RUELLE ZETA FUNCTION AT ZERO FOR SURFACES

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Abstract. We show that the Ruelle zeta function for a negatively curved oriented surface vanishes at zero to the order given by the absolute value of the Euler characteristic. This result was previously known only in constant curvature.

1. introduction

Let (Σ, g) be a compact oriented Riemannian surface of negative curvature and denote by $\mathcal{G}$ the set of primitive closed geodesics on Σ (counted with multiplicity). For $\gamma \in \mathcal{G}$ denote by ℓ_γ its length. The Ruelle zeta function [Rue] is defined by the analogy with the Riemann zeta function, $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$, replacing primes p by primitive closed geodesics:

$$\zeta_R(s) := \prod_{\gamma \in \mathcal{G}} (1 - e^{-s\ell_\gamma}). \tag{1.1}$$

The infinite product converges for $\operatorname{Re} s \gg 1$ and the meromorphic continuation of ζ_R to $\mathbb{C}$ has been a subject of extensive study.

Thanks to the Selberg trace formula the order of vanishing of $\zeta_R(s)$ at 0 has been known for a long time in the case of *constant curvature* and it is given by $-\chi(\Sigma)$ where $\chi(\Sigma)$ is the Euler characteristic. We show that the same result remains true for *any* negatively curved oriented surface:

Theorem. *Let $\zeta_R(s)$ be the Ruelle zeta function for an oriented negatively curved C^∞ Riemannian surface (Σ, g) and let $\chi(\Sigma)$ be its Euler characteristic. Then $s^{\chi(\Sigma)}\zeta_R(s)$ is holomorphic at $s = 0$ and*

$$s^{\chi(\Sigma)}\zeta_R(s)|_{s=0} \neq 0. \tag{1.2}$$

Remarks. 1. The condition that the surface is C^∞ can be replaced by C^k for a sufficiently large k – that is an automatic consequence of our microlocal methods.

2. As was pointed out to us by Yuya Takeuchi, our proof gives a stronger result in which the cosphere bundle $S^*\Sigma = \{(x, \xi) \in T^*\Sigma : |\xi|_g = 1\}$ is replaced by a connected contact 3-manifold M whose contact flow has the Anosov property with orientable stable and unstable bundles (see §§2.3,2.4). If $\mathbf{b}_1(M)$ denotes the first Betti number

of M (see (2.4)) then $s^{2-\mathbf{b}_1(M)}\zeta_R(s)$ is holomorphic at 0 and

$$s^{2-\mathbf{b}_1(M)}\zeta_R(s)|_{s=0} \neq 0. \tag{1.3}$$

Theorem above follows from the fact that for negatively curved surfaces $2-\mathbf{b}_1(S^*\Sigma) = \chi(\Sigma)$ (see Lemma 2.4 for the review of this standard fact). For the existence of contact Anosov flows on 3-manifolds which do not arise from geodesic flows see [FoHa].

3. Our result implies that for a negatively curved connected oriented Riemannian surface, its length spectrum (that is, lengths of closed geodesics counted with multiplicity) determines its genus. This appears to be a previously unknown inverse result – we refer the reader to reviews [Me, Wi, Ze] for more information.

For (Σ, g) of constant curvature the meromorphy of ζ_R follows from its relation to the Selberg zeta function:

$$\zeta_S(s) := \prod_{\gamma\in\mathcal{G}}\prod_{m=0}^{\infty}(1-e^{-(m+s)\ell_\gamma}), \quad \zeta_R(s) = \frac{\zeta_S(s)}{\zeta_S(s+1)},$$

see for instance [Ma, Theorem 5] for a self-contained presentation. In this case the behaviour at $s = 0$ was analysed by Fried [Fr1, Corollary 2] who showed that

$$\zeta_R(s) = \pm(2\pi s)^{|\chi(\Sigma)|}(1 + \mathcal{O}(s)), \tag{1.4}$$

where $\chi(\Sigma)$ is the Euler characteristic of M. A far reaching generalization of this result to locally symmetric manifolds has recently been provided by Shen [Sh, Theorem 4.1] following earlier contributions by Bismut [Bi], Fried [Fr2], and Moskovici–Stanton [MoSt].

For real analytic metrics the meromorphic continuation of $\zeta_R(s)$ is more recent and follows from results of Rugh [Rug] and Fried [Fr3] proved twenty years ago. In the C^∞ case (or C^k for sufficiently large k) that meromorphic continuation is very recent. For Anosov flows on compact manifolds it was first established by Giulietti–Liverani–Pollicott [GLP] and then by Dyatlov–Zworski [DyZw1]. See these papers for references about the background and many other contributions, and also Dyatlov–Guillarmou [DyGu] who considered the more complicated non-compact case and essentially settled the original conjecture of Smale [Sm].

The value at zero of the dynamical zeta function for certain two-dimensional hyperbolic open billiards was computed by Morita [Mo] using Markov partitions. It is possible that similar methods could work in our setting because of the better regularity of stable/unstable foliations in dimensions 2. However, our spectral approach is more direct and, as it does not rely on regularity of the stable/unstable foliations, can be applied in higher dimensions.

The first step of our proof is the standard factorization of ζ_R which shows that the multiplicity of the zero (or pole) of ζ_R can be computed from the multiplicities of Pollicott–Ruelle resonances of the generator of the flow, X, acting on differential forms – see §§2.3,3.1. The resonances are defined as eigenvalues of X acting on microlocally weighted spaces – see (2.9) which we recall from the work of Faure–Sjöstrand [FaSj] and [DyZw1]. The key fact, essentially from [FaSj] – see [DFG, Lemma 5.1] and Lemma 2.2 below – is that the generalized eigenvalue problem is equivalent to solving the equation $(X+s)^k u = 0$ under a *wavefront set condition*. We should stress that the origins of this method lie in the works on anisotropic Banach spaces by Baladi [Ba], Baladi–Tsujii [BaTs], Blank–Keller–Liverani [BKL], Butterley–Liverani [BuLi], Gouëzel–Liverani [GoLi], and Liverani [Li1, Li2].

Hence we need to show that the multiplicities of generalized eigenvalues at $s = 0$ are the same as in the case of constant curvature surfaces (for detailed analysis of Pollicott–Ruelle resonances in that case we refer to [DFG] and [GHW]). For functions and 2-forms that is straightforward. For 1-forms the dimension of the eigenspace turns out to be easily computable using the behaviour of $(X+s)^{-1}$ near 0 acting on scalars and is given by the first Betti number. That is done in §3.3 and it works for any contact Anosov flow on a 3-manifold. In the case of orientable stable and unstable manifolds that gives holomorphy of $s^{2-\mathbf{b}_1(M)}\zeta(s)$ at $s = 0$.

To show (1.3), that is to see that the order of vanishing is exactly $2 - \mathbf{b}_1(M)$, we need to show that zero is a semisimple eigenvalue, that is its algebraic and geometric multiplicities are equal. The key ingredient is a regularity result given in Lemma 2.3. It holds for any Anosov flow preserving a smooth density and could be of independent interest. It is applied in Lemma 3.5 to show that

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2. Ingredients

2.1. **Microlocal analysis.** Our proofs rely on microlocal analysis, and we briefly describe microlocal tools used in this paper providing detailed references to [HöI–II, HöIII–IV, Zw, DyZw1] and [DyZw2, Appendix E].

Let M be a compact smooth manifold and $\mathcal{E}, \mathcal{F}$ smooth vector bundles over M. For $k \in \mathbb{R}$, denote by $\Psi^k(M; \mathrm{Hom}(\mathcal{E}, \mathcal{F}))$ the class of pseudodifferential operators of order k on M with values in homomorphisms $\mathcal{E} \to \mathcal{F}$ and symbols in the class S^k; see for instance [HöIII–IV, §18.1] and [DyZw1, §C.1]. These operators act

$$C^\infty(M; \mathcal{E}) \to C^\infty(M; \mathcal{F}), \quad \mathcal{D}'(M; \mathcal{E}) \to \mathcal{D}'(M; \mathcal{F}) \tag{2.1}$$

where $C^\infty(M;\mathcal{E})$ denotes the space of smooth sections and $\mathcal{D}'(M;\mathcal{E})$ denotes the space of distributional sections [HöI–II, §6.3]. For $k\in\mathbb{N}_0$, the class Ψ^k includes all smooth differential operators of order k. To each $\mathbf{A}\in\Psi^k(M;\mathrm{Hom}(\mathcal{E};\mathcal{F}))$ we associate its principal symbol

$$\sigma(\mathbf{A})\in S^k(M;\mathrm{Hom}(\mathcal{E};\mathcal{F}))/S^{k-1}(M;\mathrm{Hom}(\mathcal{E};\mathcal{F}))$$

and its wavefront set $\mathrm{WF}(\mathbf{A})\subset T^*M\setminus 0$, which is a closed conic set. Here $T^*M\setminus 0$ denotes the cotangent bundle of M without the zero section. In the case of $\mathcal{E}=\mathcal{F}$ we use the notation $\mathrm{End}(\mathcal{E})=\mathrm{Hom}(\mathcal{E};\mathcal{E})$. For a distribution $\mathbf{u}\in\mathcal{D}'(M;\mathcal{E})$, its wavefront set

$$\mathrm{WF}(\mathbf{u})\subset T^*M\setminus 0$$

is a closed conic set defined as follows: a point $(x,\xi)\in T^*M\setminus 0$ does *not* lie in $\mathrm{WF}(\mathbf{u})$ if and only if there exists an open conic neighborhood U of (x,ξ) such that $\mathbf{Au}\in C^\infty(M;\mathcal{E})$ for each $\mathbf{A}\in\Psi^k(M;\mathrm{End}(\mathcal{E}))$ satisfying $\mathrm{WF}(\mathbf{A})\subset U$. See [HöIII–IV, Theorem 18.1.27] for more details. The wavefront set is preserved by pseudodifferential operators: that is,

$$\mathbf{A}\in\Psi^k(M;\mathrm{Hom}(\mathcal{E},\mathcal{F})),\ \mathbf{u}\in\mathcal{D}'(M;\mathcal{E})\implies \mathrm{WF}(\mathbf{Au})\subset\mathrm{WF}(\mathbf{A})\cap\mathrm{WF}(\mathbf{u}). \tag{2.2}$$

Following [HöI–II, §8.2], for a closed conic set $\Gamma\subset T^*M\setminus 0$ we consider the space

$$\mathcal{D}'_\Gamma(M;\mathcal{E})=\{\mathbf{u}\in\mathcal{D}'(M;\mathcal{E})\colon\ \mathrm{WF}(\mathbf{u})\subset\Gamma\} \tag{2.3}$$

and note that by (2.2) this space is preserved by pseudodifferential operators.

We also consider the class $\Psi_h^k(M;\mathrm{Hom}(\mathcal{E};\mathcal{F}))$ of semiclassical pseudodifferential operators with symbols in class S_h^k. The elements of this class are families of operators on (2.1) depending on a small parameter $h>0$. To each $\mathbf{A}\in\Psi_h^k(M;\mathrm{Hom}(\mathcal{E};\mathcal{F}))$ correspond its semiclassical principal symbol and wavefront set

$$\sigma_h(\mathbf{A})\in S_h^k(M;\mathrm{Hom}(\mathcal{E};\mathcal{F}))/hS_h^{k-1}(M;\mathrm{Hom}(\mathcal{E};\mathcal{F})),\quad \mathrm{WF}_h(\mathbf{A})\subset\overline{T}^*M$$

where $\overline{T}^*M$ is the fiber-radially compactified cotangent bundle, see for instance [DyZw2, §E.1]. For an tempered h-dependent family of distributions $\mathbf{u}(h)\in\mathcal{D}'(M;\mathcal{E})$, we can define its wavefront set $\mathrm{WF}_h(\mathbf{u})\subset\overline{T}^*M$.

We denote by $\Psi_h^{\mathrm{comp}}(M)\subset\bigcap_k\Psi_h^k(M)$ the class of compactly microlocalized semiclassical pseudodifferential operators, see [DyZw2, Definition E.29].

2.2. **Differential forms.** Let M be a compact oriented manifold. Denote by Ω^k the complexified vector bundle of differential k-forms on M. The de Rham cohomology spaces are defined as the quotients of the spaces of closed forms by the spaces of exact forms, that is

$$\mathbf{H}^k(M;\mathbb{C})=\frac{\{\mathbf{u}\in C^\infty(M;\Omega^k)\colon d\mathbf{u}=0\}}{\{d\mathbf{v}\colon\mathbf{v}\in C^\infty(M;\Omega^{k-1})\}}.$$

These are finite dimensional vector spaces over $\mathbb{C}$, with the dimensions

$$\mathbf{b}_k(M) := \dim \mathbf{H}^k(M;\mathbb{C}) \tag{2.4}$$

called k-th *Betti numbers*. (It is convenient for us to study cohomology over $\mathbb{C}$, which is of course just the complexification of the cohomology over $\mathbb{R}$.)

De Rham cohomology is typically formulated in terms of smooth differential forms. However, the next lemma shows that one can use instead the classes $\mathcal{D}'_\Gamma$:

Lemma 2.1. *Let $\Gamma \subset T^*M \setminus 0$ be a closed conic set. Using the notation (2.3), assume that $\mathbf{u} \in \mathcal{D}'_\Gamma(M;\Omega^k)$, $d\mathbf{u} \in C^\infty(M;\Omega^{k+1})$.*

Then there exist $\mathbf{v} \in C^\infty(M;\Omega^k)$ and $\mathbf{w} \in \mathcal{D}'_\Gamma(M;\Omega^{k-1})$ such that $\mathbf{u} = \mathbf{v} + d\mathbf{w}$.

Proof. Fix a smooth Riemannian metric on M. We use Hodge theory, in particular the fact that the Hodge Laplacian $\mathbf{\Delta}_k := d\delta + \delta d : \mathcal{D}'(M;\Omega^k) \to \mathcal{D}'(M;\Omega^k)$ is a second order differential operator with scalar principal symbol $\sigma(\mathbf{\Delta}_k)(x,\xi) = |\xi|_g^2$. By the elliptic parametrix construction (see [HöIII–IV, Theorem 18.1.24]) there exists a pseudodifferential operator $\mathbf{Q}_k \in \Psi^{-2}(M;\mathrm{End}(\Omega^k))$ such that

$$\mathbf{Q}_k\mathbf{\Delta}_k - I,\ \mathbf{\Delta}_k\mathbf{Q}_k - I : \mathcal{D}'(M;\Omega^k) \to C^\infty(M;\Omega^k). \tag{2.5}$$

Using (2.2) we now take $\mathbf{w} := \delta\mathbf{Q}_k\mathbf{u} \in \mathcal{D}'_\Gamma(M;\Omega^{k-1})$.

Then by (2.5)

$$\mathbf{u} - \delta d\mathbf{Q}_k\mathbf{u} - d\mathbf{w} = \mathbf{u} - \mathbf{\Delta}_k\mathbf{Q}_k\mathbf{u} \in C^\infty(M;\Omega^k).$$

Since $d\mathbf{u} \in C^\infty(M;\Omega^{k+1})$, we have

$$\mathbf{\Delta}_{k+1}(d\mathbf{Q}_k\mathbf{u}) = d(\mathbf{\Delta}_k\mathbf{Q}_k\mathbf{u}) \in C^\infty(M;\Omega^{k+1}).$$

By (2.5) this implies that $d\mathbf{Q}_k\mathbf{u} \in C^\infty(M;\Omega^{k+1})$ and thus $\delta d\mathbf{Q}_k\mathbf{u} \in C^\infty(M;\Omega^k)$, giving $\mathbf{v} := \mathbf{u} - d\mathbf{w} \in C^\infty(M;\Omega^k)$. ∎

2.3. **Pollicott–Ruelle resonances.** We now follow [FaSj, DyZw1] and recall a microlocal approach to Pollicott–Ruelle resonances. Let M be a compact manifold and X be a smooth vector field on M. We assume that e^{tX} is an Anosov flow, that is each tangent space T_xM admits a stable/unstable decomposition

$$T_xM = \mathbb{R}X(x) \oplus E_u(x) \oplus E_s(x), \quad x \in M,$$

where $E_u(x), E_s(x)$ are subspaces of T_xM depending continuously on x and invariant under the flow and for some constants $C, \nu > 0$ and a fixed smooth metric on M,

$$|de^{tX}(x)\cdot v| \le Ce^{-\nu|t|}\cdot|v|, \quad \begin{cases} t \ge 0, & v \in E_s(x), \\ t \le 0, & v \in E_u(x). \end{cases} \tag{2.6}$$

We consider the dual decomposition

$$T_x^*M = E_0^*(x) \oplus E_u^*(x) \oplus E_s^*(x),$$

where $E_0^*(x), E_u^*(x), E_s^*(x)$ are dual to $\mathbb{R}X(x), E_s(x), E_u(x)$. In particular, $E_u^*(x)$ is the annihilator of $\mathbb{R}X(x) \oplus E_u(x)$ and $E_u^* := \bigcup_{x\in M} E_u^*(x) \subset T^*M$ is a closed conic set.

Assume next that $\mathcal{E}$ is a smooth complex vector bundle over M and

$$\mathbf{P} : C^\infty(M; \mathcal{E}) \to C^\infty(M; \mathcal{E})$$

is a first order differential operator whose principal part is given by $-iX$, that is

$$\mathbf{P}(\varphi\mathbf{u}) = -(iX\varphi)\mathbf{u} + \varphi(\mathbf{P}\mathbf{u}), \quad \varphi \in C^\infty(M), \quad \mathbf{u} \in C^\infty(M; \mathcal{E}). \tag{2.7}$$

For $\lambda \in \mathbb{C}$ with sufficiently large $\operatorname{Im}\lambda$, the integral

$$\mathbf{R}(\lambda) := i\int_0^\infty e^{i\lambda t}e^{-it\mathbf{P}}\,dt : L^2(M; \mathcal{E}) \to L^2(M; \mathcal{E}) \tag{2.8}$$

converges and defines a bounded operator on L^2, holomorphic in λ; in fact, $\mathbf{R}(\lambda) = (\mathbf{P} - \lambda)^{-1}$ on L^2.

The operator $\mathbf{R}(\lambda)$ admits a meromoprhic continuation to the entire complex plane,

$$\mathbf{R}(\lambda) : C^\infty(M; \mathcal{E}) \to \mathcal{D}'(M; \mathcal{E}), \quad \lambda \in \mathbb{C}, \tag{2.9}$$

and the poles of this meromorphic continuation are the *Pollicott–Ruelle resonances*[†] of the operator $\mathbf{P}$. See for instance [DyZw1, §3.2] and [FaSj, Theorems 1.4,1.5].

To define the multiplicity of a Pollicott–Ruelle resonance λ_0, we use the Laurent expansion of $\mathbf{R}$ at λ_0 given by [DyZw1, Proposition 3.3]:

$$\mathbf{R}(\lambda) = \mathbf{R}_H(\lambda) - \sum_{j=1}^{J(\lambda_0)} \frac{(\mathbf{P} - \lambda_0)^{j-1}\Pi}{(\lambda - \lambda_0)^j}, \quad \mathbf{R}_H(\lambda), \Pi : \mathcal{D}'_{E_u^*}(M; \mathcal{E}) \to \mathcal{D}'_{E_u^*}(M; \mathcal{E}), \tag{2.10}$$

where $\mathbf{R}_H(\lambda)$ is holomorphic at λ_0, Π is a finite rank operator, and $\mathcal{D}'_{E_u^*}(M; \mathcal{E})$ is defined using (2.3). The fact that $\mathbf{R}_H(\lambda), \Pi$ can be extended to continuous operators on $\mathcal{D}'_{E_u^*}$ follows from the restrictions on their wavefront sets given in [DyZw1, (3.7)] together with [HöI–II, Theorem 8.2.13]. The multiplicity of λ_0, denoted $m_{\mathbf{P}}(\lambda_0)$, is defined as the dimension of the range of Π.

The multiplicity of a resonance can be computed using generalized resonant states. Here we only need the following special case:

Lemma 2.2. *Define the space of resonant states at* $\lambda_0 \in \mathbb{C}$*,*

$$\operatorname{Res}_{\mathbf{P}}(\lambda_0) = \{\mathbf{u} \in \mathcal{D}'_{E_u^*}(M; \mathcal{E}) \colon (\mathbf{P} - \lambda_0)\mathbf{u} = 0\}.$$

Then $m_{\mathbf{P}}(\lambda_0) \geq \dim \operatorname{Res}_{\mathbf{P}}(\lambda_0)$*. Moreover we have* $m_{\mathbf{P}}(\lambda_0) = \dim \operatorname{Res}_{\mathbf{P}}(\lambda_0)$ *under the following semisimplicity condition:*

$$\mathbf{u} \in \mathcal{D}'_{E_u^*}(M; \mathcal{E}), \quad (\mathbf{P} - \lambda_0)^2\mathbf{u} = 0 \quad \Longrightarrow \quad (\mathbf{P} - \lambda_0)\mathbf{u} = 0. \tag{2.11}$$

[†]To be consistent with [DyZw1] we use the spectral parameter $\lambda = is$ where s is the parameter used in §1. Note that $\operatorname{Re} s \gg 1$ corresponds to $\operatorname{Im}\lambda \gg 1$.

Proof. We first assume that (2.11) holds and prove that $m_{\mathbf{P}}(\lambda_0) \le \dim \operatorname{Res}_{\mathbf{P}}(\lambda_0)$. We have $(\mathbf{P}-\lambda)\mathbf{R}(\lambda) = I$ and thus $(\mathbf{P}-\lambda_0)^{J(\lambda_0)}\Pi = 0$. Take $\mathbf{u}$ in the range of Π, then $\mathbf{u} \in \mathcal{D}'_{E_u^*}(M;\mathcal{E})$ by the mapping property in (2.10) and $(\mathbf{P}-\lambda_0)^{J(\lambda_0)}\mathbf{u} = 0$. Arguing by induction using (2.11) we obtain $\mathbf{u} \in \operatorname{Res}_{\mathbf{P}}(\lambda_0)$, finishing the proof.

It remains to show that $\dim \operatorname{Res}_{\mathbf{P}}(\lambda_0) \le m_{\mathbf{P}}(\lambda_0)$. For that it suffices to prove that

$$\mathbf{u} \in \operatorname{Res}_{\mathbf{P}}(\lambda_0) \quad \Longrightarrow \quad \mathbf{u} = \Pi \mathbf{u}. \tag{2.12}$$

We recall from [DyZw1, §§3.1,3.2] that $\mathbf{R}(\lambda)$ is the restriction to C^∞ of the inverse of the operator

$$\mathbf{P}-\lambda : \{\mathbf{v} \in H_{sG}(M;\mathcal{E}) \colon \mathbf{P}\mathbf{v} \in H_{sG}(M;\mathcal{E})\} \to H_{sG}(M;\mathcal{E}), \tag{2.13}$$

where $H_{sG}(M;\mathcal{E}) \subset \mathcal{D}'(M;\mathcal{E})$ is a specially constructed anisotropic Sobolev space and we may take any $s > s_0$ where s_0 depends on λ. Take $s > s_0$ large enough so that $\mathbf{u}$ lies in the usual Sobolev space $H^{-s}(M;\mathcal{E})$. Since H_{sG} is equivalent to H^{-s} microlocally near E_u^* (see [DyZw1, (3.3),(3.4)]), we have $\mathbf{u} \in H_{sG}$. We compute $(\mathbf{P}-\lambda)^{-1}\mathbf{u} = (\lambda_0-\lambda)^{-1}\mathbf{u}$ for $\mathbf{u} \in \operatorname{Res}_{\mathbf{P}}(\lambda_0)$ and the space C^∞ is dense in $H_{sG} \cap \mathcal{D}'_{E_u^*}$, thus (2.12) follows from the Laurent expansion (2.10) applied to $\mathbf{u}$. □

We finish this section with the following analogue of Rellich's uniqueness theorem in scattering theory: vanishing of radiation patterns implies rapid decay.

Lemma 2.3. *Suppose that there exist a smooth volume form on M and a smooth inner product on the fibers of $\mathcal{E}$, for which $\mathbf{P}^* = \mathbf{P}$ on $L^2(M;\mathcal{E})$. Suppose that $\mathbf{u} \in \mathcal{D}'_{E_u^*}(M;\mathcal{E})$ satisfies*

$$\mathbf{P}\mathbf{u} \in C^\infty(M;\mathcal{E}), \quad \operatorname{Im} \langle \mathbf{P}\mathbf{u}, \mathbf{u}\rangle_{L^2} \ge 0.$$

Then $\mathbf{u} \in C^\infty(M;\mathcal{E})$.

Remark. Lemma 2.3 applies in particular when $\mathbf{u}$ is a resonant state at some $\lambda \in \mathbb{R}$ (replacing $\mathbf{P}$ by $\mathbf{P}-\lambda$), showing that all such resonant states are smooth. This represents a borderline case since for $\operatorname{Im}\lambda > 0$ the integral (2.8) converges and thus there are no resonances.

Proof. We introduce the semiclassical parameter $h > 0$ and use the following statement relating semiclassical and nonsemiclassical wavefront sets of an h-independent distribution $\mathbf{v}$, see [DyZw1, (2.6)]:

$$\operatorname{WF}(\mathbf{v}) = \operatorname{WF}_h(\mathbf{v}) \cap (T^*M \setminus 0). \tag{2.14}$$

Since $\mathbf{u} \in \mathcal{D}'_{E_u^*}$ and $\mathbf{P}\mathbf{u} \in C^\infty$ we have

$$\operatorname{WF}_h(\mathbf{u}) \cap (T^*M \setminus 0) \ \subset \ E_u^*, \quad \operatorname{WF}_h(\mathbf{P}\mathbf{u}) \cap (\overline{T}^*M \setminus 0) \ = \ \emptyset. \tag{2.15}$$

(The last statement uses the fiber-radially compactified cotangent bundle and it follows immediately from the proof of [DyZw1, (2.6)] in [DyZw1, §C.2].)

It suffices to prove that for each $A \in \Psi_h^{\mathrm{comp}}(M)$ with $\mathrm{WF}_h(A) \subset T^*M \setminus 0$, there exists $B \in \Psi_h^{\mathrm{comp}}(M)$ with $\mathrm{WF}_h(B) \subset T^*M \setminus 0$ such that

$$\|A\mathbf{u}\|_{L^2} \leq Ch^{1/2}\|B\mathbf{u}\|_{L^2} + \mathcal{O}(h^\infty). \tag{2.16}$$

Indeed, fix $N > 0$ such that $\mathbf{u} \in H^{-N}$, then $\|A\mathbf{u}\|_{L^2} \leq Ch^{-N}$ for all $A \in \Psi_h^{\mathrm{comp}}(M)$. By induction (2.16) implies that $\|A\mathbf{u}\|_{L^2} = \mathcal{O}(h^\infty)$. This gives $\mathrm{WF}_h(\mathbf{u}) \cap (T^*M \setminus 0) = \emptyset$ and thus by (2.14) $\mathrm{WF}(\mathbf{u}) = \emptyset$, that is $\mathbf{u} \in C^\infty$.

To show (2.16), note that $h\mathbf{P} \in \Psi_h^1(M; \mathrm{End}(\mathcal{E}))$ and its principal symbol is scalar and given by

$$\sigma_h(h\mathbf{P}) = p, \quad p(x,\xi) = \langle \xi, X(x) \rangle.$$

We now claim that there exists $\chi \in C_c^\infty(T^*M; [0,1])$ such that

$$\mathrm{supp}(1-\chi) \subset T^*M \setminus 0, \quad H_p\chi \leq 0 \text{ near } E_u^*, \quad H_p\chi < 0 \text{ on } E_u^* \cap \mathrm{WF}_h(A).$$

To construct χ, we first use part 2 of [DyZw1, Lemma C.1] (applied to $L := E_u^*$ which is a radial source for $-p$) to construct $f_1 \in C^\infty(T^*M \setminus 0; [0,\infty))$ homogeneous of degree 1, satisfying $f_1(x,\xi) \geq c|\xi|$ and $H_p f_1 \geq c f_1$ in a conic neighborhood of E_u^*, for some $c > 0$. Next we put $\chi := \chi_1 \circ f_1$ where $\chi_1 \in C_c^\infty(\mathbb{R}; [0,1])$ satisfies

$$\chi_1 = 1 \text{ near } 0, \quad \chi_1' \leq 0 \text{ on } [0,\infty), \quad \chi_1' < 0 \text{ on } f_1(\mathrm{WF}_h(A)).$$

It is then straightforward to see that χ has the required properties.

We now quantize χ to obtain an operator

$$F \in \Psi_h^{\mathrm{comp}}(M), \quad \sigma_h(F) = \chi, \quad \mathrm{WF}_h(I-F) \subset \overline{T}^*M \setminus 0, \quad F^* = F.$$

Since $H_p\chi \leq 0$ near E_u^* and $H_p\chi < 0$ on $E_u^* \cap \mathrm{WF}_h(A)$ there exists

$$A_1 \in \Psi_h^{\mathrm{comp}}(M), \quad \mathrm{WF}_h(A_1) \subset T^*M \setminus 0, \quad \mathrm{WF}_h(A_1) \cap E_u^* = \emptyset,$$

such that

$$-\tfrac{1}{2}H_p\chi + |\sigma_h(A_1)|^2 \geq C^{-1}|\sigma_h(A)|^2 \tag{2.17}$$

where $C > 0$ is some constant.

Fix $B \in \Psi_h^{\mathrm{comp}}(M)$ with $\mathrm{WF}_h(B) \subset T^*M \setminus 0$ so that

$$\big(\mathrm{WF}_h\left([\mathbf{P}, F]\right) \cup \mathrm{WF}_h(A_1) \cup \mathrm{WF}_h(A)\big) \cap \mathrm{WF}_h(I-B) = \emptyset. \tag{2.18}$$

By the second part of (2.15) we have $(I-F)\mathbf{P}\mathbf{u} = \mathcal{O}(h^\infty)_{C^\infty}$. Since $\mathrm{Im}\langle \mathbf{P}\mathbf{u}, \mathbf{u}\rangle_{L^2} \geq 0$ this gives

$$-\mathrm{Im}\langle F\mathbf{P}\mathbf{u}, \mathbf{u}\rangle_{L^2} \leq \mathcal{O}(h^\infty). \tag{2.19}$$

On the other hand, since $\mathbf{P}$ and F are both symmetric, we get

$$-\mathrm{Im}\langle F\mathbf{P}\mathbf{u}, \mathbf{u}\rangle_{L^2} = \tfrac{1}{2i}\langle [\mathbf{P}, F]\mathbf{u}, \mathbf{u}\rangle_{L^2}. \tag{2.20}$$

We now observe that

$$\tfrac{1}{2i}[\mathbf{P}, F] \in \Psi_h^{\mathrm{comp}}(M; \mathcal{E}), \quad \sigma_h\left(\tfrac{1}{2i}[\mathbf{P}, F]\right) = -\tfrac{1}{2}H_p\chi.$$

Using (2.17) we can apply the sharp Gårding inequality (see for instance [Zw, Theorem 9.11]) to the operator $\frac{1}{2i}[\mathbf{P},F]+A_1^*A_1-C^{-1}A^*A$ and the section $B\mathbf{u}$ to obtain

$$\|AB\mathbf{u}\|_{L^2}^2\leq C\|A_1B\mathbf{u}\|_{L^2}^2+\tfrac{C}{2i}\langle B^*[\mathbf{P},F]B\mathbf{u},\mathbf{u}\rangle_{L^2}+Ch\|B\mathbf{u}\|_{L^2}^2.$$

From (2.18) we see that $AB\mathbf{u}\equiv A\mathbf{u}$, $A_1B\mathbf{u}\equiv A_1\mathbf{u}$ and $B^*[\mathbf{P},F]B\mathbf{u}\equiv[\mathbf{P},F]\mathbf{u}$, modulo $\mathcal{O}(h^\infty)_{C^\infty}$. Also, the first part of (2.15) shows that $A_1\mathbf{u}=\mathcal{O}(h^\infty)_{L^2}$. Using (2.19) and (2.20) we obtain (2.16), finishing the proof. □

2.4. **Contact flows and geodesic flows.** Assume that M is a compact three-dimensional manifold and $\alpha\in C^\infty(M;\Omega^1)$ is a contact form, that is

$$d\operatorname{vol}_M:=\alpha\wedge d\alpha\neq 0\quad\text{everywhere.}$$

Then $d\operatorname{vol}_M$ fixes a volume form and an orientation on M. The form α determines uniquely the *Reeb vector field* X on M satisfying the conditions (with ι denoting the interior product)

$$\iota_X\alpha=1\,,\quad \iota_X(d\alpha)=0. \tag{2.21}$$

We record for future use the following identity which can be checked by applying both sides to a frame containing X:

$$\mathbf{u}\wedge d\alpha=(\iota_X\mathbf{u})\,d\operatorname{vol}_M\quad\text{for all }\mathbf{u}\in\mathcal{D}'(M;\Omega^1). \tag{2.22}$$

We now consider the special case when M is the unit cotangent bundle of a compact Riemannian surface (Σ,g):

$$M=S^*\Sigma=\{(x,\xi)\in T^*\Sigma\colon |\xi|_g=1\}. \tag{2.23}$$

Let $j:S^*\Sigma\hookrightarrow T^*\Sigma$ and put $\alpha:=j^*(\xi dx)$. Then α is a contact form and the corresponding vector field X is the generator of the geodesic flow.

We recall a standard topological fact which will be used in passing from the Betti number of $M=S^*\Sigma$ to the Euler characteristic of Σ. It is an immediate consequence of the Gysin long exact sequence; we provide a direct proof for the reader's convenience:

Lemma 2.4. *Assume that (Σ,g) is a compact connected oriented Riemannian surface of nonzero Euler characteristic, M is given by (2.23), and $\pi:M\to\Sigma$ is the projection map. Then for any $\mathbf{u}\in C^\infty(M;\Omega^1)$ with $d\mathbf{u}=0$ there exist $\mathbf{v},\varphi$ such that*

$$\mathbf{u}=\pi^*\mathbf{v}+d\varphi,\quad \mathbf{v}\in C^\infty(\Sigma;\Omega^1),\quad d\mathbf{v}=0,\quad \varphi\in C^\infty(M). \tag{2.24}$$

In particular, $\pi^*:\mathbf{H}^1(\Sigma;\mathbb{C})\to\mathbf{H}^1(M;\mathbb{C})$ is an isomorphism.

Proof. For computations below, we will use positively oriented local coordinates (x_1,x_2) on Σ in which the metric has the form $g=e^{2\psi}(dx_1^2+dx_2^2)$, for some smooth real-valued function ψ. The corresponding coordinates on M are (x_1,x_2,θ) with the covector given by $\xi=e^{\psi}(\cos\theta,\sin\theta)$. Let V be the vector field on M which generates rotations in the fibers of π. In local coordinates, we have $V = \partial_\theta$. To show (2.24) it suffices to find $\varphi \in C^\infty(M)$ such that

$$V\varphi = \iota_V \mathbf{u}. \tag{2.25}$$

Indeed, put $\mathbf{w} := \mathbf{u} - d\varphi$. Then $d\mathbf{w} = 0$ and $\iota_V \mathbf{w} = 0$. A calculation in local coordinates shows that $\mathbf{w} = \pi^*\mathbf{v}$ for some $\mathbf{v} \in C^\infty(\Sigma; \Omega^1)$ such that $d\mathbf{v} = 0$.

A smooth solution to (2.25) exists if $\mathbf{u}$ integrates to 0 on each fiber of π. Since $\mathbf{u}$ is closed and all fibers are homotopic to each other, the integral of $\mathbf{u}$ along each fiber is equal to some constant $c \in \mathbb{C}$, thus it remains to show that $c = 0$.

Let $K \in C^\infty(\Sigma)$ be the Gaussian curvature of Σ and $d\,\mathrm{vol}_\Sigma \in C^\infty(\Sigma; \Omega^2)$ the volume form of (Σ, g), written in local coordinates as $d\,\mathrm{vol}_\Sigma = e^{2\psi} dx_1 \wedge dx_2$. With $\chi(\Sigma) \neq 0$ denoting the Euler characteristic of Σ, we have by Gauss–Bonnet theorem

$$\int_M \mathbf{u} \wedge \pi^*(K\, d\,\mathrm{vol}_\Sigma) = 2\pi\chi(\Sigma) \cdot c.$$

It then remains to prove that $\int_M \mathbf{u} \wedge \pi^*(K\, d\,\mathrm{vol}_\Sigma) = 0$. This follows via integration by parts from the identity $\pi^*(K\, d\,\mathrm{vol}_\Sigma) = -dV^*$, where $V^* \in C^\infty(M; \Omega^1)$ is the connection form, namely the unique 1-form satisfying the relations

$$\iota_V V^* = 1, \quad d\alpha = V^* \wedge \beta, \quad d\beta = -V^* \wedge \alpha,$$

where α is the contact form and β is the pullback of α by the $\pi/2$ rotation in the fibers of π. This can be checked in local coordinates using the formulas $\alpha = e^\psi(\cos\theta\, dx_1 + \sin\theta\, dx_2)$, $\beta = e^\psi(-\sin\theta\, dx_1 + \cos\theta\, dx_2)$, $V^* = \partial_{x_1}\psi\, dx_2 - \partial_{x_2}\psi\, dx_1 + d\theta$, $K = -e^{-2\psi}\Delta\psi$; see also [GuKa, §3].

Having established (2.24), we see immediately that $\pi^* : \mathbf{H}^1(\Sigma; \mathbb{C}) \to \mathbf{H}^1(M; \mathbb{C})$ is onto. To show that π^* is one-to-one, assume that $\mathbf{v} \in C^\infty(\Sigma; \Omega^1)$ satisfies $\pi^*\mathbf{v} = d\varphi$ for some $\varphi \in C^\infty(M)$. Then $V\varphi = \iota_V d\varphi = 0$, therefore $\varphi = \pi^*\chi$ for some $\chi \in C^\infty(\Sigma)$ and $\mathbf{v} = d\chi$ is exact. □

3. Proof

In this section we prove the main theorem in a slightly more general setting – see Proposition 3.1. We assume throughout that M is a three-dimensional connected compact manifold, α is a contact form on M, and X is the Reeb vector field of α generating an Anosov flow (see §§2.3,2.4). For the application to zeta functions we also assume that the corresponding stable/unstable bundles E_u, E_s are orientable.

3.1. Zeta function and Pollicott–Ruelle resonances.

For $k = 0, 1, 2$, let $\Omega_0^k \subset \Omega^k$ be the bundle of exterior k-forms $\mathbf{u}$ on M such that $\iota_X \mathbf{u} = 0$. Consider the following operator satisfying (2.7):

$$\mathbf{P}_k := -i\mathcal{L}_X : \mathcal{D}'(M; \Omega_0^k) \to \mathcal{D}'(M; \Omega_0^k).$$

Note that by Cartan's formula

$$\mathbf{P}_k\mathbf{u} = -i\,\iota_X(d\mathbf{u}), \quad \mathbf{u} \in \mathcal{D}'(M;\Omega_0^k).$$

As discussed in §2.3 we may consider Pollicott–Ruelle resonances associated to the operators $\mathbf{P}_k$, denoting their multiplicities as follows:

$$m_k(\lambda) := m_{\mathbf{P}_k}(\lambda) \in \mathbb{N}_0, \quad \lambda \in \mathbb{C}.$$

The connection with the Ruelle zeta function comes from the following standard formula (see [DyZw1, (2.5) and §4]) for the meromorphic continuation of ζ_R:

$$\zeta_R(s) = \frac{\zeta_1(s)}{\zeta_0(s)\zeta_2(s)}, \quad s \in \mathbb{C}.$$

(It is here that we the assumption that the stable and unstable bundle are orientable.) Here each $\zeta_k(s)$ is an entire function having a zero of multiplicity $m_k(is)$ at each $s \in \mathbb{C}$. Therefore, $\zeta_R(s)$ has a zero at $s = 0$ of multiplicity

$$m_R(0) := m_1(0) - m_0(0) - m_2(0). \tag{3.1}$$

By Lemma 2.2 the multiplicity $m_k(0)$ can be calculated as

$$m_k(0) = \dim \operatorname{Res}_k(0), \tag{3.2}$$

where $\operatorname{Res}_k(0)$ is the space of resonant states at zero,

$$\operatorname{Res}_k(0) = \{\mathbf{u} \in \mathcal{D}'_{E_u^*}(M;\Omega^k) \colon \iota_X\mathbf{u} = 0,\ \iota_X(d\mathbf{u}) = 0\} \tag{3.3}$$

provided that the semisimplicity condition (2.11) is satisfied:

$$\mathbf{u} \in \mathcal{D}'_{E_u^*}(M;\Omega^k), \quad \iota_X\mathbf{u} = 0, \quad \iota_X(d\mathbf{u}) \in \operatorname{Res}_k(0) \quad \Longrightarrow \quad \iota_X(d\mathbf{u}) = 0. \tag{3.4}$$

The main result of this section is

Proposition 3.1. *In the notation of* (3.3) *we have*

(1) $\dim \operatorname{Res}_0(0) = \dim \operatorname{Res}_2(0) = 1$;
(2) $\dim \operatorname{Res}_1(0)$ *is equal to the Betti number* $\mathbf{b}_1(M)$ *defined in* (2.4)*;*
(3) *the condition* (3.4) *holds for* $k = 0, 1, 2$.

It is direct to see that Proposition 3.1 implies the main theorem when applied to the case $M = S^*\Sigma$ discussed in §2.4 (strictly speaking, to each connected component of Σ). Indeed, X generates an Anosov flow since Σ is negatively curved (see for example [Kl, Theorem 3.9.1]), the stable/unstable bundles are orientable since Σ is orientable and $m_R(0) = \mathbf{b}_1(M) - 2$ equals to $-\chi(\Sigma)$ by Lemma 2.4.

3.2. **Scalars and 2-forms.** We start the proof of Proposition 3.1 by considering the cases $k=0$ and $k=2$:

Lemma 3.2. *We have*

$$\operatorname{Res}_0(0)=\{c\colon c\in\mathbb{C}\},\quad \operatorname{Res}_2(0)=\{c\,d\alpha\colon c\in\mathbb{C}\},\tag{3.5}$$

and (3.4) *holds for $k=0,2$, that is the resonance at 0 for $k=0,2$ is simple.*

Remark. The argument for $\operatorname{Res}_0(0)$ applies to any contact Anosov flow on a compact connected manifold. It can be generalized to show that $\operatorname{Res}_0(0)$ consists of constant functions and $\operatorname{Res}_0(\lambda)$ is trivial for all $\lambda\in\mathbb{R}\setminus 0$. This in particular implies that the flow is mixing.

Proof. We first handle the case of $\operatorname{Res}_0(0)$. Clearly this space contains constant functions, therefore we need to show that

$$u\in\mathcal{D}'_{E_u^*}(M),\quad Xu=0\quad\Longrightarrow\quad u=c\quad\text{for some } c\in\mathbb{C}.\tag{3.6}$$

By Lemma 2.3 we have $u\in C^\infty(M)$. Since $Xu=0$ we have $u=u\circ e^{tX}$ and thus

$$\langle du(x),v\rangle=\langle du(e^{tX}(x)),de^{tX}(x)\cdot v\rangle\quad\text{for all } (x,v)\in TM,\ t\in\mathbb{R}.$$

Now if $v\in E_s(x)$ then taking the limit as $t\to\infty$ and using (2.6) we obtain $\langle du(x),v\rangle=0$. Similarly if $v\in E_u(x)$ then the limit $t\to-\infty$ gives $\langle du(x),v\rangle=0$. Therefore $du|_{E_u\oplus E_s}=0$. However $E_u\oplus E_s=\ker\alpha$, thus we have for some $\varphi\in C^\infty(M)$,

$$du=\varphi\alpha.$$

We have $0=\alpha\wedge d(\varphi\alpha)=\varphi\alpha\wedge d\alpha$, thus $du=0$, implying (3.6) since M is connected.

Next, (3.4) holds for $k=0$. Indeed, if $u\in\mathcal{D}'_{E_u^*}(M)$ then

$$\int_M (Xu)\,d\operatorname{vol}_M=0,$$

implying that Xu cannot be a nonzero element of $\operatorname{Res}_0(0)$.

Now, assume that $\mathbf{u}\in\mathcal{D}'_{E_u^*}(M;\Omega^2)$ satisfies $\iota_X\mathbf{u}=0$. Then $\mathbf{u}$ can be written as

$$\mathbf{u}=u\,d\alpha,\quad u\in\mathcal{D}'_{E_u^*}(M);\quad \iota_X(d\mathbf{u})=(Xu)d\alpha.$$

Therefore the case of $\operatorname{Res}_2(0)$ follows immediately from that of $\operatorname{Res}_0(0)$. □

Lemma 3.2 implies solvability of the equation $Xu=f$ in the class $\mathcal{D}'_{E_u^*}$:

Proposition 3.3. *Assume that $f\in C^\infty(M)$ and $\int_M f\,d\operatorname{vol}_M=0$. Then there exists $u\in\mathcal{D}'_{E_u^*}(M)$ such that $Xu=f$.*

Proof. It follows from Lemma 3.2 and the proof of Lemma 2.2 that the resolvent $\mathbf{R}_0(\lambda)$ of the operator $\mathbf{P}_0 = -iX$ defined in (2.9) has the expansion

$$\mathbf{R}_0(\lambda) = \mathbf{R}_H(\lambda) - \frac{\Pi}{\lambda}$$

where $\mathbf{R}_H(\lambda)$ is holomorphic at $\lambda = 0$ and the range of Π consists of constant functions. By analytic continuation from (2.8), we see that $\mathbf{R}_0(\lambda)^* = -\mathbf{R}_{-\mathbf{P}_0}(-\bar{\lambda})$ where $\mathbf{R}_{-\mathbf{P}_0}(\lambda)$ is the resolvent of $-\mathbf{P}_0$. Applying Lemma 3.2 to the field $-X$, we see that the range of Π^* also consists of constant functions. By (2.12) we have $\Pi(1) = 1$, therefore

$$\Pi u = \frac{1}{\mathrm{vol}(M)} \int_M u \, d\,\mathrm{vol}_M \, .$$

Now, put $u := -i\mathbf{R}_H(0)f$, then $u \in \mathcal{D}'_{E_u^*}(M)$ by (2.10). Since $\Pi f = 0$ and $(\mathbf{P}_0 - \lambda)\mathbf{R}_0(\lambda) = I$, we have $Xu = f$. □

3.3. **1-forms.** It remains to show Proposition 3.1 for the case $k = 1$, that is to analyse the space

$$\mathrm{Res}_1(0) = \{\mathbf{u} \in \mathcal{D}'_{E_u^*}(M, \Omega^1) \colon \iota_X \mathbf{u} = 0,\ \iota_X(d\mathbf{u}) = 0\}.$$

The next lemma shows that the $\dim \mathrm{Res}_1(0) = \mathbf{b}_1(M)$:

Lemma 3.4. *Assume that* $\mathbf{u} \in \mathrm{Res}_1(0)$. *Then there exists* $\varphi \in \mathcal{D}'_{E_u^*}(M)$ *such that*

$$\mathbf{u} - d\varphi \in C^\infty(M; \Omega^1), \quad d(\mathbf{u} - d\varphi) = 0. \tag{3.7}$$

The cohomology class $[\mathbf{u} - d\varphi] \in \mathbf{H}^1(M; \mathbb{C})$ *is independent of the choice of* φ. *The map*

$$\mathrm{Res}_1(0) \ni \mathbf{u} \ \mapsto \ [\mathbf{u} - d\varphi] \in \mathbf{H}^1(M; \mathbb{C}) \tag{3.8}$$

is a linear isomorphism.

Proof. We first show that

$$\mathbf{u} \in \mathrm{Res}_1(0) \implies d\mathbf{u} = 0. \tag{3.9}$$

Definition (3.3) shows that $d\mathbf{u} \in \mathrm{Res}_2(0)$ and therefore by Lemma 3.2 we have $d\mathbf{u} = c\, d\alpha$ for some $c \in \mathbb{C}$. From (2.22) and $\iota_X \mathbf{u} = 0$ we also have $\mathbf{u} \wedge d\alpha = 0$, thus Stokes's theorem gives (3.9):

$$c\,\mathrm{vol}(M) = \int_M \alpha \wedge d\mathbf{u} = \int_M \mathbf{u} \wedge d\alpha = 0.$$

Lemma 2.1 and (3.9) imply the existence of $\varphi \in \mathcal{D}'_{E_u^*}(M)$ such that (3.7) holds. Moreover, if $\tilde{\varphi} \in \mathcal{D}'_{E_u^*}(M)$ is another function satisfying (3.7) then $d(\varphi - \tilde{\varphi}) \in C^\infty(M; \Omega^1)$, implying by Lemma 2.1 that $\varphi - \tilde{\varphi} \in C^\infty(M)$. Therefore $\mathbf{u} - d\varphi$ and $\mathbf{u} - d\tilde{\varphi}$ belong to the same de Rham cohomology class, thus the map (3.8) is well-defined.

Next, assume that (3.7) holds and $\mathbf{u} - d\varphi$ is exact. By changing φ we may assume that $\mathbf{u} = d\varphi$. Since $\iota_X \mathbf{u} = 0$ we have $X\varphi = 0$, which by Lemma 3.2 implies that φ is constant and thus $\mathbf{u} = 0$. This shows that (3.8) is injective.

It remains to show that (3.8) is surjective. For that, take a closed $\mathbf{v} \in C^\infty(M; \Omega^1)$. We need to find $\varphi \in \mathcal{D}'_{E_u^*}(M)$ such that $\mathbf{v} + d\varphi \in \operatorname{Res}_1(0)$. This is equivalent to $\iota_X(\mathbf{v} + d\varphi) = 0$, that is $X\varphi = -\iota_X \mathbf{v}$. A solution φ to the above equation exists by Lemma 3.3 since (2.22) implies

$$\int_M \iota_X \mathbf{v}\, d\operatorname{vol}_M = \int_M \mathbf{v} \wedge d\alpha = \int_M \alpha \wedge d\mathbf{v} = 0.$$

This finishes the proof. □

To prove Proposition 3.1 it remains to show the semisimplicity condition:

Lemma 3.5. *Suppose that*

$$\mathbf{u} \in \mathcal{D}'_{E_u^*}(M; \Omega^1), \quad \iota_X \mathbf{u} = 0, \quad \iota_X(d\mathbf{u}) = \mathbf{v} \in \operatorname{Res}_1(0).$$

Then $\mathbf{v} = 0$, that is, condition (3.4) holds for $k = 1$.

Proof. We have $\alpha \wedge d\mathbf{u} = a\, d\operatorname{vol}_M$ for some $a \in \mathcal{D}'_{E_u^*}(M)$. By (2.22),

$$\int_M a\, d\operatorname{vol}_M = \int_M \mathbf{u} \wedge d\alpha = \int_M \iota_X \mathbf{u}\, d\operatorname{vol}_M = 0.$$

Moreover since $\mathcal{L}_X(\alpha) = 0$, $\mathcal{L}_X(d\alpha) = 0$, and $d\mathbf{v} = 0$ by (3.9), we have

$$(Xa)\, d\operatorname{vol}_M = \mathcal{L}_X(\alpha \wedge d\mathbf{u}) = \alpha \wedge d\mathbf{v} = 0.$$

Thus $Xa = 0$ and Lemma 3.2 gives that $a = 0$ and thus $\alpha \wedge d\mathbf{u} = 0$. This implies $d\mathbf{u} = \alpha \wedge \iota_X d\mathbf{u} = \alpha \wedge \mathbf{v}$. Now by Lemma 3.4 there exist

$$\mathbf{w} \in C^\infty(M; \Omega^1), \quad \varphi \in \mathcal{D}'_{E_u^*}(M), \quad \mathbf{v} = \mathbf{w} + d\varphi, \quad d\mathbf{w} = 0.$$

Since $\iota_X \mathbf{v} = 0$ we have $X\varphi = -\iota_X \mathbf{w}$. Integration by parts together with (2.22) gives

$$\begin{aligned} 0 = \operatorname{Re} \int_M d\mathbf{u} \wedge \overline{\mathbf{w}} &= \operatorname{Re} \int_M \alpha \wedge d\varphi \wedge \overline{\mathbf{w}} \\ &= \operatorname{Re} \int_M \varphi\, \overline{\mathbf{w}} \wedge d\alpha = -\operatorname{Re} \langle X\varphi, \varphi \rangle_{L^2}. \end{aligned} \tag{3.10}$$

By Lemma 2.3 with $\mathbf{P} = -iX$ this implies $\varphi \in C^\infty(M)$ and thus $\mathbf{v} \in C^\infty(M; \Omega^1)$.

We can now use the same argument as in the proof of Lemma 3.2: $(e^{tX})^* \mathbf{v} = \mathbf{v}$ and hence

$$\langle \mathbf{v}(x), z \rangle = \langle \mathbf{v}(e^{tX} x), de^{tX}(x) \cdot z \rangle, \quad (x, z) \in TM, \quad t \in \mathbb{R}.$$

The right hand side tends to zero as $t \to \infty$ for $z \in E_s(x)$, and as $t \to -\infty$ for $z \in E_u(x)$. Since $\iota_X \mathbf{v} = 0$ it follows that $\mathbf{v} = 0$. □

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