $N(T)$ be the number of closed trajectories of φ_t of period no more than T. Then*

$$N(T) \le Ce^{(2n-1)LT}. \tag{2.3}$$

2.2. **Trace identities.** Let $\varphi_t = e^{tV}$ be as in §2.1 and $\mathbf{P} : C^\infty(X;\mathcal{E}) \to C^\infty(X;\mathcal{E})$ be defined by $\mathbf{P} = \frac{1}{i}\mathcal{L}_V$ on the vector bundle of differential forms of all orders on X, see (3.1). Let $\mathcal{E}_0^k$ be the smooth invariant subbundle of $\mathcal{E}$ given by all differential k-forms $\mathbf{u}$ satisfying $\iota_V\mathbf{u} = 0$, where ι denotes the contraction operator by a vector field – see also [GiLiPo, (3.5)]. We recall the trace formula of Guillemin [Gu, Theorem 8, (II.22)] which is valid for any flow with nondegenerate periodic trajectories – see Appendix B for a self-contained proof in the Anosov case. In our notation it says that

$$\operatorname{tr}^\flat e^{-it\mathbf{P}}|_{C^\infty(X;\mathcal{E}_0^k)} = \sum_\gamma \frac{T_\gamma^\# \operatorname{tr}(\wedge^k\mathcal{P}_\gamma)\,\delta(t-T_\gamma)}{|\det(I-\mathcal{P}_\gamma)|}, \quad t > 0, \tag{2.4}$$

where γ's are periodic orbits, $\mathcal{P}_\gamma := d\varphi_{-T_\gamma}|_{E_s\oplus E_u}$ is the linearized Poincaré map, T_γ is the period of γ, and $T_\gamma^\#$ is the primitive period. See §2.4 for definition and properties of the flat trace $\mathrm{tr}^\flat$. By the Anosov property, the eigenvalues of $\mathcal{P}_\gamma|_{E_u}$ satisfy $|\mu|<1$, therefore $\det(I-\mathcal{P}_\gamma|_{E_u})>0$. Similarly $\det(I-\mathcal{P}_\gamma^{-1}|_{E_s})>0$. If E_s is orientable, then $\det(\mathcal{P}_\gamma|_{E_s})=\det(d\varphi_{-T_\gamma}|_{E_s})>0$; since $\det(I-\mathcal{P}_\gamma|_{E_s})=\det(-\mathcal{P}_\gamma|_{E_s})\det(I-\mathcal{P}_\gamma^{-1}|_{E_s})$,

$$|\det(I-\mathcal{P}_\gamma)|=(-1)^{\frac{n-1}{2}}\det(I-\mathcal{P}_\gamma),$$

that is (1.1) holds with $q=(n-1)/2$. We now assume (1.1) for some integer q.

Consequently we relate the expressions on the right hand side of (2.4) to the Ruelle zeta function using

$$\det(I-\mathcal{P}_\gamma)=\sum_{k=0}^{n-1}(-1)^k\,\mathrm{tr}\wedge^k\mathcal{P}_\gamma.$$

This is a standard argument going back to Ruelle [Ru76] but the particular determinants here seem to be rather different than the one related to his transfer operators:

$$\begin{aligned}\zeta_{\mathrm{R}}(\lambda)&=\prod_{\gamma^\#}(1-e^{i\lambda T_\gamma^\#})=\exp\Bigg(-\sum_{\gamma^\#}\sum_{m=1}^\infty\frac{1}{m}e^{i\lambda mT_\gamma^\#}\Bigg)\\&=\exp\Bigg(-\sum_\gamma T_\gamma^\# e^{i\lambda T_\gamma}/T_\gamma\Bigg)=\prod_{k=0}^{n-1}\exp\Bigg(-\sum_\gamma\frac{T_\gamma^\# e^{i\lambda T_\gamma}\,\mathrm{tr}\wedge^k\mathcal{P}_\gamma}{T_\gamma|\det(I-\mathcal{P}_\gamma)|}\Bigg)^{(-1)^{k+q}}\end{aligned}\tag{2.5}$$

We note that thanks to Lemma 2.2 the sums on the right hand side converge for $\operatorname{Im}\lambda\gg 0$.

2.3. Microlocal and semiclassical analyses.

In this section we present concepts and facts from microlocal/semiclassical analysis which are needed in the proofs. Their proofs and detailed references are provided in Appendix C.

Let X be a manifold. For a distribution $u\in\mathcal{D}'(X)$, a phase space description of its singularities is given by the wave front set $\mathrm{WF}(u)$, a closed conic subset of $T^*X\setminus 0$. A more general object is the semiclassical wave front set defined using a (small) asymptotic parameter h for h-tempered families of distributions $\{u(h)\}_{0<h<1}$: $\mathrm{WF}_h(u)\subset\overline{T}^*X$ where $\overline{T}^*X$ is the fiber-radially compactified cotangent bundle, a manifold with interior T^*X and boundary $\partial\overline{T}^*X=S^*X=(T^*X\setminus 0)/\mathbb{R}^+$, the cosphere bundle. In addition to singularities, WF_h measures oscillations on the h-scale. The relation of the two wave front sets is the following: if u is an h-independent distribution, then

$$\mathrm{WF}(u)=\mathrm{WF}_h(u)\cap(T^*X\setminus 0),\tag{2.6}$$

see §C.2 and for a more general statement, [Zw, (8.4.8)].

For operators we define the wave front set $\mathrm{WF}'(B)$ (or $\mathrm{WF}'_h(B)$ for h-dependent families of operators) using the Schwartz kernel – see (C.2). This way $\mathrm{WF}'(I)=$

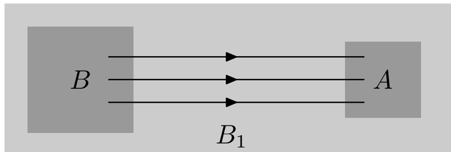

FIGURE 1. The assumptions of Proposition 2.5, displaying the wave front sets of A, B, B_1 and the flow lines of H_p.

$\Delta(T^*X)$, the diagonal in $T^*X \times T^*X$, rather than $N^*\Delta(X)$, the conormal bundle to the diagonal in $X \times X$.

The following result, proved in §C.2, will allow us to calculate $\mathrm{WF}'_h((P-\lambda)^{-1})$, and thus, by (2.6), $\mathrm{WF}'((P-\lambda)^{-1})$. It states that away from the fiber infinity, the semiclassical wave front set of an operator is characterized using its action on distributions:

Lemma 2.3. *Let $B : C_c^\infty(X) \to \mathcal{D}'(Y)$ be an h-tempered family of operators. A point $(y, \eta, x, \xi) \in T^*(Y \times X)$ does not lie in $\mathrm{WF}'_h(B)$ if and only if there exist neighbourhoods U of (x, ξ) and V of (y, η) such that*

$$\mathrm{WF}_h(f) \subset U \implies \mathrm{WF}_h(Bf) \cap V = \emptyset \tag{2.7}$$

for each h-tempered family of functions $f(h) \in C_c^\infty(X)$.

We next state several semiclassical estimates used in §3. To be able to work with differential forms, we consider a semiclassical pseudodifferential operator $\mathbf{P} \in \Psi_h^k(X; \mathrm{Hom}(\mathcal{E}))$ acting on h-tempered families of distributions $\mathbf{u}(h) \in \mathcal{D}'(X; \mathcal{E})$ with values in a vector bundle $\mathcal{E}$ over X. For simplicity, we assume below that X is a compact manifold. We provide estimates in semiclassical Sobolev spaces $H_h^m(X, \mathcal{E})$ (denoted H_h^m for simplicity) and the corresponding restrictions on wave front sets. Each of the estimates (2.8), (2.10), (2.13), (2.15) is understood as follows: if the right-hand side is well-defined, then for h small enough, the left-hand side is well-defined and the estimate holds. For example, in the case of (2.10), if $\mathbf{Pu} \in H_h^m$ and $B\mathbf{u} \in H_h^m$, then we have $A\mathbf{u} \in H_h^m$. See §C.3 for the proofs.

Proposition 2.4. *(Elliptic estimate) Let $\mathbf{u}(h) \in \mathcal{D}'(X; \mathcal{E})$ be h-tempered. Then:*

1. If $A \in \Psi_h^0(X)$ (acting on $\mathcal{D}'(X; \mathcal{E})$ diagonally) and $\mathbf{P}$ is elliptic on $\mathrm{WF}_h(A)$, then for each m,

$$\|A\mathbf{u}\|_{H_h^m(X;\mathcal{E})} \le C\|\mathbf{Pu}\|_{H_h^{m-k}(X;\mathcal{E})} + \mathcal{O}(h^\infty). \tag{2.8}$$

2. If $\mathrm{ell}_h(\mathbf{P}) \subset \overline{T}^*X$ denotes the elliptic set of $\mathbf{P}$, then

$$\mathrm{WF}_h(\mathbf{u}) \cap \mathrm{ell}_h(\mathbf{P}) \subset \mathrm{WF}_h(\mathbf{Pu}). \tag{2.9}$$

Proposition 2.5. *(Propagation of singularities) Assume that* $\mathbf{P} \in \Psi_h^1(X; \mathrm{Hom}(\mathcal{E}))$ *and the semiclassical principal symbol*

$$\sigma_h(\mathbf{P}) \in S_h^1(X; \mathrm{Hom}(\mathcal{E}))/hS_h^0(X; \mathrm{Hom}(\mathcal{E}))$$

is diagonal with entries[1] $p - iq$, *with* $p \in S^1(X; \mathbb{R})$ *independent of* h *and* $q \geq 0$ *everywhere. Assume also that* p *is homogeneous of degree* 1 *in* ξ, *for* $|\xi|$ *large enough. Let* e^{tH_p} *be the Hamiltonian flow of* p *on* $\overline{T}^*X$ *and* $\mathbf{u}(h) \in \mathcal{D}'(X; \mathcal{E})$ *be an* h*-tempered family of distributions. Then (see Figure 1):*

1. *Assume that* $A, B, B_1 \in \Psi_h^0(X)$ *and for each* $(x, \xi) \in \mathrm{WF}_h(A)$, *there exists* $T \geq 0$ *with* $e^{-TH_p}(x, \xi) \in \mathrm{ell}_h(B)$ *and* $e^{tH_p}(x, \xi) \in \mathrm{ell}_h(B_1)$ *for* $t \in [-T, 0]$. *Then for each* m,

$$\|A\mathbf{u}\|_{H_h^m(X;\mathcal{E})} \leq C\|B\mathbf{u}\|_{H_h^m(X;\mathcal{E})} + Ch^{-1}\|B_1\mathbf{P}\mathbf{u}\|_{H_h^m(X;\mathcal{E})} + \mathcal{O}(h^\infty). \tag{2.10}$$

2. *If* $\gamma(t)$ *is a flow line of* H_p, *then for each* $T > 0$,

$$\gamma(-T) \notin \mathrm{WF}_h(\mathbf{u}),\ \gamma([-T, 0]) \cap \mathrm{WF}_h(\mathbf{P}\mathbf{u}) = \emptyset \implies \gamma(0) \notin \mathrm{WF}_h(\mathbf{u}). \tag{2.11}$$

Propagation of singularities states in particular that if $\mathbf{P}\mathbf{u} = \mathcal{O}(h^\infty)_{C^\infty}$ and $\mathbf{u} = \mathcal{O}(1)_{H_h^m}$ microlocally near some $(x, \xi) \in \overline{T}^*X$, then $\mathbf{u} = \mathcal{O}(1)_{H_h^m}$ microlocally near $e^{tH_p}(x, \xi)$ for $t \geq 0$; in other words, regularity can be propagated forward along the Hamiltonian flow lines. (If $q \leq 0$ instead, then regularity could be propagated backward.) We next state less standard estimates guaranteeing regularity of $\mathbf{u}$ near sources/sinks, provided that $\mathbf{u}$ lies in a sufficiently high Sobolev space.

Denote by $\kappa : T^*X \setminus 0 \to S^*X = \partial\overline{T}^*X$ the natural projection map. Let p be a real-valued function on T^*X; for simplicity, we assume that it is homogeneous of degree 1 in ξ. Assume that $L \subset T^*X \setminus 0$ is a closed conic set invariant under the flow e^{tH_p} and there exists an open conic neighbourhood U of L with the following properties for some constant $\theta > 0$:

$$\begin{aligned} d\big(\kappa(e^{-tH_p}(U)), \kappa(L)\big) &\to 0 \ \text{ as } t \to +\infty; \\ (x, \xi) \in U \implies |e^{-tH_p}(x, \xi)| &\geq C^{-1}e^{\theta t}|\xi|, \ \text{ for any norm on the fibers.} \end{aligned} \tag{2.12}$$

We call L a *radial source.* A *radial sink* is defined analogously, reversing the direction of the flow. The following propositions come essentially from the work of Melrose [Me94, Propositions 9,10] and Vasy [Va, Propositions 2.3,2.4]. The first one shows that for sufficiently regular distributions the wave front set at radial sources is controlled.

Proposition 2.6. *Assume that* $\mathbf{P} \in \Psi_h^1(X; \mathrm{Hom}(\mathcal{E}))$ *is as in Proposition 2.5 and* $L \subset T^*X \setminus 0$ *is a radial source. Then there exists* $m_0 > 0$ *such that (see Figure 2(a))*

[1] Strictly speaking, this means that $p - iq$ is some representative of the equivalence class $\sigma_h(\mathbf{P})$ satisfying the specified conditions.

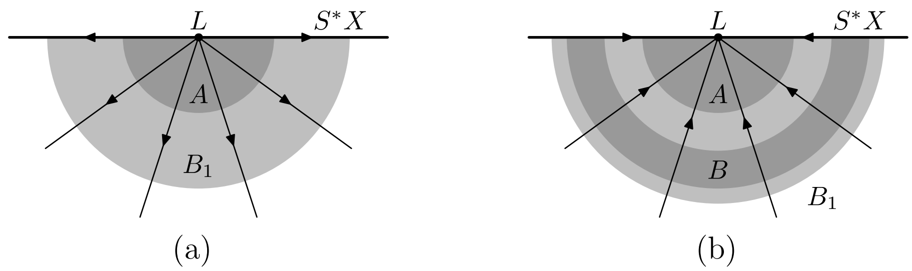

Figure 2. (a) The assumptions of Proposition 2.6. (b) The assumptions of Proposition 2.7. Here S^*X is the boundary of $\overline{T}^*X$ and the flow lines of H_p are pictured.

1. *For each $B_1 \in \Psi_h^0(X)$ elliptic on $\kappa(L) \subset S^*X = \partial\overline{T}^*X$, there exists $A \in \Psi_h^0(X)$ elliptic on $\kappa(L)$ such that if $\mathbf{u}(h) \in \mathcal{D}'(X;\mathcal{E})$ is h-tempered, then for each $m \geq m_0$,*

$$A\mathbf{u} \in H_h^{m_0} \implies \|A\mathbf{u}\|_{H_h^m} \leq Ch^{-1}\|B_1\mathbf{P}\mathbf{u}\|_{H_h^m} + \mathcal{O}(h^\infty). \tag{2.13}$$

2. *If $\mathbf{u}(h) \in \mathcal{D}'(X;\mathcal{E})$ is h-tempered and $B_1 \in \Psi_h^0(X)$ is elliptic on $\kappa(L)$, then*

$$B_1\mathbf{u} \in H_h^{m_0},\ \mathrm{WF}_h(\mathbf{P}\mathbf{u}) \cap \kappa(L) = \emptyset \implies \mathrm{WF}_h(\mathbf{u}) \cap \kappa(L) = \emptyset. \tag{2.14}$$

The second result shows that for sufficiently low regularity we have a propagation result at radial sinks analogous to (2.10).

Proposition 2.7. *Assume that $\mathbf{P} \in \Psi_h^1(X;\mathrm{Hom}(\mathcal{E}))$ is as in Proposition 2.5 and $L \subset T^*X \setminus 0$ is a radial sink. Then there exists $m_0 > 0$ such that for each $B_1 \in \Psi_h^0(X)$ elliptic on $\kappa(L)$, there exists $A \in \Psi_h^0(X)$ elliptic on $\kappa(L)$ and $B \in \Psi_h^0(X)$ with $\mathrm{WF}_h(B) \subset \mathrm{ell}_h(B_1) \setminus \kappa(L)$, such that if $\mathbf{u}(h) \in \mathcal{D}'(X;\mathcal{E})$ is h-tempered, then for each $m \leq -m_0$ (see Figure 2(b))*

$$\|A\mathbf{u}\|_{H_h^m} \leq C\|B\mathbf{u}\|_{H_h^m} + Ch^{-1}\|B_1\mathbf{P}\mathbf{u}\|_{H_h^m} + \mathcal{O}(h^\infty). \tag{2.15}$$

Remark. The precise value of the threshold m_0 can be computed by being slightly more careful in the proofs (using a regularizer $\langle\varepsilon\xi\rangle^{-\delta}$ for small $\delta > 0$ in place of $\langle\varepsilon\xi\rangle^{-1}$ and an additional regularization procedure to justify (C.9)) – see for example [Va, Propositions 2.3,2.4].

2.4. **The flat trace.** We now consider an operator $B : C^\infty(X) \to \mathcal{D}'(X)$ satisfying

$$\mathrm{WF}'(B) \cap \Delta(T^*X) = \emptyset, \quad \Delta(T^*X) := \{(x,\xi,x,\xi) \mid (x,\xi) \in T^*X\}, \tag{2.16}$$

on a compact manifold X, and define the flat trace

$$\mathrm{tr}^\flat B := \int_X (\iota^*K_B)(x)\,dx, \quad \iota : x \mapsto (x,x). \tag{2.17}$$

Here K_B is the Schwartz kernel of X with respect to the density dx on X; the trace $\mathrm{tr}^\flat B$ does not depend on the choice of the density. The pullback $\iota^*K_B \in \mathcal{D}'(X)$ of the

Schwartz kernel $K_B \in \mathcal{D}'(X \times X)$ is defined under the condition (2.16) as in [HöI–II, Theorem 8.2.4].

To obtain a concrete expression for $\operatorname{tr}^\flat B$ we use traces of regularized operators. For that we introduce a family of mollifiers. Let $d(x, y)$ be the geodesic distance for (x, y) in a neighbourhood of $\Delta(X) \subset X \times X$ with respect to some fixed Riemannian metric. Let $\psi \in C^\infty_c(\mathbb{R}, [0, 1])$ be equal to 1 near 0. We define $E_\varepsilon : \mathcal{D}'(X) \to C^\infty(X)$,

$$E_\varepsilon u(x) = \int_X E_\varepsilon(x, y)u(y)\,dy, \quad E_\varepsilon(x, y) = \frac{1}{F_\varepsilon(x)}\,\psi\left(\frac{d(x, y)}{\varepsilon}\right), \tag{2.18}$$

where $F_\varepsilon(x)$ is chosen so that $E_\varepsilon(1) = 1$ and satisfies $\varepsilon^n/C \le F_\varepsilon(x) \le C\varepsilon^n$. We have

$$E_\varepsilon \in \Psi^{-\infty}(X), \quad E_\varepsilon \longrightarrow I \ \text{ in } \Psi^{0+}(X). \tag{2.19}$$

The next lemma shows that the flat trace is well approximated by regular traces – see §C.1 for a proof.

Lemma 2.8. *For B satisfying (2.16) and E_ε given by (2.18) we have*

$$\operatorname{tr}^\flat B = \lim_{\varepsilon\to 0} \operatorname{tr} E_\varepsilon B E_\varepsilon \tag{2.20}$$

where the trace on the right hand side is well-defined since $E_\varepsilon B E_\varepsilon$ is smoothing and thus trace class on $L^2(X)$.

If an operator $\mathbf{B}$ instead acts on sections of a smooth vector bundle, $\mathbf{B} : C^\infty(X; \mathcal{E}) \to \mathcal{D}'(X; \mathcal{E})$, and satisfies (2.16), then we can define the trace of $\mathbf{B}$ by the formula

$$\operatorname{tr}^\flat \mathbf{B} = \operatorname{tr}^\flat \sum_{j=1}^r B_{jj}, \quad \mathbf{B}(f\mathbf{e}_l) = \sum_{j=1}^r (B_{jl}f)\mathbf{e}_j, \ f \in C^\infty(X),$$

if $\mathbf{e}_1, \ldots, \mathbf{e}_r$ is a local frame of $\mathcal{E}$ and $\mathbf{B}$ is supported in the domain of the local frame – the general case is handled by a partition of unity and the independence of the choice of the frame is easily verified.

3. Properties of the resolvent

In this section we use the anisotropic Sobolev spaces H_{sG} and the propagation results recalled in §2.3 to describe the microlocal structure of the meromorphic continuation of the resolvent. Our proof is different that the argument in [FaSj] in the sense that we use a less refined weight to define anisotropic Sobolev spaces and derive the Fredholm property of $\mathbf{P} - \lambda$ from propagation of singularities.

Anisotropic Sobolev spaces appeared in the study of Anosov flows in the works of Baladi [Ba], Baladi–Tsujii [BaTs], Gouëzel–Liverani [GoLi], Liverani [Li], and other authors. However, the use of microlocally defined exponential weights allows a more direct study using PDE methods.

3.1. **Anisotropic Sobolev spaces.** Let (X, φ_t) be as in §2.1 and consider the vector bundle, $\mathcal{E}$, of differential forms of all orders on X. (The resolvents on forms of different degree are decoupled from each other, however we treat them as a single resolvent to simplify notation.) Consider the first order differential operator

$$\mathbf{P} : C^\infty(X; \mathcal{E}) \to C^\infty(X; \mathcal{E}), \quad \mathbf{P}(\mathbf{u}) = \frac{1}{i}\mathcal{L}_V \mathbf{u}, \quad \mathcal{E} := \bigoplus_{j=0}^{n} \Lambda^j(T^*X), \tag{3.1}$$

where V is the generator of the flow φ_t, $\mathcal{L}$ denotes the Lie derivative, and $\mathbf{u}$ is a differential form on X.

The principal symbol $\sigma(\mathbf{P}) = p \in S^1(X; \mathbb{R})$, as defined in §C.1, is diagonal and homogeneous of degree 1: $p(x, \xi) = \xi(V(x))$, $(x, \xi) \in T^*X$. This follows immediately from the fact that for any basis $\mathbf{e}_1, \dots, \mathbf{e}_r$ of $\mathcal{E}$, and all $u_1, \dots, u_r \in C^\infty(X)$,

$$\mathcal{L}_V \sum_{j=1}^{r} u_j \mathbf{e}_j = \sum_{j=1}^{r} V u_j \, \mathbf{e}_j + \sum_{j=1}^{r} u_j \, \mathcal{L}_V \mathbf{e}_j,$$

where the second term in the sum is a differential operator of order 0.

The Hamilton flow is $e^{tH_p}(x, \xi) = (\varphi_t(x), ({}^T d\varphi_t(x))^{-1}\xi)$. Define the decomposition

$$T_x^* X = E_0^*(x) \oplus E_s^*(x) \oplus E_u^*(x),$$

where $E_0^*(x), E_s^*(x), E_u^*(x)$ are dual to $E_0(x), E_u(x), E_s(x)$. From (2.1) it follows that

$$\begin{aligned} \xi \notin E_0^*(x) \oplus E_s^*(x) &\implies d\big(\kappa(e^{tH_p}(x,\xi)), \kappa(E_u^*)\big) \to 0 \text{ as } t \to +\infty, \\ \xi \notin E_0^*(x) \oplus E_u^*(x) &\implies d\big(\kappa(e^{tH_p}(x,\xi)), \kappa(E_s^*)\big) \to 0 \text{ as } t \to -\infty. \end{aligned} \tag{3.2}$$

Here $\kappa : T^*X \setminus 0 \to S^*X$ is the projection defined before (2.12). Moreover, under the assumptions of (3.2) we have $|e^{tH_p}(x, \xi)| \geq C^{-1} e^{\theta|t|}|\xi|$, and the convergence in (3.2) and the constant C are locally uniform in (x, ξ). In particular (3.2) implies that, in the sense of definition (2.12), the closed conic sets E_s^* and E_u^* are a radial source and a radial sink, respectively – see Figure 3 below.

To define anisotropic Sobolev spaces on which $\mathbf{P} - \lambda$ is a Fredholm operator, we use a function $m_G \in C^\infty(T^*X \setminus 0; [-1, 1])$, homogeneous of degree 0 and such that

$$\begin{gathered} m_G = 1 \quad \text{near } E_s^*, \quad m_G = -1 \quad \text{near } E_u^*, \\ H_p m_G \leq 0 \quad \text{everywhere}. \end{gathered} \tag{3.3}$$

A function with these properties, supported in a small neighbourhood of $E_s^* \cup E_u^*$, can be constructed using part 1 of Lemma C.1. A more refined version, not needed here, can be found in [FaSj, Lemma 1.2]. With m_G in place we choose a pseudodifferential operator $G \in \Psi^{0+}(X)$ satisfying

$$\sigma(G)(x, \xi) = m_G(x, \xi) \log |\xi|, \tag{3.4}$$

where $|\cdot|$ is any smooth norm on the fibers of T^*X. Then, using [Zw, §§8.3,9.3,14.2] as in [DDZ, (3.9)], $\exp(\pm sG)\in\Psi^s(X)$ for any $s>0$. The anisotropic Sobolev spaces are defined using this exponential weight:

$$H_{sG}:=\exp(-sG)(L^2(X)),\quad \|\mathbf{u}\|_{H_{sG}}:=\|\exp(sG)\mathbf{u}\|_{L^2}.$$

Note that $H^s(X)\subset H_{sG}\subset H^{-s}(X)$. Define the domain, D_{sG}, of $\mathbf{P}$ as the set of $\mathbf{u}\in H_{sG}$ such that the distribution $\mathbf{P}u$ is in H_{sG}. The Hilbert space norm on D_{sG} is given by $\|\mathbf{u}\|^2_{D_{sG}}:=\|\mathbf{u}\|^2_{H_{sG}}+\|\mathbf{P}\mathbf{u}\|^2_{H_{sG}}$.

3.2. **Ruelle–Pollicott resonances for forms.** Here we state the properties of the resolvent of $\mathbf{P}$:

Proposition 3.1. *Fix a constant $C_0>0$. Then for $s>0$ large enough depending on C_0, $\mathbf{P}-\lambda:D_{sG}\to H_{sG}$ is a Fredholm operator of index 0 in the region $\{\operatorname{Im}\lambda>-C_0\}$.*

Proposition 3.2. *Let $s>0$ be fixed as in Proposition 3.1. Then there exists a constant C_1 depending on s, such that for $\operatorname{Im}\lambda>C_1$, the operator $\mathbf{P}-\lambda:D_{sG}\to H_{sG}$ is invertible and*

$$(\mathbf{P}-\lambda)^{-1}=i\int_0^\infty e^{i\lambda t}\varphi^*_{-t}\,dt, \tag{3.5}$$

where $\varphi^*_{-t}:C^\infty(X;\mathcal{E})\to C^\infty(X;\mathcal{E})$ is the pullback operator by φ_{-t} on differential forms and the integral on the right-hand side converges in operator norm $H^s\to H^s$ and $H^{-s}\to H^{-s}$.

The Fredholm property and the invertibility of $\mathbf{P}-\lambda$ for large $\operatorname{Im}\lambda$ show that the resolvent $\mathbf{R}(\lambda)=(\mathbf{P}-\lambda)^{-1}:H_{sG}\to H_{sG}$ is a meromorphic family of operators with poles of finite rank – see for example [Zw, Proposition D.4]. Note that Ruelle–Pollicott resonances, the poles of $\mathbf{R}(\lambda)$ in the region $\operatorname{Im}\lambda>-C_0$, are then the poles of the meromorphic continuation of the Schwartz kernel of the operator given by the right-hand side of (3.5), and thus are independent of the choice of s and the weight G. Microlocal structure of $\mathbf{R}(\lambda)$ is described in

Proposition 3.3. *Let C_0 and s be as in Proposition 3.1 and assume $\operatorname{Im}\lambda_0>-C_0$. Then for λ near λ_0,*

$$\mathbf{R}(\lambda)=\mathbf{R}_H(\lambda)-\sum_{j=1}^{J(\lambda_0)}\frac{(\mathbf{P}-\lambda_0)^{j-1}\Pi}{(\lambda-\lambda_0)^j} \tag{3.6}$$

where $\mathbf{R}_H(\lambda)$ holomorphic near λ_0, $\Pi:H_{sG}\to H_{sG}$ is the commuting projection onto the kernel of $(\mathbf{P}-\lambda_0)^{J(\lambda_0)}$, and

$$\operatorname{WF}'(\mathbf{R}_H(\lambda))\subset\Delta(T^*X)\cup\Omega_+\cup(E^*_u\times E^*_s),\quad \operatorname{WF}'(\Pi)\subset E^*_u\times E^*_s, \tag{3.7}$$

where $\Delta(T^*X)$ is the diagonal and Ω_+ is the positive flow-out of e^{tH_p} on $\{p=0\}$:

$$\Omega_+=\{(e^{tH_p}(x,\xi),x,\xi)\mid t\geq 0,\ p(x,\xi)=0\}.$$

In §3.3, we construct a semiclassical nontrapping parametrix and study its h-wave front set. In §3.4, we express $\mathbf{R}(\lambda)$ via the parametrix and use the results of §3.3 to finish the proofs of Propositions 3.1–3.3.

3.3. **Complex absorbing potential near the zero section.** We will modify $\mathbf{P}-\lambda$ by a complex absorbing potential which will eliminate trapping and guarantee invertibility of the modified operator.

It is convenient now to introduce a semiclassical parameter h and use the algebra Ψ_h of semiclassical pseudodifferential operators, see §C.2. If $\mathbf{P}$ is defined in (3.1), then $h\mathbf{P} \in \Psi_h^1(X; \mathrm{Hom}(\mathcal{E}))$ is a semiclassical differential operator with principal symbol $p = \sigma_h(h\mathbf{P})$.

We need a semiclassical adaptation, $G(h) \in \Psi_h^{0+}(X)$, of the operator G, such that

$$\sigma_h(G(h))(x,\xi) = (1-\chi(x,\xi)) m_G(x,\xi) \log|\xi|, \tag{3.8}$$

where $\chi \in C_0^\infty(T^*X)$ is equal to 1 near the zero section, and $\mathrm{WF}_h(G(h))$ does not intersect the zero section. Note that, since $H_p \log|\xi|$ is homogeneous of degree zero,

$$H_p \sigma_h(G(h))(x,\xi) = (H_p m_G(x,\xi)) \log|\xi| + \mathcal{O}(1)_{S_h^0}. \tag{3.9}$$

Define the space $H_{sG(h)} = \exp(-sG(h))(L^2(X))$. For each fixed $h > 0$, the operator $G(h)$ lies in $\Psi^{0+}(X)$ and $\sigma(G(h))(x,\xi) = \sigma_h(G(h))(x, h\xi)$; therefore, $\sigma(G(h) - G)$ is bounded as $|\xi| \to \infty$. By [Zw, Theorem 8.8], $H_{sG(h)} = H_{sG}$ and the norms are equivalent, with the constant depending on h. We also use the semiclassical analogue of the space D_{sG}, with the norm

$$\|\mathbf{u}\|^2_{D_{sG(h)}} := \|\mathbf{u}\|^2_{H_{sG(h)}} + \|h\mathbf{P}\mathbf{u}\|^2_{H_{sG(h)}}.$$

We modify $h\mathbf{P}$ by adding an h-pseudodifferential *complex absorbing potential* $-iQ_\delta \in \Psi_h^0(X)$, which provides a localization to a neighbourhood of the zero section:

$$\mathrm{WF}_h(Q_\delta) \subset \{|\xi| < \delta\}, \quad \sigma_h(Q_\delta) > 0 \text{ on } \{|\xi| \leq \delta/2\}, \quad \sigma_h(Q_\delta) \geq 0 \text{ everywhere},$$

here $|\cdot|$ is a fixed norm on the fibers of T^*X. The action of

$$\mathbf{P}_\delta(z) := h\mathbf{P} - iQ_\delta - z$$

on H_{sG} is equivalent to the action on L^2 of the conjugated operator

$$\mathbf{P}_{\delta,s}(z) := e^{sG(h)} \mathbf{P}_\delta(z) e^{-sG(h)} = \mathbf{P}_\delta(z) + s[G(h), h\mathbf{P}] + \mathcal{O}(h^2)_{\Psi_h^{-1+}},$$

where the asymptotic expansion follows from [Zw, §§8.3,9.3,14.2] – see [DDZ, (3.11)]. We note that $[G(h), Q_\delta] = \mathcal{O}(h^\infty)_{\Psi^{-\infty}}$ for small enough δ, because $\mathrm{WF}_h(G(h))$ does not intersect the zero section.

We now use the propagation of semiclassical singularities and the elimination of trapping due to the complex absorbing potential to establish existence and properties of the inverse of $\mathbf{P}_\delta(z)$. The relation between propagation and solvability has a long

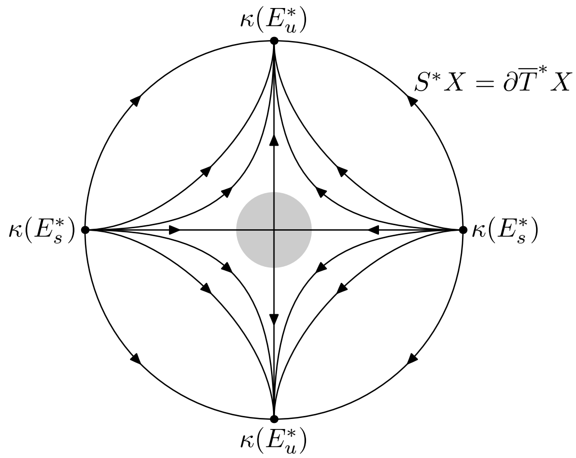

FIGURE 3. Dynamics of the flow e^{tH_p} on $\{p=0\} = \overline{E_s^* \oplus E_u^*} \subset \overline{T}^*X$, projected onto the fibers of $\overline{T^*X}$. The shaded region is the wave front set of Q_δ.

tradition – see [HöIII–IV, §26.1] Although the details below may look complicated the idea is simple and natural, given the dynamics of the flow pictured on Figure 3: given bounds on $\|\mathbf{P}_\delta(z)\mathbf{u}\|_{H_{sG(h)}}$, we first establish bounds on $\mathbf{u}$ microlocally near the sources $\kappa(E_s^*)$ by Proposition 2.6. By ellipticity (Proposition 2.4) we can also estimate $\mathbf{u}$ on $\{p \neq 0\}$ and in $\{|\xi| < \delta/2\}$, where the latter is made possible by the potential Q_δ. The resulting estimates can be propagated forward along the flow e^{tH_p}, using Proposition 2.5, to the whole $\overline{T}^*X \setminus \kappa(E_u^*)$; finally, to bound $\mathbf{u}$ microlocally near $\kappa(E_u^*)$, we use Proposition 2.7. The spaces $H_{sG(h)}$ provide the correct regularity for Propositions 2.6 and 2.7.

Proposition 3.4. *Fix a constant $C_0 > 0$ and $\varepsilon > 0$. Then for $s > 0$ large enough depending on C_0 and h small enough, the operator*

$$\mathbf{P}_\delta(z) : D_{sG(h)} \to H_{sG(h)}, \quad -C_0 h \le \operatorname{Im} z \le 1, \quad |\operatorname{Re} z| \le h^\varepsilon,$$

is invertible, and the inverse, $\mathbf{R}_\delta(z)$, satisfies

$$\|\mathbf{R}_\delta(z)\|_{H_{sG(h)} \to H_{sG(h)}} \le Ch^{-1}, \quad \mathrm{WF}'_h(\mathbf{R}_\delta(z)) \cap T^*(X \times X) \subset \Delta(T^*X) \cup \Omega_+,$$

with $\Delta(T^*X), \Omega_+$ defined in Proposition 3.3, and $\mathrm{WF}'_h(\bullet) \subset \overline{T}^*(X \times X)$ is defined in §C.2.

Proof. We first prove the bound

$$\|\mathbf{u}\|_{H_{sG(h)}} \le Ch^{-1}\|\mathbf{f}\|_{H_{sG(h)}}, \quad \mathbf{u} \in D_{sG(h)}, \quad \mathbf{f} = \mathbf{P}_\delta(z)\mathbf{u}. \tag{3.10}$$

Without loss of generality, we assume that $\|\mathbf{u}\|_{H_{sG(h)}} \le 1$. By a microlocal partition of unity, it suffices to obtain bounds on $A\mathbf{u}$, where $A \in \Psi_h^0(X)$ falls into one of the following five cases:

Case 1: $\mathrm{WF}_h(A)\cap\{p=0\}\cap\{|\xi|\geq\delta/2\}=\emptyset$. Then $\mathbf{P}_{\delta,s}(z)$ is elliptic on $\mathrm{WF}_h(A)$. We have $\|A\mathbf{u}\|_{H_{sG(h)}}=\|A^s e^{sG(h)}\mathbf{u}\|_{L^2}$, where $A^s=e^{sG(h)}Ae^{-sG(h)}\in\Psi_h^0$ and $\mathrm{WF}_h(A^s)\subset\mathrm{WF}_h(A)$. By Proposition 2.4,

$$\|A^s e^{sG(h)}\mathbf{u}\|_{L^2}\leq C\|B_1^s\mathbf{P}_{\delta,s}(z)e^{sG(h)}\mathbf{u}\|_{L^2}+\mathcal{O}(h^\infty),$$

where $B_1^s\in\Psi_h^0(X)$ is microlocalized in a neighbourhood of $\mathrm{WF}_h(A)$. Putting $B_1:=e^{-sG(h)}B_1^s e^{sG(h)}$, we obtain

$$\|A\mathbf{u}\|_{H_{sG(h)}}\leq C\|B_1\mathbf{f}\|_{H_{sG(h)}}+\mathcal{O}(h^\infty). \tag{3.11}$$

Case 2: $\mathrm{WF}_h(A)$ is contained in a small neighbourhood of $\kappa(E_s^*)$, where $\kappa:T^*X\setminus 0\to S^*X=\partial\overline{T}^*X$ is the natural projection. By [Zw, Theorem 8.6], $\exp(sG(h))\in\Psi_h^s(X)$ and $\sigma_h(\exp(sG(h)))=\exp(s\sigma_h(G(h)))=|\xi|^s$ near $\kappa(E_s^*)$. Therefore, $H_{sG(h)}$ is microlocally equivalent to the space $H_h^s(X;\mathcal{E})$ near $\kappa(E_s^*)$ in the sense that

$$\|B\mathbf{v}\|_{H_h^s}\leq C\|\mathbf{v}\|_{H_{sG(h)}}+\mathcal{O}(h^\infty),\quad \|B\mathbf{v}\|_{H_{sG(h)}}\leq C\|\mathbf{v}\|_{H_h^s}+\mathcal{O}(h^\infty), \tag{3.12}$$

for each $B\in\Psi_h^0(X)$ with $\mathrm{WF}_h(B)$ contained in a neighbourhood of $\kappa(E_s^*)$ and each h-tempered $\mathbf{v}$.

Since $\operatorname{Im} z\geq -C_0h$, we get $\operatorname{Im}\sigma_h(\mathbf{P}_\delta(z))\leq 0$. The set E_s^* is a radial source (see the discussion following (3.2)) and we can apply Proposition 2.6 and (3.12) to obtain, for s sufficiently large,

$$\|A\mathbf{u}\|_{H_{sG(h)}}\leq Ch^{-1}\|B_1\mathbf{f}\|_{H_{sG(h)}}+\mathcal{O}(h^\infty), \tag{3.13}$$

where $B_1\in\Psi_h^0(X)$ is some operator with $\mathrm{WF}_h(B_1)$ in a neighbourhood of $\kappa(E_s^*)$.

Case 3: $\mathrm{WF}_h(A)$ is contained in a small neighbourhood of some $(x_0,\xi_0)\in\{p=0\}\setminus\overline{E_u^*}$, where $\overline{E_u^*}=E_u^*\cup\kappa(E_u^*)$ is the closure of E_u^* in $\overline{T}^*X$. Then by (3.2) and the discussion following it, $d(e^{tH_p}(x_0,\xi_0),\kappa(E_s^*))\to 0$ in $\overline{T}^*X$ as $t\to-\infty$. Therefore, for any fixed neighbourhood U of $\kappa(E_s^*)$, there exists $B\in\Psi_h^0(X)$ with $\mathrm{WF}_h(B)\subset U$ and $T>0$ such that $e^{-TH_p}(\mathrm{WF}_h(A))\subset\mathrm{ell}_h(B)$.

From (3.3),(3.9) and the fact that $\operatorname{Im} z\geq -C_0h$,

$$\operatorname{Im}\sigma_h(\mathbf{P}_{\delta,s}(z))=-\sigma_h(Q_\delta)-\operatorname{Im} z+shH_p\sigma_h(G(h))\leq 0,\quad\text{in } S_h^1(X)/hS_h^0(X).$$

Applying Proposition 2.5 to the operator $\mathbf{P}_{\delta,s}(z)$ and arguing similarly to Case 1, we get $\|A\mathbf{u}\|_{H_{sG(h)}}\leq C\|B\mathbf{u}\|_{H_{sG(h)}}+Ch^{-1}\|B_2\mathbf{f}\|_{H_{sG(h)}}+\mathcal{O}(h^\infty)$, where $B_2\in\Psi_h^0$ is microlocalized in a small neighbourhood of $\bigcup_{t\in[-T,0]}e^{tH_p}(\mathrm{WF}_h(A))$. Now, $\|B\mathbf{u}\|_{H_{sG(h)}}$ can be estimated by Case 2, yielding

$$\|A\mathbf{u}\|_{H_{sG(h)}}\leq Ch^{-1}(\|B_1\mathbf{f}\|_{H_{sG(h)}}+\|B_2\mathbf{f}\|_{H_{sG(h)}})+\mathcal{O}(h^\infty), \tag{3.14}$$

where $B_1\in\Psi_h^0(X)$ is microlocalized in a small neighbourhood of $\kappa(E_s^*)$.

Case 4: $\mathrm{WF}_h(A)$ is contained in a small neighbourhood of some $(x_0,\xi_0)\in E_u^*$. Then $e^{tH_p}(x_0,\xi_0)$ converges to the zero section as $t\to-\infty$; therefore, there exists

$T > 0$ such that $e^{-TH_p}(\mathrm{WF}_h(A)) \subset \{|\xi| < \delta/2\}$. Similarly to Case 3, by propagation of singularities we find $\|A\mathbf{u}\|_{H_{sG(h)}} \le C\|B\mathbf{u}\|_{H_{sG(h)}} + Ch^{-1}\|B_2\mathbf{f}\|_{H_{sG(h)}} + \mathcal{O}(h^\infty)$, where $\mathrm{WF}_h(B) \subset \{|\xi| < \delta/2\}$ and $\mathrm{WF}_h(B_2)$ is contained in a small neighbourhood of $\bigcup_{t\in[-T,0]} e^{tH_p}(\mathrm{WF}_h(A))$. Estimating $\|B\mathbf{u}\|_{H_{sG(h)}}$ by Case 1, we get

$$\|A\mathbf{u}\|_{H_{sG(h)}} \le Ch^{-1}(\|B_1\mathbf{f}\|_{H_{sG(h)}} + \|B_2\mathbf{f}\|_{H_{sG(h)}}) + \mathcal{O}(h^\infty), \tag{3.15}$$

where B_2 is microlocalized in a small neighbourhood of $e^{-TH_p}(\mathrm{WF}_h(A))$.

Case 5: $\mathrm{WF}_h(A)$ is contained in a small neighbourhood of $\kappa(E_u^*)$. Note that the space $H_{sG(h)}$ is microlocally equivalent to the space $H_h^{-s}(X)$ near $\kappa(E_u^*)$, similarly to Case 2. Since E_u^* is a radial sink, by Proposition 2.7 we get, for s sufficiently large, $\|A\mathbf{u}\|_{H_{sG(h)}} \le C\|B\mathbf{u}\|_{H_{sG(h)}} + Ch^{-1}\|B_1\mathbf{f}\|_{H_{sG(h)}} + \mathcal{O}(h^\infty)$, where $B, B_1 \in \Psi_h^0(X)$ are microlocalized in a small neighbourhood of $\kappa(E_u^*)$ and $\mathrm{WF}_h(B) \cap \kappa(E_u^*) = \emptyset$. Then $\|B\mathbf{u}\|_{H_{sG(h)}}$ can be estimated by a combination of the preceding cases, using a microlocal partition of unity; this gives

$$\|A\mathbf{u}\|_{H_{sG(h)}} \le Ch^{-1}\|\mathbf{f}\|_{H_{sG(h)}} + \mathcal{O}(h^\infty). \tag{3.16}$$

Combining (3.11), (3.13)–(3.16), we get (3.10).

For for the dynamics of $-H_p$, E_s^* is a sink and E_u^* a source. Hence the proof of (3.10) applies to $-\mathbf{P}_\delta(z)^* = -(h\mathbf{P} - iQ_\delta - z)^*$, and we obtain the adjoint bound

$$\|\mathbf{v}\|_{H_{-sG(h)}} \le Ch^{-1}\|\mathbf{P}_\delta(z)^*\mathbf{v}\|_{H_{-sG(h)}}, \quad \mathbf{v} \in D_{-sG(h)}. \tag{3.17}$$

We now show that $\mathbf{P}_\delta(z)$ is invertible $D_{sG(h)} \to H_{sG(h)}$. Injectivity follows immediately from (3.10); we also get the bound on the inverse once surjectivity is proved. To see surjectivity, note first that (3.10) implies that if $\mathbf{u}_j \in D_{sG(h)}$ and $\mathbf{P}_\delta(z)\mathbf{u}_j$ is a Cauchy sequence in $H_{sG(h)}$, then $\mathbf{u}_j$ is a Cauchy sequence in $H_{sG(h)}$ as well; since the operator $\mathbf{P}_\delta(z)$ is closed on $H_{sG(h)}$ with domain $D_{sG(h)}$, we see that the image of $\mathbf{P}_\delta(z)$ is a closed subspace of $H_{sG(h)}$. Now, $H_{-sG(h)}$ is the dual to $H_{sG(h)}$ under the L^2 pairing (fixing an inner product on the fibers of $\mathcal{E}$) – see [Zw, (8.3.11)]. Therefore, it suffices to show that if $\mathbf{v} \in H_{-sG(h)}$ and $\langle \mathbf{P}_\delta(z)\mathbf{u}, \mathbf{v}\rangle_{L^2} = 0$ for all $\mathbf{u} \in D_{sG(h)}$, then $\mathbf{v} = 0$. Taking $\mathbf{u} \in C^\infty$, we see that $\mathbf{P}_\delta(z)^*\mathbf{v} = 0$; it remains to use (3.17).

To show the restriction on the wave front set of $\mathbf{R}_\delta(z)$, by Lemma 2.3 it is enough to show that for each $(y,\eta,x,\xi) \in T^*(X\times X)\setminus(\Delta(T^*X)\cup\Omega_+)$, there exist neighbourhoods U of (x,ξ) and V of (y,η) such that for each h-tempered $\mathbf{u} \in H_{sG(h)}$ and $\mathbf{f} := (h\mathbf{P} - iQ_\delta - z)\mathbf{u}$, if $\mathrm{WF}_h(\mathbf{f}) \subset U$, then $\mathrm{WF}_h(\mathbf{u}) \cap V = \emptyset$. This follows similarly to the proof of part 2 of Proposition 2.4 from the estimates (3.11),(3.14),(3.15), keeping in mind that $\kappa(E_s^* \cup E_u^*) \cap T^*X = \emptyset$. □

3.4. **Proofs of Propositions 3.1–3.3.** We assume that λ varies in some compact subset of $\{\operatorname{Im}\lambda > -C_0\}$ and choose h small enough so that $z = h\lambda$ satisfies $-C_0 h \le \operatorname{Im} z \le 1$, $|\operatorname{Re} z| \le h^{1/2}$.

Proposition 3.1 follows immediately from Proposition 3.4, given that H_{sG}, D_{sG} are topologically isomorphic to $H_{sG(h)}, D_{sG(h)}$ and $Q_\delta : D_{sG} \to H_{sG}$ is smoothing and thus compact.

To show Proposition 3.2, we first note that since derivatives of the flow φ_t are bounded exponentially in t, we have $\varphi_t^* = \mathcal{O}(e^{C_1|t|})_{H^{\pm s}\to H^{\pm s}}$, where C_1 is a constant depending on s. Therefore, if $\operatorname{Im}\lambda > C_1$, $\mathbf{u} \in H_{sG} \subset H^{-s}$, and $(\mathbf{P}-\lambda)\mathbf{u} = \mathbf{f} \in H_{sG}$, then integrating by parts, we see $\mathbf{u} = i\int_0^\infty e^{i\lambda t}\varphi_{-t}^*\mathbf{f}\,dt$, where the integral on the right-hand side converges in H^{-s}. This implies that $\mathbf{P}-\lambda$ is injective $D_{sG} \to H_{sG}$ and thus invertible, and (3.5) holds.

For (3.6) in Proposition 3.3 we note that the Fredholm property shows that, near a pole λ_0, $\mathbf{R}(\lambda) = \mathbf{R}_H(\lambda) + \sum_{j=1}^{J(\lambda_0)} A_j/(\lambda-\lambda_0)^j$, where A_j are operators of finite rank – see for instance [Zw, §D.3]. We have

$$\Pi := -A_1 = \frac{1}{2\pi i}\oint_{\lambda_0}(\lambda-\mathbf{P})^{-1}d\lambda, \tag{3.18}$$

$[\Pi, \mathbf{P}] = 0$ and, using Cauchy's theorem, $\Pi^2 = \Pi$. Equating powers of $\lambda - \lambda_0$ in the equation $(\mathbf{P}-\lambda)\mathbf{R}(\lambda) = I_{H_{sG}}$ shows that $A_j = -(\mathbf{P}-\lambda_0)^{j-1}\Pi$, and $(\mathbf{P}-\lambda_0)^{J(\lambda_0)}\Pi = 0$.

Finally, to show (3.7) we use the formula

$$\mathbf{R}(\lambda) = h\big(\mathbf{R}_\delta(z) - i\mathbf{R}_\delta(z)Q_\delta\mathbf{R}_\delta(z)\big) - \mathbf{R}_\delta(z)Q_\delta\mathbf{R}(\lambda)Q_\delta\mathbf{R}_\delta(z), \tag{3.19}$$

where $\mathbf{R}(\lambda) = (\mathbf{P}-\lambda)^{-1}$, $\mathbf{R}_\delta(z) = (h\mathbf{P} - z - iQ_\delta)^{-1}$, and $z = h\lambda$. Now, by Proposition 3.4, and since Q_δ is pseudodifferential, we get

$$\operatorname{WF}'_h(\mathbf{R}_\delta(z) - i\mathbf{R}_\delta(z)Q_\delta\mathbf{R}_\delta(z)) \cap T^*(X\times X) \subset \Delta(T^*X)\cup\Omega_+.$$

To handle the remaining term in (3.19), we first assume that λ is not a pole of $\mathbf{R}$. Applying again Proposition 3.4, we see that

$$\operatorname{WF}'_h(\mathbf{R}_\delta(z)Q_\delta\mathbf{R}(\lambda)Q_\delta\mathbf{R}_\delta(z)) \cap T^*(X\times X) \subset \Upsilon_\delta,$$
$$\Upsilon_\delta := \{(\rho',\rho) \mid \exists t, s \geq 0 : e^{tH_p}(\rho) \in \operatorname{WF}_h(Q_\delta),\ e^{-sH_p}(\rho') \in \operatorname{WF}_h(Q_\delta)\}.$$

Therefore, $\operatorname{WF}'_h(\mathbf{R}(\lambda)) \cap T^*(X\times X) \subset \Delta(T^*X)\cup\Omega_+\cup\Upsilon_\delta$. Since $\mathbf{R}(\lambda)$ does not depend on δ and h, by (2.6),

$$\operatorname{WF}'(\mathbf{R}(\lambda)) \subset \Delta(T^*X)\cup\Omega_+\cup\bigcap_{\delta>0}\Upsilon_\delta = \Delta(T^*X)\cup\Omega_+\cup(E_u^*\times E_s^*),$$

as claimed.

In a neighbourhood of a pole λ_0 of $\mathbf{R}$, we replace $\mathbf{R}(\lambda)$ in (3.19) by $(\lambda-\lambda_0)^{J(\lambda_0)}\mathbf{R}(\lambda)$. Arguing as before, we get $\operatorname{WF}'((\lambda-\lambda_0)^{J(\lambda_0)}\mathbf{R}(\lambda)) \subset \Delta(T^*X)\cup\Omega_+\cup(E_u^*\times E_s^*)$ uniformly in λ near λ_0. By taking $J(\lambda_0)$ derivatives at $\lambda = \lambda_0$ we obtain the first part of (3.7). By taking $J(\lambda_0)-1$ derivatives at $\lambda = \lambda_0$, we get $\Pi = -\mathbf{R}_\delta(z_0)Q_\delta\Pi Q_\delta\mathbf{R}_\delta(z_0)$, which implies the second part of (3.7).

4. Proof of the main theorem

The proof is based on (3.5) which relates the resolvent and the propagator. The description of the wave front set of $(\mathbf{P}-\lambda)^{-1}$ allows us to take the flat trace of the left hand side composed with $\varphi^*_{-t_0}$ and that formally gives the meromorphic continuation.

To justify this we first use the mollifiers E_ε to obtain trace class operators to which Lemma 2.8 can be applied:

Lemma 4.1. *Suppose that E_ε is given by (2.18) and that $T \geq t_0 > 0$. Then there exists a constant C, independent of ε, t, T such that*

$$\|E_\varepsilon \varphi^*_{-t} E_\varepsilon\|_{\mathrm{tr}} \leq Ce^{CT}\varepsilon^{-n-2} \quad \text{and} \quad \int_T^{T+1} |\operatorname{tr} E_\varepsilon \varphi^*_{-t} E_\varepsilon|\, dt \leq Ce^{CT}. \tag{4.1}$$

Proof. We replace φ^*_{-t} by φ^*_t (considering the flow in the opposite time direction). The first estimate follows from

$$\begin{aligned}\|E_\varepsilon \varphi^*_t E_\varepsilon\|_{\mathrm{tr}} &\leq \|E_\varepsilon\|_{\mathrm{tr}} \cdot \|\varphi^*_t\|_{L^2\to L^2} \cdot \|E_\varepsilon\|_{L^2\to L^2} \\ &\leq Ce^{CT}\|E_\varepsilon\|_{\mathrm{tr}} \leq Ce^{CT}\|(-\Delta_g+1)^{-k}(-\Delta_g+1)^k E_\varepsilon\|_{\mathrm{tr}} \\ &\leq Ce^{CT}\|(-\Delta_g+1)^{-k}\|_{\mathrm{tr}} \cdot \|(-\Delta_g+1)^k E_\varepsilon\|_{L^2\to L^2} \leq C'e^{CT}\varepsilon^{-2k},\end{aligned}$$

provided $2k > n$. Here g is any fixed Riemannian metric on X. For the second estimate in (4.1) we use the definition of E_ε:

$$\begin{aligned}\int_T^{T+1} |\operatorname{tr} E_\varepsilon \varphi^*_t E_\varepsilon|\, dt &= \int_T^{T+1}\int_{X\times X} E_\varepsilon(x,y)E_\varepsilon(\varphi_t(y),x)\, dxdydt \\ &\leq C\varepsilon^{-2n}\int_T^{T+1}\int_{X\times X} \mathbb{1}_{\{d(x,y)<c_1\varepsilon\}}\mathbb{1}_{\{d(x,\varphi_t(y))<c_1\varepsilon\}}\, dx\, dy\, dt \\ &\leq C\varepsilon^{-n}\int_T^{T+1}\int_X \mathbb{1}_{\{d(y,\varphi_t(y))<2c_1\varepsilon\}}\, dy\, dt \leq C'e^{nLT},\end{aligned}$$

where the last estimate comes from Lemma 2.1. □

We now complete the proof of the meromorphic continuation of $\zeta_R(\lambda)$. Thanks to formula (2.5) we need to show that

$$f_k(\lambda) := \frac{1}{i}\sum_\gamma \frac{T^\#_\gamma e^{i\lambda T_\gamma}\operatorname{tr}\wedge^k\mathcal{P}_\gamma}{|\det(I-\mathcal{P}_\gamma)|} = \frac{\partial}{\partial\lambda}\log\exp\left(-\sum_\gamma \frac{T^\#_\gamma e^{i\lambda T_\gamma}\operatorname{tr}\wedge^k\mathcal{P}_\gamma}{T_\gamma|\det(I-\mathcal{P}_\gamma)|}\right) \tag{4.2}$$

has a meromorphic continuation to $\operatorname{Im}\lambda > -C_0$ for any C_0, with poles that are simple and residues which are integral.

Fix t_0 such that $0 < t_0 < T_\gamma$ for all γ and put $\mathbf{P}_k := \mathbf{P}|_{C^\infty(X;\mathcal{E}^k_0)}$ where $\mathcal{E}^k_0$ is defined in §2.2. Then (2.4) shows that

$$f_k(\lambda) = \frac{1}{i}\int_{t_1}^\infty e^{i\lambda t}\operatorname{tr}^\flat e^{-it\mathbf{P}_k}\, dt, \quad 0 \leq t_1 \leq t_0,$$

with the convergence guaranteed by the bound on the number of closed geodesics given in Lemma 2.2.

Using (2.20) (applied to the operator $\int_0^\infty \chi(t)e^{i\lambda t}e^{-it\mathbf{P}_k}\,dt$ where $\chi\in C_0^\infty(t_0-T^{-1},T+1)$, $\chi=1$ near $[t_0,T]$, and T is large), and (4.1) we have for $\operatorname{Im}\lambda\gg 0$,

$$\begin{aligned} f_k(\lambda) &= \frac{1}{i}\lim_{\varepsilon\to 0}\int_{t_0}^\infty e^{i\lambda t}\operatorname{tr}E_\varepsilon e^{-it\mathbf{P}_k}E_\varepsilon\,dt \\ &= \frac{1}{i}e^{i\lambda t_0}\lim_{\varepsilon\to 0}\int_0^\infty e^{i\lambda t}\operatorname{tr}E_\varepsilon e^{-it_0\mathbf{P}_k}e^{-it\mathbf{P}_k}E_\varepsilon\,dt \\ &= \frac{1}{i}e^{i\lambda t_0}\lim_{\varepsilon\to 0}\operatorname{tr}\int_0^\infty E_\varepsilon e^{i\lambda t}e^{-it_0\mathbf{P}_k}e^{-it\mathbf{P}_k}E_\varepsilon\,dt \\ &= -e^{i\lambda t_0}\lim_{\varepsilon\to 0}\operatorname{tr}E_\varepsilon e^{-it_0\mathbf{P}_k}\mathbf{R}_k(\lambda)E_\varepsilon, \end{aligned}$$

where the last equality follows from Proposition 3.2. Here $\mathbf{R}_k(\lambda)=\mathbf{R}(\lambda)|_{H_{sG}(X;\mathcal{E}_0^k)}$, where $\mathbf{R}(\lambda)$ is the inverse of $\mathbf{P}-\lambda$ on the anisotropic Sobolev space $H_{sG}(X;\mathcal{E})$, studied in §3.2, where s is large depending on C_0.

Because of the choice of t_0 ($0<t_0<T_\gamma$ for all γ), and $\operatorname{WF}'(e^{-it_0\mathbf{P}_k})$ is contained in the graph of $e^{t_0H_p}$, Proposition 3.3 shows that $e^{-it_0\mathbf{P}_k}\mathbf{R}_k(\lambda)$ satisfies the assumptions of Lemma 2.8 with the poles handled as in (3.6). Hence, by another application of (2.20),

$$f_k(\lambda)=-e^{i\lambda t_0}\operatorname{tr}^\flat\left(e^{-it_0\mathbf{P}_k}\mathbf{R}_k(\lambda)\right),$$

which is a meromorphic function. Finally, to see that f_k has simple poles and integral residues, we use the following elementary fact:

Lemma 4.2. *Suppose that that a linear map $A:\mathbb{C}^m\to\mathbb{C}^m$ satisfies $(A-\lambda_0)^J=0$ for some $\lambda_0\in\mathbb{C}$. Then for φ holomorphic near λ_0 we have*

$$\lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)\operatorname{tr}\left(\varphi(A)\sum_{j=1}^J\frac{(A-\lambda_0)^{j-1}}{(\lambda-\lambda_0)^j}\right)=m\varphi(\lambda_0),$$

where $\varphi(A)$ is defined by the power series expansion at λ_0 (which is finite).

From (3.6) we have near a pole λ_0 of $\mathbf{R}_k$,

$$e^{it_0\lambda}e^{-it_0\mathbf{P}_k}\mathbf{R}_k(\lambda)=e^{it_0\lambda}\mathbf{R}_{H,k}(\lambda)-e^{it_0\lambda}\sum_{j=1}^{J(\lambda_0,k)}\frac{e^{-it_0\mathbf{P}_k}(\mathbf{P}_k-\lambda_0)^{j-1}\Pi_k}{(\lambda-\lambda_0)^j},$$

where $\mathbf{R}_{H,k}$ is holomorphic near λ_0 and Π_k is given by (3.18):

$$\Pi_k:=\frac{1}{2\pi i}\oint_{\lambda_0}(\lambda-\mathbf{P}_k)^{-1}d\lambda,\quad \operatorname{tr}^\flat\Pi_k=\operatorname{tr}_{H_{sG}}\Pi_k\in\mathbb{N}.$$

Here we use the fact that $\operatorname{tr}^\flat$ and $\operatorname{tr}_{H_{sG}}$ agree on finite rank operators. We now apply Lemma 4.2 with $\varphi(\mu)=e^{-it_0\mu}$ and $A=\mathbf{P}_k|_{\ker(\mathbf{P}_k-\lambda_0)^J}$.

Appendix A. Estimates on recurrence

In this Appendix we provide proofs of statements made in §2.1.

It follows immediately from the Anosov property (2.1) that (with I denoting the identity operator)

$$t \neq 0,\ \varphi_t(x) = x \implies (d\varphi_t(x) - I)|_{E_u(x)\oplus E_s(x)} \quad \text{is invertible.} \tag{A.1}$$

Indeed, if $v \in E_u(x) \oplus E_s(x)$ and $d\varphi_t(x)v = v$, then $d\varphi_{Nt}(x)v = v$ for all $N \in \mathbb{Z}$, implying by (2.1) that $v = 0$.

The following lemma is a generalization of (A.1) to the case when $\varphi_t(x)$ is close to x. We fix a smooth distance function $d(\cdot,\cdot)$ on X and a smooth norm $|\cdot|$ on the fibers of TX.

Lemma A.1. *Let $\delta_0 > 0$ and $\mathcal{T}_{x,y} : T_xX \to T_yX$, $d(x,y) < \delta_0$, be a continuous family of invertible linear transformations such that $\mathcal{T}_{x,x} = I$ and $\mathcal{T}_{x,y}$ maps $E_u(x), E_s(x), \mathbb{R}V(x)$ onto $E_u(y), E_s(y), \mathbb{R}V(y)$. Fix $t_e > 0$. Then there exist $\delta \in (0,\delta_0)$ and C such that*

$$|v| \leq C|(d\varphi_t(x) - \mathcal{T}_{x,\varphi_t(x)})v| \quad \text{if } d(x,\varphi_t(x)) < \delta,\ t \geq t_e,\ v \in E_u(x)\oplus E_s(x). \tag{A.2}$$

Proof. We first note that it suffices to prove (A.2) for sufficiently large t. Indeed, if N is a large fixed integer, $v \in E_u(x)\oplus E_s(x)$, and $d(x,\varphi_t(x))$ and $|(d\varphi_t(x) - \mathcal{T}_{x,\varphi_t(x)})v|$ are both small, then $d(x,\varphi_{Nt}(x))$ and $|(d\varphi_{Nt}(x) - \mathcal{T}_{x,\varphi_{Nt}(x)})v|$ are small as well; applying (A.2) for Nt in place of t, we get that $|v|$ is small.

Assume that the conditions of (A.2) are satisfied and put $v = v_u + v_s$, where $v_u \in E_u(x), v_s \in E_s(x)$. For t large enough, the Anosov property (2.1) implies

$$|v_u| \leq \tfrac{1}{2}|d\varphi_t(x)v_u|, \quad |d\varphi_t(x)v_s| \leq \tfrac{1}{2}|v_s|;$$

since for δ small enough, $\|\mathcal{T}_{x,\varphi_t(x)}\|, \|\mathcal{T}^{-1}_{x,\varphi_t(x)}\|$ are close to 1, we get

$$\begin{aligned}|v| \leq |v_u| + |v_s| &\leq 3\big(|(d\varphi_t(x) - \mathcal{T}_{x,\varphi_t(x)})v_u| + |(d\varphi_t(x) - \mathcal{T}_{x,\varphi_t(x)})v_s|\big)\\ &\leq C|(d(\varphi_t(x) - \mathcal{T}_{x,\varphi_t(x)})v|,\end{aligned}$$

where the last inequality is due to the fact that $(d\varphi_t(x) - \mathcal{T}_{x,\varphi_t(x)})v_u \in E_u(\varphi_t(x))$, $(d\varphi_t(x) - \mathcal{T}_{x,\varphi_t(x)})v_s \in E_s(\varphi_t(x))$. ∎

Fix a constant $L > 0$ such that for some choice of the norm on the space $C^2(X)$ of twice differentiable functions on X, there exists a constant C such that

$$\|f\circ\varphi_t\|_{C^2(X)} \leq Ce^{L|t|}\|f\|_{C^2(X)}, \quad f \in C^2(X). \tag{A.3}$$

Such L exists since X is compact and φ_t is a one-parameter group. As a consequence of (A.3), we get

$$d(\varphi_t(x),\varphi_t(x')) \leq Ce^{L|t|}d(x,x'). \tag{A.4}$$

The next lemma in particular implies (by letting $\varepsilon \to 0$) that two different closed trajectories of nearby periods t, t' have to be at least δe^{-Lt} away from each other, where δ is a small constant.

Lemma A.2. *Fix $t_e > 0$. Then there exist $C, \delta > 0$ such that for each $\varepsilon > 0$,*

$$\begin{aligned} d(x,\varphi_t(x)) \leq \varepsilon,\ d(x',\varphi_{t'}(x')) \leq \varepsilon,\ t,t' \geq t_e,\ |t-t'| \leq \delta,\ d(x,x') \leq \delta e^{-Lt} \\ \Longrightarrow\ |t-t'| \leq C\varepsilon,\ \exists s \in (-1,1) : d(x,\varphi_s(x')) \leq C\varepsilon. \end{aligned} \tag{A.5}$$

Proof. Without loss of generality, we may assume that ε is small depending on δ. By (A.4), we see that $d(\varphi_t(x),\varphi_t(x'')) \leq C\delta$ whenever $d(x,x'') \leq \delta e^{-Lt}$. Therefore, we may operate in a coordinate neighbourhood containing $x, x', \varphi_t(x), \varphi_{t'}(x')$, identified with a ball in $\mathbb{R}^n$. We replace x' with $\varphi_s(x')$ for some $|s| < 1$ so that

$$x' - x \in E_u(x) \oplus E_s(x). \tag{A.6}$$

By (A.3), we have for all j, k,

$$|\partial^2_{x_j x_k}\varphi_t(x'')| \leq Ce^{Lt} \quad \text{for } d(x,x'') \leq \delta e^{-Lt};$$

using the Taylor expansion of $\varphi_t(x)$ in x, we see that

$$|\varphi_t(x') - \varphi_t(x) - d\varphi_t(x)(x'-x)| \leq Ce^{Lt}|x'-x|^2 \leq C\delta|x'-x|.$$

Next, $|\partial_t^2\varphi_t(x')| \leq C$; by Taylor expanding $\varphi_t(x')$ in t, we get

$$|\varphi_{t'}(x') - \varphi_t(x') - V(\varphi_t(x'))(t'-t)| \leq C|t'-t|^2 \leq C\delta|t'-t|.$$

Together, these give

$$|\varphi_{t'}(x') - \varphi_t(x) - d\varphi_t(x)(x'-x) - V(\varphi_t(x'))(t'-t)| \leq C\delta(|x'-x| + |t'-t|).$$

Since $d(x,\varphi_t(x)) \leq \varepsilon$ and $d(x',\varphi_{t'}(x')) \leq \varepsilon$, we get

$$|(d\varphi_t(x) - I)(x'-x) + V(\varphi_t(x'))(t'-t)| \leq C\delta(|x'-x| + |t'-t|) + C\varepsilon.$$

The map $\mathcal{T}_{x,y}$ in the statement of the lemma maps $E_u(x) \oplus E_s(x)$ onto $E_u(y) \oplus E_s(y)$. Since $d(x,\varphi_t(x)) \leq \varepsilon$, we get for ε small enough depending on δ, $|(I - \mathcal{T}_{x,\varphi_t(x)})(x'-x)| \leq \delta|x-x'|$. Since $|\varphi_t(x') - \varphi_t(x)| \leq C\delta$, we find $|V(\varphi_t(x')) - V(\varphi_t(x))| \leq C\delta$. Then

$$|(d\varphi_t(x) - \mathcal{T}_{x,\varphi_t(x)})(x'-x) + V(\varphi_t(x))(t'-t)| \leq C\delta(|x'-x| + |t'-t|) + C\varepsilon.$$

Now, by (A.6), $(d\varphi_t(x) - \mathcal{T}_{x,\varphi_t(x)})(x'-x) \in E_u(\varphi_t(x)) \oplus E_s(\varphi_t(x))$; since this space is transverse to $V(\varphi_t(x))$, and by Lemma A.1, we get

$$|x'-x| + |t'-t| \leq C(|(d\varphi_t(x) - \mathcal{T}_{x,\varphi_t(x)})(x'-x)| + |t'-t|) \leq C\delta(|x'-x| + |t'-t|) + C\varepsilon.$$

It remains to choose δ small enough so that $C\delta < 1/2$. □

We now give a volume bound on the set of nearly closed trajectories:

Proof of Lemma 2.1. First of all, we can replace the range of values of t in (2.2) by $|t-T| \leq \delta/2$, where δ is the constant from Lemma A.2. Next, let $x_1, \dots, x_N$, with N depending on T, be a maximal set of points in X such that $d(x_j, x_k) \geq \delta e^{-LT}/2$. Since the metric balls of radius $\delta e^{-LT}/4$ centered at x_j are disjoint, by calculating the volume of their union we find $N \leq Ce^{nLT}$. Now,

$$\{(x,t) \mid |t-T| \leq \delta/2,\ d(x, \varphi_t(x)) \leq \varepsilon\} \subset \bigcup_{j=1}^N A_j,$$
$$A_j := \{(x,t) \mid |t-T| \leq \delta/2,\ d(x, x_j) \leq \delta e^{-LT}/2,\ d(x, \varphi_t(x)) \leq \varepsilon\}.$$

Take some j such that A_j is nonempty and fix $(x', t') \in A_j$. Then for each $(x,t) \in A_j$, we have $|t-t'| \leq \delta$, $d(x,x') \leq \delta e^{-LT}$. By Lemma A.2, A_j is contained in an $\mathcal{O}(\varepsilon)$ sized tubular neighbourhood of the trajectory $\{(\varphi_s(x'), t') \mid |s| < 1\}$. Therefore, we get $\tilde\mu(A_j) \leq C\varepsilon^n$, finishing the proof. □

Proof of Lemma 2.2. Let $\gamma(t) = \varphi_t(x_0)$ be a closed trajectory of period t_0. Then for each $\varepsilon > 0$, we have by (A.4),

$$d(x, \varphi_t(x)) \leq C\varepsilon \quad \text{if } |t-t_0| \leq \varepsilon \text{ and } d(x, \gamma(s)) \leq \varepsilon e^{-Lt_0} \quad \text{for some } s. \tag{A.7}$$

Moreover, for $t_0 \leq T$ and ε small enough depending on T, the tubular neighbourhoods (A.7) for different closed trajectories do not intersect. The volume (in x, t) of each tubular neighbourhood is bounded from below by $C^{-1}\varepsilon^n e^{-(n-1)Lt_0}$; it remains to let $\varepsilon \to 0$ and apply Lemma 2.1. □

Appendix B. Proof of Guillemin's trace formula

In this appendix, we give a self-contained proof of Guillemin's trace formula (2.4) (including the special case (1.5)) in the case of Anosov flow $\varphi_t = e^{tV}$ on a compact manifold X. The proof is somewhat simplified by the fact that $E_u(x) \oplus E_s(x)$ is a subbundle of TX transversal to $\mathbb{R}V$ and invariant under the flow.

If $\gamma(t) = \varphi_t(x_0)$ is a closed trajectory with period $t_0 \neq 0$ (here t_0 need not be the *primitive* period), then the linearized Poincaré map is defined by

$$\mathcal{P}_\gamma := d\varphi_{-t_0}(x_0)|_{E_u(x_0) \oplus E_s(x_0)}. \tag{B.1}$$

Note that $I - \mathcal{P}_\gamma$ is invertible by (A.1). The maps $d\varphi_{-t_0}(\varphi_s(x_0))$ are conjugate to each other by $d\varphi_s(x_0)$ for all s, therefore the expressions $\det(I - P_\gamma)$ and $\operatorname{tr}(\wedge^k P_\gamma)$, used in (2.4), are independent of the choice of the base point on γ.

Fix a density dx on X and let $K(t, y, x)$ be the Schwartz kernel of $\varphi^*_{-t} = e^{-itP}$ with respect to this density, that is for $f \in C^\infty(X)$,

$$f(\varphi_{-t}(y)) = \int_X K(t, y, x) f(x)\, dx. \tag{B.2}$$

To be able to define the flat trace of φ_{-t}^* as a distribution in $t \in \mathbb{R} \setminus 0$, we need to take some $\chi(t) \in C_c^\infty(\mathbb{R} \setminus 0)$ and show that the operator

$$T_\chi := \int_{\mathbb{R}} \chi(t) \varphi_{-t}^* \, dt$$

satisfies the condition (2.16), that is $\mathrm{WF}'(T_\chi)$ does not intersect the diagonal. By the formula for the wave front set of a pushforward [HöI–II, Theorem 8.2.12], we know that

$$\mathrm{WF}'(T_\chi) \subset \{(y, \eta, x, -\xi) \mid \exists t \in \operatorname{supp} \chi : (t, 0, y, \eta, x, \xi) \in \mathrm{WF}(K)\},$$

and thus it suffices to show that

$$\mathrm{WF}(K) \cap \{(t, 0, x, \xi, x, -\xi) \mid t \neq 0,\ (x, \xi) \in T^*X \setminus 0\} = \emptyset. \tag{B.3}$$

Note that (B.3) is exactly the condition under which one can define the pullback $K(t, x, x) \in \mathcal{D}'((\mathbb{R} \setminus 0) \times X)$ of K by the map $(t, x) \mapsto (t, x, x)$, and

$$\mathrm{tr}^\flat(T_\chi) = \int_{\mathbb{R} \times X} \chi(t) K(t, x, x) \, dx dt.$$

Now, $K(t, y, x)$ is a delta function on the surface $\{y = \varphi_t(x)\}$, therefore by [HöI–II, Theorem 8.2.4] its wave front set is contained in the conormal bundle to that surface:

$$\mathrm{WF}(K) \subset \{(t, -V(x) \cdot \eta, \varphi_t(x), \eta, x, -{}^T d\varphi_t(x) \cdot \eta) \mid t \in \mathbb{R},\ x \in X,\ \eta \in T^*_{\varphi_t(x)} X \setminus 0\}.$$

Then to prove (B.3), we need to show that if $t \neq 0$, $\varphi_t(x) = x$, $V(x) \cdot \eta = 0$, and $(I - {}^T d\varphi_t(x)) \cdot \eta = 0$, then $\eta = 0$; this follows immediately from (A.1).

The principal component of the proof of the trace formula (2.4) is the following

Lemma B.1. *Let $x_0 \in X$ and $t_0 \neq 0$ be such that $\varphi_{t_0}(x_0) = x_0$. Then there exists $\varepsilon > 0$ and a neighborhood $U \subset X$ of x_0 such that $\varphi_s(x_0) \in U$ for $|s| < \varepsilon$ and for each $\chi(t, x) \in C_c^\infty((t_0 - \varepsilon, t_0 + \varepsilon) \times U)$, we have*

$$\int_{\mathbb{R} \times X} \chi(t, x) K(t, x, x) \, dx = \frac{1}{|\det(I - \mathcal{P}_\gamma)|} \int_{-\varepsilon}^{\varepsilon} \chi(t_0, \varphi_s(x_0)) \, ds, \tag{B.4}$$

where $\mathcal{P}_\gamma$ is defined in (B.1).

Proof. We choose a local coordinate system $w = \psi(x)$, $\psi : U_1 \to B(0, \varepsilon_1) \subset \mathbb{R}^n$, where U_1 is a neighborhood of x_0, such that

$$\psi(x_0) = 0, \quad \psi_* V = \partial_{w_1}, \quad d\psi(x_0)\big(E_u(x_0) \oplus E_s(x_0)\big) = \{dw_1 = 0\}.$$

We next choose small $\varepsilon \in (0, \varepsilon_1)$ such that for $U := \psi^{-1}(B(0, \varepsilon))$ and $|t - t_0| < \varepsilon$, we have $\varphi_{-t}(U) \subset U_1$. We define the maps $A : B_{\mathbb{R}^{n-1}}(0, \varepsilon) \to B_{\mathbb{R}^{n-1}}(0, \varepsilon_1)$ and $F : B_{\mathbb{R}^{n-1}}(0, \varepsilon) \to (-\varepsilon_1, \varepsilon_1)$ by the formulas

$$\varphi_{-t_0}(\psi^{-1}(0, w')) = \psi^{-1}(F(w'), A(w')), \quad w' \in \mathbb{R}^{n-1},\ |w'| < \varepsilon.$$

Then for $|t-t_0|<\varepsilon$ and $(w_1,w')\in B(0,\varepsilon)$, we have

$$\varphi_{-t}(\psi^{-1}(w_1,w'))=\psi^{-1}(-t+t_0+w_1+F(w'),A(w')).$$

Moreover, $F(0)=0$ and $A(0)=0$.

Since the flat trace does not depend on the choice of density on X, we may choose the density dx so that ψ_*dx is the standard density on $\mathbb{R}^n$. Then for $|t-t_0|<\varepsilon$ and $(z_1,z'),(w_1,w')\in B(0,\varepsilon)$, we have

$$K(t,\psi^{-1}(z_1,z'),\psi^{-1}(w_1,w'))=\delta(w'-A(z'))\delta(w_1+t-t_0-z_1-F(z')).$$

The left-hand side of (B.4) is

$$\int_{\mathbb{R}\times B(0,\varepsilon)}\chi(t,\psi^{-1}(w_1,w'))\delta(w'-A(w'))\delta(t-t_0-F(w'))\,dw_1dw'dt.$$

Integrating out t, we get

$$\int_{B(0,\varepsilon)}\chi(t_0-F(w'),\psi^{-1}(w_1,w'))\delta(w'-A(w'))\,dw_1dw'.$$

Now, $dA(0)$ is conjugated by the map $d\psi(x_0)$ to the Poincaré map $\mathcal{P}_\gamma$, therefore $I-dA(0)$ is invertible and for ε small enough and $|w'|<\varepsilon$, the equation $w'=A(w')$ has exactly one root at $w'=0$. We then integrate out w' to get

$$\frac{1}{|\det(I-dA(0))|}\int_{-\varepsilon}^{\varepsilon}\chi(t_0,\psi^{-1}(w_1,0))\,dw_1=\frac{1}{|\det(I-\mathcal{P}_\gamma)|}\int_{-\varepsilon}^{\varepsilon}\chi(t_0,\varphi_s(x_0))\,ds,$$

which finishes the proof. □

By Lemma B.1 and a partition of unity, we see that for each $\chi(t,x)\in C_c^\infty((\mathbb{R}\backslash 0)\times X)$, we have

$$\int_{\mathbb{R}\times X}\chi(t,x)K(t,x,x)\,dx=\sum_\gamma\frac{1}{|\det(I-\mathcal{P}_\gamma)|}\int_\gamma\chi(T_\gamma,x)\,dL(x) \tag{B.5}$$

where the sum is over all closed trajectories γ with period T_γ and dL refers to the measure dt on $\gamma(t)=\varphi_t(x_0)$. By taking $\chi(t,x)=\chi(t)$, we obtain (1.5).

To show the more general (2.4), it suffices to prove a local version similar to (B.4):

$$\int_{\mathbb{R}\times X}\chi(t,x)K^k(t,x,x)\,dx=\frac{\operatorname{tr}(\wedge^k\mathcal{P}_\gamma)}{|\det(I-\mathcal{P}_\gamma)|}\int_{-\varepsilon}^{\varepsilon}\chi(t_0,\varphi_s(x_0))\,ds, \tag{B.6}$$

where K^k is the Schwartz kernel of the operator $\sum_{j=1}^r B_{jj}$, $r=\dim\mathcal{E}_0^k$, and $B_{jl}:C_0^\infty(U)\to C^\infty(U)$ are the operators defined by

$$\varphi_{-t}^*(f\mathbf{e}_l)=\sum_{j=1}^r(B_{jl}(t)f)\mathbf{e}_j,$$

here $\mathbf{e}_1,\dots,\mathbf{e}_r$ is a local frame of $\mathcal{E}_0^k$ defined near x_0. Define the functions b_{jl} on $(t_0-\varepsilon,t_0+\varepsilon)\times U$ by

$$\varphi_{-t}^*\mathbf{e}_l=\sum_{j=1}^r b_{jl}(t)\mathbf{e}_j.$$

Then $B_{jl}(t)f=b_{jl}(t)(\varphi_{-t}^*f)$, which means that

$$K^k(t,x,y)=\sum_j b_{jj}(t,y)K(t,x,y),$$

with $K(t,x,y)$ defined in (B.2). Then by Lemma B.1,

$$\int_{\mathbb{R}\times X}\chi(t,x)K^k(t,x,x)=\frac{1}{|\det(I-\mathcal{P}_\gamma)|}\int_{-\varepsilon}^{\varepsilon}\chi(t_0,\varphi_s(x_0))\sum_j b_{jj}(t_0,\varphi_s(x_0))\,ds.$$

It remains to note that

$$\sum_j b_{jj}(t,\varphi_s(x_0))=\operatorname{tr}\wedge^k({}^T d\varphi_{-t_0}(x_0)|_{E_s^*(x_0)\oplus E_u^*(x_0)})=\operatorname{tr}\wedge^k\mathcal{P}_\gamma.$$

Appendix C. Review of microlocal and semiclassical analysis

In this Appendix, we provide details and references for the concepts and facts listed in §2.3. All the proofs are essentially well known but we include them for the reader's convenience.

C.1. **Microlocal calculus.** Let X be a manifold with a fixed volume form. We use the algebra of pseudodifferential operators $\Psi^k(X)$, $k\in\mathbb{R}$, with symbols lying in the class $S^k(X)\subset C^\infty(T^*X)$:

$$a\in S^k(X)\iff\sup_{x\in K}\langle\xi\rangle^{|\beta|-k}|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|\le C_{\alpha\beta K},\quad K\Subset X. \tag{C.1}$$

See for example [HöIII–IV, §18.1] for the basic properties of operators in Ψ^k. In particular, each $A\in\Psi^k(X)$ is bounded between Sobolev spaces $H^m_{\mathrm{comp}}(X)\to H^{m-k}_{\mathrm{loc}}(X)$, or simply $H^m(X)\to H^{m-k}(X)$ if X is compact. The wave front set $\mathrm{WF}(A)$ of $A\in\Psi^k(X)$ is a closed conic subset of $T^*X\setminus 0$, with 0 denoting the zero section; the complement of $\mathrm{WF}(A)$ consists of points in whose conic neighbourhoods the full symbol of A is $\mathcal{O}(\langle\xi\rangle^{-\infty})$, see the discussion following [HöIII–IV, Proposition 18.1.26].

The wave front set $\mathrm{WF}(u)\subset T^*X\setminus 0$ of a distribution $u\in\mathcal{D}'(X)$ is defined as follows: a point $(x,\xi)\in T^*X\setminus 0$ does not lie in $\mathrm{WF}(u)$ if there exists a conic neighbourhood U of (x,ξ) such that $Au\in C^\infty(X)$ for each $A\in\Psi^0(X)$ with $\mathrm{WF}(A)\subset U$ – see [HöIII–IV, (18.1.35) and Theorem 18.1.27]. An equivalent definition (see [HöI–II, Definition 8.1.2]) is given in terms of the Fourier transform: $(x,\xi)\notin\mathrm{WF}(u)$ if and only if there exists $\chi\in C_c^\infty(X)$ with $\operatorname{supp}\chi$ contained in some coordinate neighbourhood and $\chi(x)\neq 0$ such that $\widehat{\chi u}(\xi')=\mathcal{O}(\langle\xi'\rangle^{-\infty})$ for ξ' in a conic neighbourhood of ξ; here χu is considered a function on $\mathbb{R}^n$ using some coordinate system and ξ is accordingly considered as vector in $\mathbb{R}^n$.

The wave front set $\mathrm{WF}'(B) \subset T^*(Y \times X)$ of an operator $B : C_c^\infty(X) \to \mathcal{D}'(Y)$ is defined using its Schwartz kernel $K_B(y,x) \in \mathcal{D}'(Y \times X)$:

$$\mathrm{WF}'(B) := \{(y,\eta,x,-\xi) \mid (y,\eta,x,\xi) \in \mathrm{WF}(K_B)\}. \tag{C.2}$$

Here we use the fixed smooth density on X to define the Schwartz kernel as a distribution on $Y \times X$; however, this choice does not affect the wave front set. If $B \in \Psi^k(X)$, then the set defined in (C.2) is the image of the wave front set $\mathrm{WF}(B) \subset T^*X$ of B as a pseudodifferential operator under the diagonal embedding $T^*X \to T^*(X \times X)$, see [HöIII–IV, (18.1.34)].

The concept of the wave front plays a crucial role in the definition of the flat trace. Before proving Lemma 2.8 we give

Proof of (2.19). We first show that $E_\varepsilon \in \Psi^{0+}(X)$ with seminorm estimates independent of ε. For that we use Melrose's characterization of pseudodifferential operators [HöIII–IV, §18.2]: it is enough to show that for any set of vector fields $V_j \in C^\infty(X \times X; T(X \times X))$ tangent to the diagonal, we have $V_1 \cdots V_N K_{E_\varepsilon} \in H^{-n/2-}(X \times X)$ with norm bounded uniformly in ε. This can be done in local coordinates, writing $\psi(d(x,y)/\varepsilon) = \Psi(x,(x-y)/\varepsilon,\varepsilon)$, where Ψ is a smooth function on $\mathbb{R}^n \times \mathbb{R}^n \times [0,\infty)$, compactly supported in the second argument. We have $F_\varepsilon(x) = \int_{\mathbb{R}^n} \Psi(x,(x-y)/\varepsilon,\varepsilon) J(y)\, dy$, where J is the Jacobian, and the support of the integrand lies $\mathcal{O}(\varepsilon)$ close to x. Then $\partial_x^\alpha F_\varepsilon(x) = \mathcal{O}_\alpha(\varepsilon^n)$; indeed, one can rewrite the x derivatives falling on the second argument of Ψ as derivatives in y and integrate by parts. This implies that $\partial_x^\alpha(1/F_\varepsilon(x)) = \mathcal{O}_\alpha(\varepsilon^{-n})$. Locally, vector fields tangent to the diagonal are generated by $\partial_{x_j} + \partial_{y_j}$ and $(x_j - y_j)\partial_{x_k}$ and we see that they preserve the class of smooth functions of $x, (x-y)/\varepsilon, \varepsilon$. Therefore, for $|\alpha| = |\beta|$,

$$(x-y)^\alpha \partial_x^\beta (\partial_x + \partial_y)^\gamma K_{E_\varepsilon}(x,y) = \varepsilon^{-n} F_{\alpha\beta\gamma}(x,(x-y)/\varepsilon,\varepsilon),$$

where $F_{\alpha\beta\gamma} \in C^\infty(\mathbb{R}^{2n} \times [0,\infty))$ are smooth functions. The right hand side is in $H^{-n/2-}(\mathbb{R}^{2n})$ uniformly in ε which proves the claim. To obtain[2] $E_\varepsilon \to I$ in $\Psi^{0+}(X)$ we apply the same argument to $K_{E_\varepsilon} - K_I$. □

Proof of Lemma 2.8. Let $\Delta(X) = \{(x,x)\} \subset X \times X$ and let Γ be the complement of a small conic neighbourhood of the conormal bundle $N^*\Delta(X) \subset T^*(X \times X)$. Since $\mathrm{WF}(K_B) \cap N^*\Delta(X) = \emptyset$ by (2.16) we can choose Γ so that $\mathrm{WF}(K_B) \subset \Gamma$. This means that $K_B \in \mathcal{D}'_\Gamma(X \times X)$ where the last space consists of all distributions $u \in \mathcal{D}'(X \times X)$ with $\mathrm{WF}(u) \subset \Gamma$. If we write $B_\varepsilon := E_\varepsilon B E_\varepsilon$ then $B_\varepsilon : \mathcal{D}'(X) \to C^\infty(X)$, and hence

[2]This specific statement is not used in the paper: all we need is $E_\varepsilon\varphi \to \varphi$ in C^∞ for $\varphi \in C^\infty(X)$, and that E_ε is uniformly bounded in *some* $\Psi^k(X)$.

$K_{B_\varepsilon} \in C^\infty(X \times X)$, $\operatorname{tr} B_\varepsilon = \int_X K_{B_\varepsilon}(x,x)dx = \int_X \iota^* K_{B_\varepsilon}\, dx$. Since $E_\varepsilon \to I$ in Ψ^{0+}, $E_\varepsilon \varphi \to \varphi$ in $C^\infty(X)$ for $\varphi \in C^\infty(X)$. Hence $K_{B_\varepsilon}(\varphi_1 \otimes \varphi_2) \to K_B(\varphi_1 \otimes \varphi_2)$, $\varphi_j \in C^\infty(X)$, and consequently $K_{B_\varepsilon} \to K_B$ in $\mathcal{D}'(X \times X)$. To show that $K_{B_\varepsilon} \to K_B$ in $\mathcal{D}'_\Gamma(X \times X)$, we adapt [HöI–II, Definition 8.2.2] and it suffices to show that for each $A \in \Psi^0(X \times X)$ with $\operatorname{WF}(A) \cap \Gamma = \emptyset$, AK_{B_ε} is bounded in $C^\infty(X \times X)$ uniformly in ε. In fact,

$$AK_{B_\varepsilon} = AE^t_{\varepsilon,x} E_{\varepsilon,y} K_B,$$

where $E_{\varepsilon,x}$ and $E_{\varepsilon,y}$ denote the operator E_ε acting on x and y variables in $X \times X$, and the superscript t denotes the transpose. Since E_ε is uniformly bounded in $\Psi^{0+}(X)$ and $\operatorname{WF}(A)$ is contained in a small neighbourhood of $N^*\Delta(X)$, $C_\varepsilon := AE^t_{\varepsilon,x}E_{\varepsilon,y}$ is in $\Psi^{0+}(X \times X)$ with seminorms uniformly bounded with respect to ε, and with $\operatorname{WF}(C_\varepsilon) \cap \Gamma = \emptyset$.[3] Hence $C_\varepsilon K_B \in C^\infty(X \times X)$ uniformly in ε and thus $K_{B_\varepsilon} \to K_B$ in $\mathcal{D}'_\Gamma(X \times X)$. We now invoke [HöI–II, Theorem 8.2.4] to conclude that $\iota^* K_{B_\varepsilon} \to \iota^* K_B$ in $\mathcal{D}'(X)$. Hence $\int_X \iota^* K_{B_\varepsilon}\, dx \to \int_X \iota^* K_B\, dx$ as $\varepsilon \to 0$, proving the lemma. □

If $\mathcal{E}$ is a smooth r-dimensional vector bundle over X (see for example [HöI–II, Definition 6.4.2]), then we can consider distributions $\mathbf{u} \in \mathcal{D}'(X;\mathcal{E})$ with values in $\mathcal{E}$. The wave front set $\operatorname{WF}(\mathbf{u})$, a closed conic subset of $T^*X \setminus 0$, is defined as follows: $(x,\xi) \notin \operatorname{WF}(\mathbf{u})$ if and only if for each local basis $\mathbf{e}_1, \dots, \mathbf{e}_r \in C^\infty(U;\mathcal{E})$ of $\mathcal{E}$ defined in a neighbourhood U of x, and for $\mathbf{u}|_U = \sum_{j=1}^r u_j \mathbf{e}_j$, $u_j \in \mathcal{D}'(U)$, we have $(x,\xi) \notin \operatorname{WF}(u_j)$ for all j. Similarly, one can define $\operatorname{WF}'(\mathbf{B})$ for an operator $\mathbf{B}$ with values in some smooth vector bundle over $Y \times X$.

An operator $\mathbf{A} : \mathcal{D}'(X;\mathcal{E}) \to \mathcal{D}'(X;\mathcal{E})$ is said to be pseudodifferential in the class $\Psi^k(X)$, denoted $\mathbf{A} \in \Psi^k(X; \operatorname{Hom}(\mathcal{E}))$, if $\operatorname{WF}(\mathbf{A}\mathbf{u}) \subset \operatorname{WF}(\mathbf{u})$ for all $u \in \mathcal{D}'(X;\mathcal{E})$ and, for each local basis $\mathbf{e}_1, \dots, \mathbf{e}_r \in C^\infty(U;\mathcal{E})$ over some open $U \subset X$, we have on U,

$$\mathbf{A}(f\mathbf{e}_l) = \sum_{j=1}^r (A_{jl} f)\mathbf{e}_j, \quad \text{for each } f \in \mathcal{D}'(X;\mathcal{E}),\ \operatorname{supp} f \Subset U,$$

where $A_{jk} \in \Psi^k(U)$. As before, the wave front set $\operatorname{WF}(\mathbf{A})$ on U is defined as the union of $\operatorname{WF}(A_{jl})$ over all j, l. The principal symbol

$$\sigma(\mathbf{A}) \in S^k(X; \operatorname{Hom}(\mathcal{E}))/S^{k-1}(X; \operatorname{Hom}(\mathcal{E}))$$

is defined using the standard notion of the principal symbol $\sigma(A_{jl}) \in S^k(X)/S^{k-1}(X)$ (see the discussion following [HöIII–IV, Definition 18.1.20]) as follows:

$$\sigma(\mathbf{A})\mathbf{e}_l = \sum_{j=1}^r \sigma(A_{jl})\mathbf{e}_j \quad \text{on } U.$$

[3]The slight subtlety here lies in the fact that $E_{\varepsilon,x}, E_{\varepsilon,y}$ are *not* pseudifferential operators on $X \times X$. However, the localization to a region where $|\xi|$ and $|\eta|$ are comparable makes the composition into a pseudodifferential operator.

The operator $\mathbf{A}$ is called *elliptic* in the class Ψ^k at some point $(x,\xi)\in T^*X\setminus 0$, if $\langle\xi'\rangle^{-k}\sigma(\mathbf{A})(x',\xi')$ is invertible (as a homomorphism $\mathcal{E}\to\mathcal{E}$) uniformly as $\xi'\to\infty$ for (x',ξ') in a conic neighbourhood of (x,ξ); equivalently, $|\det(\langle\xi'\rangle^{-k}\sigma(\mathbf{A}))|\geq c>0$ in a conic neighbourhood of (x,ξ). The (open conic) set of all elliptic points of $\mathbf{A}$ is denoted $\operatorname{ell}(\mathbf{A})$.

C.2. **Semiclassical calculus.** We now introduce the algebra $\Psi^k_h(X)$ of *semiclassical* pseudodifferential operators, depending on a parameter $h>0$ tending to zero [Zw, §14.2]. The corresponding symbols $a(x,\xi;h)$ (denoted $a\in S^k_h(X)$) satisfy $a(\cdot,\cdot;h)\in S^k(X)$ uniformly in h as $h\to 0$, with the class S^k defined in (C.1). Each $A\in\Psi^k_h(X)$ has a semiclassical wave front set $\operatorname{WF}_h(A)$, a closed (and not necessarily conic) subset of the fiber-radially compactified cotangent bundle $\overline{T}^*X$ (see [Va, §2.1]); a point $(x,\xi)\in\overline{T}^*X$ does not lie in $\operatorname{WF}_h(A)$ if and only if the full symbol a of A satisfies $a(x',\xi')=\mathcal{O}(h^\infty\langle\xi'\rangle^{-\infty})$ for h small enough and $(x',\xi')\in T^*X$ in a neighbourhood of (x,ξ) in $\overline{T}^*X$. The elements of $\Psi^k_h(X)$ act between semiclassical Sobolev spaces $H^m_{h,\mathrm{comp}}(X)\to H^{m-k}_{h,\mathrm{loc}}(X)$ with norm $\mathcal{O}(1)$, see [Zw, §14.2.4].

Using operators in $\Psi^k_h(X)$, we define the semiclassical wave front set $\operatorname{WF}_h(u)\subset\overline{T}^*X$ for an h-tempered family of distributions $u=u(h)$, see for example [Zw, §8.4.2], [DaDy, §3.1]. Similarly to $\operatorname{WF}(u)$, the set $\operatorname{WF}_h(u)$ can be characterized using the Fourier transform as follows: $(x,\xi)\notin\operatorname{WF}_h(u)$ if and only if there exists $\chi\in C^\infty_c(X)$ supported in some coordinate neighbourhood, with $\chi(x)\neq 0$, and a neighbourhood U_ξ of ξ in $\overline{T}^*X$, such that $\mathcal{F}_h(\chi u)(\xi'):=\widehat{\chi u}(\xi'/h)=\mathcal{O}(h^\infty\langle\xi'\rangle^{-\infty})$ for $\xi'\in U_\xi$. This characterization immediately implies (2.6). Similarly, one can define the wave front set $\operatorname{WF}'_h(B)\subset\overline{T}^*(Y\times X)$ of an h-tempered family of operators $B(h):C^\infty_c(X)\to\mathcal{D}'(Y)$.

The semiclassical principal symbol of $A\in\Psi^k_h(X)$, denoted $\sigma_h(A)$, lies in the space $S^k_h(X)/hS^{k-1}_h(X)$ – see [Zw, Theorem 14.1]. Note that this encodes the behaviour of the full symbol of A at $h=0$ everywhere on $\overline{T}^*X$, as well as the behaviour at the fiber infinity $\partial\overline{T}^*X$ for small, but positive, values of h – see [Va, §2.1]. We cannot use the more convenient space of classical operators, whose principal symbol is just a function on T^*X (see [DaDy, §3.1]) because the symbol of the operator $e^{sG(h)}\mathbf{P}e^{-sG(h)}$ (see §3) has the form $p+ishH_pG$, with $p\in S^1(X)$ and $H_pG=\mathcal{O}(\log(2+|\xi|))$ narrowly missing the class $S^0(X)$. The (open) elliptic set $\operatorname{ell}_h(A)\subset\overline{T}^*X$ is defined as follows: $(x,\xi)\in\operatorname{ell}_h(A)$ if $\langle\xi'\rangle^{-k}|\sigma_h(A)(x',\xi';h)|\geq c>0$ for h small enough and all $(x',\xi')\in T^*X$ in a neighbourhood of (x,ξ) in $\overline{T}^*X$. Similarly to §C.1, we can study operators and distributions with values in smooth vector bundles over X.

Proof of Lemma 2.3. Using local coordinates, we reduce to the case $X=\mathbb{R}^n$, $Y=\mathbb{R}^m$. Assume first that there exist neighbourhoods U,V such that (2.7) holds. Take $\chi_x\in C^\infty_c(X),\chi_y\in C^\infty_c(Y)$ with $\chi_x(x)\neq 0,\chi_y(y)\neq 0$, and neighbourhoods U_ξ,V_η of ξ,η, such that $\operatorname{supp}\chi_x\times U_\xi\subset U,\quad \operatorname{supp}\chi_y\times V_\eta\subset V$.

Let $K'_B(y',x')=\chi_y(y')K_B(y',x')\chi_x(x')$, and take arbitrary $\xi'\in U_\xi,\eta'\in V_\eta$ (depending on h). Then

$$\mathcal{F}_hK'_B(\eta',-\xi')=\mathcal{F}_h(\chi_yBf)(\eta'),\quad f(x'):=\chi_x(x')e^{ix'\cdot\xi'/h}.$$

where $\mathcal{F}_h$ denotes the semiclassical Fourier transform [Zw, §3.3]. We have $\mathrm{WF}_h(f)\subset U$ (see [Zw, (8.4.7)]) and thus by (2.7), $\mathrm{WF}_h(Bf)\cap V=\emptyset$. It follows that $\mathrm{WF}_h(\chi_yBf)\cap(\mathbb{R}^n\times V_\eta)=\emptyset$ and thus by the semiclassical analog of [HöI–II, Proposition 8.1.3], $\mathcal{F}_h(\chi_yBf)(\eta')=\mathcal{O}(h^\infty)$ for $\eta'\in V_\eta$, yielding, by the characterization of WF_h via the Fourier transform, $(y,\eta,x,\xi)\notin\mathrm{WF}'_h(B)$.

Now, assume that $(y,\eta,x,\xi)\notin\mathrm{WF}'_h(B)$. Take $\chi_x\in C_c^\infty(X),\chi_y\in C_c^\infty(Y)$ such that $\chi_x=1$ on a neighbourhood U_x of x, $\chi_y=1$ on a neighbourhood V_y of y, and neighbourhoods U_ξ,V_η of ξ,η, such that

$$(\operatorname{supp}\chi_y\times\overline{V}_\eta\times\operatorname{supp}\chi_x\times\overline{U}_\xi)\cap\mathrm{WF}'_h(B)=\emptyset. \tag{C.3}$$

Put $U:=U_x\times U_\xi$, $V:=V_y\times V_\eta$, and assume that f is an h-tempered family of distributions on X such that $\mathrm{WF}_h(f)\subset U$. By Fourier inversion formula together with the characterization of WF_h via the Fourier transform,

$$\begin{aligned}f(x')&=\chi_x(x')(2\pi h)^{-n}\int_{U_\xi}e^{ix'\cdot\xi'/h}\mathcal{F}_hf(\xi')\,d\xi'+(1-\chi_x(x'))f(x')\\&\quad+\chi_x(x')(2\pi h)^{-n}\int_{\mathbb{R}^n\setminus U_\xi}e^{ix'\cdot\xi'/h}\mathcal{F}_hf(\xi')\,d\xi'\\&=(2\pi h)^{-n}\int_{U_\xi}\chi_x(x')e^{ix'\cdot\xi'/h}\mathcal{F}_hf(\xi')\,d\xi+\mathcal{O}(h^\infty)_{C_c^\infty}.\end{aligned}$$

Therefore, if $K'_B(y',x')=\chi_y(y')K_B(y',x')\chi_x(x')$, then for bounded η',

$$\mathcal{F}_h(\chi_yBf)(\eta')=(2\pi h)^{-n}\int_{U_\xi}\mathcal{F}_hK'_B(\eta',-\xi')\mathcal{F}_hf(\xi')\,d\xi'+\mathcal{O}(h^\infty)_{\mathscr{S}(\mathbb{R}^m)}.$$

However, we have by (C.3), $\mathcal{F}_hK'_B(\eta',-\xi')=\mathcal{O}(h^\infty)$ for $(\eta',\xi')\in V_\eta\times U_\xi$; therefore, $\mathcal{F}_h(\chi_yBf)(\eta')=\mathcal{O}(h^\infty)$ for $\eta'\in V_\eta$, implying that $\mathrm{WF}_h(Bf)\cap V=\emptyset$. ∎

C.3. **Proofs of semiclassical estimates.** In this subsection, we denote by boldface letters distributions with values in $\mathcal{E}$ or operators acting on such distributions, and with regular letters, scalar distributions and operators. Note that any $A\in\Psi_h^k(X)$ can be viewed as an element of $\Psi_h^k(X;\mathrm{Hom}(\mathcal{E}))$ via the diagonal action.

Proof of Proposition 2.4. Part 2 follows immediately from part 1 and the definition of WF_h. Indeed, assume that $(x,\xi)\in\mathrm{ell}_h(\mathbf{P})\setminus\mathrm{WF}_h(\mathbf{Pu})$; it suffices to prove that $(x,\xi)\notin\mathrm{WF}_h(\mathbf{u})$. Take a neighbourhood U of (x,ξ) such that $U\Subset\mathrm{ell}_h(\mathbf{P})\setminus\mathrm{WF}_h(\mathbf{Pu})$, and choose $B\in\Psi_h^0(X)$ such that $U\subset\mathrm{ell}_h(B)$ and $\mathrm{WF}_h(B)\cap\mathrm{WF}_h(\mathbf{Pu})=\emptyset$. Then $B\mathbf{P}$ is elliptic on U and $\|B\mathbf{Pu}\|_{H_h^{m-k}}=\mathcal{O}(h^\infty)$ for all m; by part 1, applied to the operator $B\mathbf{P}$ in place of $\mathbf{P}$, we get $\|A\mathbf{u}\|_{H_h^m} = \mathcal{O}(h^\infty)$ for all m and all $A \in \Psi_h^0(X)$ such that $\mathrm{WF}_h(A) \subset U$, as required.

It remains to prove part 1. Similarly to the proof of [HöIII–IV, Theorem 18.1.9] (reducing to local frames of $\mathcal{E}$ and either using Cramer's rule or repeatedly differentiating the equation $\sigma_h(\mathbf{P})^{-1}\sigma_h(\mathbf{P}) = 1$), we see that the inverse $\sigma_h(\mathbf{P})^{-1}$ of $\sigma_h(\mathbf{P})$ in $C^\infty(X; \mathrm{Hom}(\mathcal{E}))$ is well-defined and lies in $S_h^{-k}(X; \mathrm{Hom}(\mathcal{E}))$ for h small enough and $(x, \xi) \in \mathrm{ell}(\mathbf{P})$. Using a cutoff function in $\overline{T}^*X$, we can then construct $\mathbf{q} \in S_h^{-k}(X; \mathrm{Hom}(\mathcal{E}))$ such that $\mathbf{q} = \sigma_h(\mathbf{P})^{-1}$ near $\mathrm{WF}_h(A)$. Take $\mathbf{Q}_0 \in \Psi_h^{-k}(X; \mathrm{Hom}(\mathcal{E}))$ such that $\sigma_h(\mathbf{Q}_0) = \mathbf{q}$, then $\mathbf{Q}_0\mathbf{P} = 1 - h\mathbf{R}$ microlocally near $\mathrm{WF}_h(A)$, where $\mathbf{R} \in \Psi_h^{-1}(X; \mathrm{Hom}(\mathcal{E}))$. Using asymptotic Neumann series exactly as in the proof of [HöIII–IV, Theorem 18.1.9] to invert $1 - h\mathbf{R}$, we construct $\mathbf{Q} \in \Psi_h^{-k}(X; \mathrm{Hom}(\mathcal{E}))$ such that

$$\mathbf{QP} = 1 + \mathcal{O}(h^\infty)_{\Psi^{-\infty}} \quad \text{microlocally near } \mathrm{WF}_h(A).$$

Then $A\mathbf{u} = A\mathbf{QPu} + \mathcal{O}(h^\infty)_{C^\infty}$, implying (2.8). □

Proof of Proposition 2.5. Similarly to Proposition 2.4, it is enough to prove part 1. Moreover, by a partition of unity, we may assume that $\mathrm{WF}_h(A)$ is contained in a small neighbourhood of some fixed $(x_0, \xi_0) \in \overline{T}^*X$. Let $\gamma(t) = \exp(tH_p)(x_0, \xi_0)$ and take $T \geq 0$ such that $\gamma(-T) \in \mathrm{ell}_h(B)$; we may then assume that

$$e^{-TH_p}(\mathrm{WF}_h(A)) \subset \mathrm{ell}_h(B), \quad e^{tH_p}(\mathrm{WF}_h(A)) \subset \mathrm{ell}_h(B_1) \quad \text{for } t \in [-T, 0]. \tag{C.4}$$

It is enough to prove the estimate

$$\|A\mathbf{u}\|_{H_h^m} \leq C\|B\mathbf{u}\|_{H_h^m} + Ch^{-1}\|B_1\mathbf{Pu}\|_{H_h^m} + \mathcal{O}(h^{1/2})\|B_1\mathbf{u}\|_{H_h^{m-1/2}} + \mathcal{O}(h^\infty). \tag{C.5}$$

Indeed, without loss of generality we may assume that each for each $(x, \xi) \in \mathrm{WF}_h(B_1)$, there exists $t \in [-T, 0]$ such that $e^{tH_p}(x, \xi) \in \mathrm{WF}_h(B)$; one can then apply (C.5) with A replaced by B_1 and replace $\mathcal{O}(h^{1/2})\|B_1\mathbf{u}\|_{H_h^{m-1/2}}$ by $\mathcal{O}(h)\|B_2\mathbf{u}\|_{H_h^{m-1}}$ for certain $B_2 \in \Psi_h^0$ microlocalized near $\gamma([-T, 0])$; repeating this process, and recalling that $\mathbf{u}$ is h-tempered, we can ultimately make this term $\mathcal{O}(h^\infty)$.

In addition to a smooth density on X, we fix a smooth inner product on the fibers of $\mathcal{E}$; this defines a Hilbert inner product $\langle \cdot, \cdot \rangle$ on $L^2(X; \mathcal{E})$. We denote

$$\mathrm{Re}\,\mathbf{P} = \frac{\mathbf{P} + \mathbf{P}^*}{2}, \quad \mathrm{Im}\,\mathbf{P} = \frac{\mathbf{P} - \mathbf{P}^*}{2i},$$

so that $\mathrm{Re}\,\mathbf{P}, \mathrm{Im}\,\mathbf{P} \in \Psi_h^1(X; \mathrm{Hom}(\mathcal{E}))$ are symmetric and $\mathbf{P} = \mathrm{Re}\,\mathbf{P} + i\,\mathrm{Im}\,\mathbf{P}$.

We will use an *escape function* $f(x, \xi) \in C^\infty(\overline{T}^*X)$, such that $\operatorname{supp} f \subset \mathrm{ell}_h(B_1)$ and

$$f \geq 0 \quad \text{everywhere}; \tag{C.6}$$

$$f > 0 \quad \text{near } \mathrm{WF}_h(A); \tag{C.7}$$

$$H_p f \leq -C_0 f \quad \text{outside of } \mathrm{ell}_h(B). \tag{C.8}$$

Here $C_0 > 0$ is a large constant to be chosen later. To construct such f, we use (C.4) and identify a tubular neighbourhood of $\gamma([-T,0])$ contained in $\mathrm{ell}_h(B_1)$ with

$$\{|\theta| < \delta\} \times (-T-\delta,\delta)_\tau \subset \mathbb{R}^{2n-1}_\theta \times \mathbb{R}_\tau,$$

for small $\delta > 0$, so that H_p is mapped to ∂_τ. We then put $f(\theta,\tau) = \chi(\theta)\psi(\tau)$, where $\chi \in C^\infty_{\mathrm{c}}(\{|\theta| < \delta\};[0,1])$ satisfies $\chi = 1$ on $\{|\theta| \leq \delta/2\}$, and $\psi \in C^\infty_{\mathrm{c}}(-T-\delta,\delta)$ satisfies $\psi \geq 0$ everywhere, $\psi(0) > 0$, and $\psi' \leq -C_0\psi$ outside of $(-T-\delta,-T+\delta)$. (To construct ψ we first choose $\psi_0 \in C^\infty_{\mathrm{c}}(-T-\delta,\delta)$ such that $\psi_0 \geq 0$, $\psi_0(0) = 1$, and $\psi' \leq 0$ on $(-T+\delta,\delta)$. We then put $\psi(\tau) := e^{-C_0\tau}\psi_0(\tau)$.)

We now prove (C.5) by a positive commutator argument, going back to [Hö]. Because $\mathrm{WF}_h(A)$ might intersect the fiber infinity $\partial\overline{T}^*X$, we have to put in regularizing pseudodifferential operators. Assume that $S_\varepsilon \in \Psi^{m-1}_h$, $\varepsilon \in (0,1)$, quantizes the symbol $\sigma_h(S_\varepsilon) := \langle\xi\rangle^m\langle\varepsilon\xi\rangle^{-1}$. Note that S_ε is bounded uniformly in Ψ^m_h for $\varepsilon > 0$. Take $F \in \Psi^0_h$ such that $\sigma_h(F) = f$ and $\mathrm{WF}_h(F) \subset \mathrm{ell}_h(B_1)$, and put $F_\varepsilon = S_\varepsilon F \in \Psi^{m-1}_h$, so that $\sigma_h(F_\varepsilon) = f_\varepsilon := \langle\xi\rangle^m\langle\varepsilon\xi\rangle^{-1}f$. Assume that $B_1\mathbf{u} \in H^{m-1/2}_h(X;\mathcal{E})$. For each $\varepsilon > 0$

$$\operatorname{Im}\langle\mathbf{Pu}, F^*_\varepsilon F_\varepsilon\mathbf{u}\rangle = \frac{i}{2}\langle[\operatorname{Re}\mathbf{P}, F^*_\varepsilon F_\varepsilon]\mathbf{u},\mathbf{u}\rangle + \frac{1}{2}\langle(F^*_\varepsilon F_\varepsilon \operatorname{Im}\mathbf{P} + (\operatorname{Im}\mathbf{P})F^*_\varepsilon F_\varepsilon)\mathbf{u},\mathbf{u}\rangle, \quad \text{(C.9)}$$

where the product on the left-hand side makes sense because $B_1\mathbf{Pu} \in H^m_h \subset H^{m-3/2}_h$, $\mathrm{WF}_h(F_\varepsilon) \subset \mathrm{ell}_h(B_1)$ and $F^*_\varepsilon F_\varepsilon\mathbf{u} \in H^{-m+3/2}_h$.

We now estimate the terms on the right-hand side of (C.9). Denote

$$\mathbf{T}_\varepsilon := \frac{i}{2h}[\operatorname{Re}\mathbf{P}, F^*_\varepsilon F_\varepsilon] \in \Psi^{2m-2}_h(X;\mathrm{Hom}(\mathcal{E})), \quad \text{(C.10)}$$

which is bounded in Ψ^{2m}_h, uniformly in ε. The principal symbol of $\mathbf{T}_\varepsilon$ in Ψ^{2m}_h is independent of h and diagonal with entries

$$f_\varepsilon\, H_p f_\varepsilon = \langle\xi\rangle^m\langle\varepsilon\xi\rangle^{-1} f_\varepsilon\, H_p f + f^2_\varepsilon\Big(\frac{m}{2}\langle\xi\rangle^{-2} - \frac{\varepsilon^2}{2}\langle\varepsilon\xi\rangle^{-2}\Big)H_p(|\xi|^2). \quad \text{(C.11)}$$

Since $H_p(|\xi|^2) = \mathcal{O}(|\xi|^2)$, we get

$$\Big(\frac{m}{2}\langle\xi\rangle^{-2} - \frac{\varepsilon^2}{2}\langle\varepsilon\xi\rangle^{-2}\Big)H_p(|\xi|^2) = \mathcal{O}(1),$$

uniformly in ε,ξ. Therefore, for C_0 large enough depending on m, and some large constant C, (C.8) implies that

$$f_\varepsilon\, H_p f_\varepsilon + \frac{C_0}{2} f^2_\varepsilon \leq C|\langle\xi\rangle^m\sigma_h(B)|^2.$$

The sharp Gårding inequality [Zw, Theorem 9.11] applied to the operator $\mathbf{T}_\varepsilon + \frac{C_0}{2}F^*_\varepsilon F_\varepsilon - C(S_0B)^*(S_0B)$, where $\sigma_h(S_0) = \langle\xi\rangle^m$, gives, uniformly in ε,

$$\langle\mathbf{T}_\varepsilon\mathbf{u},\mathbf{u}\rangle + \frac{C_0}{2}\|F_\varepsilon\mathbf{u}\|^2_{L^2} \leq C\|B\mathbf{u}\|^2_{H^m_h} + Ch\|B_1\mathbf{u}\|^2_{H^{m-1/2}_h} + \mathcal{O}(h^\infty). \quad \text{(C.12)}$$

We next claim that, uniformly in ε,

$$\frac{1}{2}\langle (F_\varepsilon^* F_\varepsilon \operatorname{Im}\mathbf{P} + (\operatorname{Im}\mathbf{P})F_\varepsilon^* F_\varepsilon)\mathbf{u},\mathbf{u}\rangle \leq C_1 h\|F_\varepsilon \mathbf{u}\|_{L^2}^2 + Ch^2\|B_1\mathbf{u}\|^2_{H_h^{m-1/2}} + \mathcal{O}(h^\infty), \quad \text{(C.13)}$$

where C_1 is a constant independent of the choice of f. Indeed, the left-hand side of (C.13) can be written as

$$\langle (\operatorname{Im}\mathbf{P})F_\varepsilon\mathbf{u}, F_\varepsilon\mathbf{u}\rangle + \frac{1}{2}\langle (F_\varepsilon^*[F_\varepsilon,\operatorname{Im}\mathbf{P}] - [F_\varepsilon^*,\operatorname{Im}\mathbf{P}]F_\varepsilon)\mathbf{u},\mathbf{u}\rangle.$$

Since $\sigma_h(\operatorname{Im}\mathbf{P}) = -q$ is diagonal and nonpositive, the first term is bounded from above by $C_1 h\|F_\varepsilon\mathbf{u}\|_{L^2}^2$ by the sharp Gårding inequality. The second term is bounded by $Ch^2\|B_1\mathbf{u}\|^2_{H_h^{m-1/2}} + \mathcal{O}(h^\infty)$, since the principal symbol calculus shows that

$$F_\varepsilon^*[F_\varepsilon,\operatorname{Im}\mathbf{P}] - [F_\varepsilon^*,\operatorname{Im}\mathbf{P}]F_\varepsilon \in h^2\Psi_h^{2m-1}$$

uniformly in ε.

Combining (C.9), (C.12), (C.13), taking $C_0 > 4C_1$, we get uniformly in ε,

$$\frac{C_0}{4}\|F_\varepsilon\mathbf{u}\|_{L^2}^2 \leq C\|B\mathbf{u}\|^2_{H_h^m} + Ch^{-1}\|B_1\mathbf{P}\mathbf{u}\|_{H_h^m}\|F_\varepsilon\mathbf{u}\|_{L^2} + Ch\|B_1\mathbf{u}\|^2_{H_h^{m-1/2}} + \mathcal{O}(h^\infty).$$

Therefore, we have uniformly in ε,

$$\|F_\varepsilon\mathbf{u}\|_{L^2} \leq C\|B\mathbf{u}\|_{H_h^m} + Ch^{-1}\|B_1\mathbf{P}\mathbf{u}\|_{H_h^m} + Ch^{1/2}\|B_1\mathbf{u}\|_{H_h^{m-1/2}} + \mathcal{O}(h^\infty).$$

Now, $F_\varepsilon = S_\varepsilon F$ and $S_\varepsilon \to S_0$ in $\Psi_h^{m+1/2}$ as $\varepsilon \to 0$; therefore, $F_\varepsilon\mathbf{u} \to S_0 F\mathbf{u}$ in H_h^{-1}. Since $\|F_\varepsilon\mathbf{u}\|_{L^2}$ is bounded uniformly in ε, by the compactness of the unit ball in L^2 in the weak topology we get $S_0F\mathbf{u} \in L^2$; therefore, $F\mathbf{u} \in H_h^m$, and

$$\|F\mathbf{u}\|_{H_h^m} \leq C\|B\mathbf{u}\|_{H_h^m} + Ch^{-1}\|B_1\mathbf{P}\mathbf{u}\|_{H_h^m} + Ch^{1/2}\|B_1\mathbf{u}\|_{H_h^{m-1/2}} + \mathcal{O}(h^\infty).$$

It remains to apply the elliptic estimate (2.8) together with (C.7). □

To prove Propositions 2.6 and 2.7 we need the following

Lemma C.1. *Suppose L is a radial source in the sense of definition (2.12). Then there exist:*

1. $f_0 \in C^\infty(T^*X \setminus 0; [0,1])$, homogeneous of degree 0 and such that $f_0 = 1$ near L, $\operatorname{supp} f_0 \subset U$, and $H_p f_0 \leq 0$;

2. $f_1 \in C^\infty(T^*X \setminus 0; [0,\infty))$, homogeneous of degree 1 and such that $f_1 \geq c|\xi|$ everywhere and $H_p f_1 \leq -cf_1$ on U, for some $c > 0$.

Proof. To obtain part 1 we adapt the proof of [FaSj, Lemma 2.1]. Let $V = \kappa_* H_p$, where $\kappa : T^*X \setminus 0 \to S^*X \simeq (T^*X \setminus 0)/\mathbb{R}_+$ is the natural projection. Since p is homogeneous of degree 1, $\kappa_* H_p$ is a smooth vector field on S^*X, and the closed set $\kappa(L)$ is invariant under the flow e^{-tV}. We will construct $F \in C^\infty(S^*X; [0,1])$ such that $V(F) \leq 0$,

$\operatorname{supp} F \subset \kappa(U)$ and $F = 1$ on a neighbourhood of $\kappa(L)$. Then $f_0 = \kappa^* F$ will be a function satisfying the condition in part 1.

To obtain F, fix $F_0 \in C^\infty(S^*X; [0,1])$ such that $F_0 = 1$ near $\kappa(L)$ and $\operatorname{supp} F_0 \subset \kappa(U)$. By the first assumption in (2.12), we have for $T > 0$ large enough,

$$e^{-tV} \operatorname{supp}(F_0) \subset \{F_0 = 1\}, \quad \text{for } t \geq T, \tag{C.14}$$

and by the invariance of $\kappa(U)$ by the flow, $\operatorname{supp}(F_0 \circ e^{tV}) \subset \kappa(U)$ for all $t \geq T$. Furthermore, $F_0(\rho) \geq F_0(e^{TV}(\rho))$ for all ρ; indeed, if $e^{TV}(\rho) \in \operatorname{supp} F_0$, then $F_0(\rho) = 1$ and otherwise $F_0(e^{TV}(\rho)) = 0$, and $0 \leq F_0 \leq 1$ everywhere. Then the function

$$F := \frac{1}{T} \int_T^{2T} F_0 \circ e^{tV}\, dt, \quad V(F) = \frac{1}{T}(F_0 \circ e^{2TV} - F_0 \circ e^{TV}),$$

satisfies the required assumptions.

The proof of part 2 is "orthogonal" to the proof of part 1 in the sense that we are concerned about the radial component of H_p. To find f_1, fix a smooth norm $|\cdot|$ of the fibers of T^*X. By the second part of (2.12), we have for T_1 large enough,

$$|e^{-tH_p}(x,\xi)| \geq 2|\xi|, \quad \text{for } (x,\xi) \in U,\ t \geq T_1.$$

Then the function

$$f_1(x,\xi) := \int_0^{T_1} |e^{-tH_p}(x,\xi)|\, dt, \quad H_p f_1(x,\xi) = |\xi| - |e^{-T_1 H_p}(x,\xi)|,$$

is homogeneous of degree 1, $0 < c|\xi| \leq f_1(x,\xi) \leq c^{-1}|\xi|$ everywhere, and $H_p f_1(x,\xi) \leq -|\xi| \leq -c f_1(x,\xi)$ for $(x,\xi) \in U$. □

Proof of Proposition 2.6. As before, it is enough to prove part 1. Similarly to (C.5), it suffices to prove that for each $B_1 \in \Psi_h^0$ elliptic on $\kappa(L)$, there exists $A \in \Psi_h^0$ elliptic on $\kappa(L)$ such that for each $m \geq m_0$,

$$\|A\mathbf{u}\|_{H_h^m} \leq Ch^{-1}\|B_1 \mathbf{P}\mathbf{u}\|_{H_h^m} + \mathcal{O}(h^{1/2})\|B_1\mathbf{u}\|_{H_h^{m-1/2}} + \mathcal{O}(h^\infty). \tag{C.15}$$

Indeed, without loss of generality we may assume that $\mathrm{WF}_h(B_1) \subset U$; then by (2.12), each backward flow line of H_p starting on $\mathrm{WF}_h(B_1)$ reaches $\mathrm{ell}_h(A)$. Combining (C.15) with propagation of singularities (Proposition 2.5), we see that for each $B_1' \in \Psi_h^0$ elliptic on $\kappa(L)$, there exists $A \in \Psi_h^0$ elliptic on $\kappa(L)$ such that for each $m \geq m_0$,

$$\|A\mathbf{u}\|_{H_h^m} \leq Ch^{-1}\|B_1' \mathbf{P}\mathbf{u}\|_{H_h^m} + \mathcal{O}(h^{1/2})\|A\mathbf{u}\|_{H_h^{m-1/2}} + \mathcal{O}(h^\infty).$$

Iterating this estimate, we arrive to

$$\|A\mathbf{u}\|_{H_h^m} \leq Ch^{-1}\|B_1' \mathbf{P}\mathbf{u}\|_{H_h^m} + \mathcal{O}(h^\infty)\|A\mathbf{u}\|_{H_h^{m_0}} + \mathcal{O}(h^\infty), \tag{C.16}$$

and the $\mathcal{O}(h^\infty)\|A\mathbf{u}\|_{H_h^{m_0}}$ error term can be trivially removed provided that $A\mathbf{u} \in H_h^{m_0}$.

To prove (C.15), we shrink the conic neighbourhood U of L so that $\kappa(U) \subset \mathrm{ell}_h(B_1)$; here $\kappa : T^*X \setminus 0 \to S^*X = \partial\overline{T}^*X$ is the natural projection to the fiber infinity. Let

f_0, f_1 be given by Lemma C.1 and consider $R > 0$ large enough so that $\operatorname{supp} f_0 \cap \{f_1 \geq R\} \subset \operatorname{ell}_h(B_1)$. Let $\chi \in C^\infty(\mathbb{R}; [0,1])$ satisfy $\operatorname{supp} \chi \subset (R, \infty)$, $\chi = 1$ on $[2R, \infty)$, and $\chi' \geq 0$ everywhere. Define $f \in C^\infty(\overline{T}^* X)$ by

$$f(x,\xi) = f_0(x,\xi)\chi(f_1(x,\xi)). \tag{C.17}$$

It follows from Lemma C.1 that $\operatorname{supp} f \subset \operatorname{ell}_h(B_1)$, $f = 1$ near $\kappa(L)$, and $H_p f \leq 0$ everywhere.

We now proceed as in the proof of Proposition 2.5, putting

$$\sigma_h(S_\varepsilon) = f_2^m \langle \varepsilon\xi \rangle^{-1}.$$

Here $f_2 \in C^\infty(\overline{T}^* X)$ is positive everywhere and is equal to f_1 for large $|\xi|$, in particular for $f_1(x,\xi) \geq R$. If $f_\varepsilon = \sigma_h(S_\varepsilon) f$, then similarly to (C.11), we find

$$f_\varepsilon H_p f_\varepsilon = f_2^m \langle \varepsilon\xi \rangle^{-1} f_\varepsilon H_p f + f_\varepsilon^2 \Big(m \frac{H_p f_2}{f_2} - \frac{\varepsilon^2 H_p |\xi|^2}{2\langle \varepsilon\xi \rangle^2} \Big) \tag{C.18}$$

Since $H_p f \leq 0$ and $H_p f_2 \leq -c f_2 < 0$ on $\operatorname{supp} f$, we see that for any fixed $C_0 > 0$, m_0 large enough depending on C_0, and $m \geq m_0$,

$$f_\varepsilon H_p f_\varepsilon + C_0 f_\varepsilon^2 \leq 0.$$

Moreover, m_0 can be chosen independently of B_1. For $\mathbf{T}_\varepsilon$ defined by (C.10), the sharp Gårding inequality gives, uniformly in ε,

$$\langle \mathbf{T}_\varepsilon \mathbf{u}, \mathbf{u} \rangle + C_0 \|F_\varepsilon \mathbf{u}\|_{L^2}^2 \leq Ch \|B_1 \mathbf{u}\|^2_{H_h^{m-1/2}} + \mathcal{O}(h^\infty).$$

Arguing as in the proof of Proposition 2.5, we obtain (C.15) with $A := F$. □

Proof of Proposition 2.7. We proceed as in the proof of Proposition 2.6, showing that for each $B_1 \in \Psi_h^0$ elliptic on $\kappa(L)$, there exists $A \in \Psi_h^0(X)$ elliptic on $\kappa(L)$ and $B \in \Psi_h^0(X)$ with $\operatorname{WF}_h(B) \subset \operatorname{ell}_h(B_1) \setminus \kappa(L)$ such that for $m \leq -m_0$,

$$\|A\mathbf{u}\|_{H_h^m} \leq C\|B\mathbf{u}\|_{H_h^m} + Ch^{-1}\|B_1 \mathbf{P}\mathbf{u}\|_{H_h^m} + \mathcal{O}(h^{1/2})\|B_1 \mathbf{u}\|_{H_h^{m-1/2}} + \mathcal{O}(h^\infty). \tag{C.19}$$

Take $f \in C^\infty(\overline{T}^* X; [0,1])$ such that $\operatorname{supp} f \subset \operatorname{ell}_h(B_1)$ and $f = 1$ near $\kappa(L)$, and define f_2 using Lemma C.1 with the sign of p reversed, so that $H_p f_2 \geq c f_2$ on $\operatorname{supp} f$. We define $S_\varepsilon, f_\varepsilon$ as in the proof of Proposition 2.7 and analyse the terms on the right-hand side of (C.18). The first term vanishes near $\kappa(L)$ since $f = 1$ there. Using the second term, we see that for each C_0, m_0 large enough depending on C_0, and $m \leq -m_0$,

$$f_\varepsilon H_p f_\varepsilon + C_0 f_\varepsilon^2 \leq |\langle \xi \rangle^m \sigma_h(B)|^2,$$

for some choice of $B \in \Psi_h^0$ with $\operatorname{WF}_h(B) \subset \operatorname{ell}_h(B_1) \setminus \kappa(L)$. By sharp Gårding inequality, we have uniformly in ε

$$\langle \mathbf{T}_\varepsilon \mathbf{u}, \mathbf{u} \rangle + C_0 \|F_\varepsilon \mathbf{u}\|_{L^2}^2 \leq C\|B\mathbf{u}\|^2_{H_h^m} + Ch\|B_1 \mathbf{u}\|^2_{H_h^{m-1/2}} + \mathcal{O}(h^\infty);$$

arguing as in the proof of Proposition 2.5, we obtain (C.19) with $A := F$. □

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References

[Ba] Viviane Baladi, *Anisotropic Sobolev spaces and dynamical transfer operators: C^∞ foliations,* Algebraic and topological dynamics, 123–135, Contemp. Math. **385**, AMS, 2005.

[BaTs] Viviane Baladi and Masato Tsujii, *Anisotropic Hölder and Sobolev spaces for hyperbolic diffeomorphisms,* Ann. Inst. Fourier **57**(2007), no. 1, 127–154.

[DaDy] Kiril Datchev and Semyon Dyatlov, *Fractal Weyl laws for asymptotically hyperbolic manifolds,* Geom. Funct. Anal. **23**(2013) 1145–1206.

[DDZ] Kiril Datchev, Semyon Dyatlov, and Maciej Zworski, *Sharp polynomial bounds on the number of Pollicott–Ruelle resonances,* Ergodic Theory Dyn. Syst., published online.

[DFG] Semyon Dyatlov, Frédéric Faure, and Colin Guillarmou, *Power spectrum of the geodesic flow on hyperbolic manifolds,* preprint, arXiv:1403.0256.

[FaSj] Frédéric Faure and Johannes Sjöstrand, *Upper bound on the density of Ruelle resonances for Anosov flows,* Comm. Math. Phys. **308**(2011), no. 2, 325–364.

[FaTs1] Frédéric Faure and Masato Tsujii, *Band structure of the Ruelle spectrum of contact Anosov flows,* Comptes rendus – Mathématique **351**(2013), 385–391.

[FaTs2] Frédéric Faure and Masato Tsujii, *The semiclassical zeta function for geodesic flows on negatively curved manifolds,* preprint, arXiv:1311.4932.

[Fr] David Fried, *Meromorphic zeta functions for Analytic flows,* Comm. Math. Phys. **174**(1995), 161–190.

[GiLiPo] Paolo Giulietti, Carlangelo Liverani, and Mark Pollicott, *Anosov Flows and Dynamical Zeta Functions*, Ann. of Math. **178**(2013), 687–773.

[GoLi] Sébastien Gouëzel and Carlangelo Liverani, *Banach spaces adapted to Anosov systems,* Ergodic Theory Dynam. Systems **26**(2006), no. 1, 189–217.

[Gu] Victor Guillemin, *Lectures on spectral theory of elliptic operators,* Duke Math. J. **44**(1977), 485–517.

[GuZw97] Laurent Guillopé and Maciej Zworski, *Scattering asymptotics for Riemann surfaces,* Ann. of Math. (2) **145**(1997), 597–660.

[GuZw99] Laurent Guillopé and Maciej Zworski, *The wave trace for Riemann surfaces,* Geom. Funct. Anal., **9**(1999), 1156–1168.

[Hö] Lars Hörmander, *On the existence and the regularity of solutions of linear pseudo-differential equations,* Enseignement Math. (2) **17**(1971), 99–163.

[HöI–II] Lars Hörmander, *The Analysis of Linear Partial Differential Operators, Volumes I and II,* Springer, 1983.

[HöIII–IV] Lars Hörmander, *The Analysis of Linear Partial Differential Operators, Volumes III and IV,* Springer, 1985.

[Le] Patricio Leboeuf, *Periodic orbit spectrum in terms of Ruelle–Pollicott resonances,* Phys. Rev. E (3) **69**(2004), no. 2, 026204.

[Li] Carlangelo Liverani, *Fredholm determinants, Anosov maps and Ruelle resonances,* Discrete Contin. Dyn. Syst. **13**(2005), no. 5, 1203–1215.

[Me82] Richard B. Melrose, *Scattering theory and the trace formula of the wave group,* J. Funct. Anal. **45**(1982), 429–440.

[Me94] Richard B. Melrose, *Spectral and scattering theory for the Laplacian on asymptotically Euclidian spaces,* in *Spectral and scattering theory* (M. Ikawa, ed.), Marcel Dekker, 1994.

[Po] Mark Pollicott, *On the rate of mixing of Axiom A flows,* Invent. Math. **81**(1986), 147–164.

[Ru76] David Ruelle, *Zeta-functions for expanding maps and Anosov flows.* Invent. Math. **34**(1976), 231–242.

[Ru86] David Ruelle, *Resonances of chaotic dynamical systems*, Phys. Rev. Lett. **56**(1986), 405–407.

[Ru] Hans Henrik Rugh, *Generalized Fredholm determinants and Selberg zeta functions for Axiom A dynamical systems,* Ergodic Theory and Dynamical Systems, **16**(1996), 805–819.

[SjZw] Johannes Sjöstrand and Maciej Zworski, *Lower bounds on the number of scattering poles II*, J. Funct. Anal. **123**(1994), 336–367.

[Sm] Steven Smale, *Differentiable dynamical systems,* Bull. Amer. Math. Soc. **73**(1967), 747–817.

[Va] András Vasy, *Microlocal analysis of asymptotically hyperbolic and Kerr–de Sitter spaces,* with an appendix by Semyon Dyatlov, Invent. Math. **194**(2013), 381–513.

[Zw96] Maciej Zworski, *Poisson formula for resonances,* Séminaire ÉDP, École Polytechnique, XIII-1-12, 1996–97.

[Zw] Maciej Zworski, *Semiclassical analysis,* Graduate Studies in Mathematics **138** AMS, 2012.

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