RESONANCES FOR ASYMPTOTICALLY HYPERBOLIC MANIFOLDS: VASY'S METHOD REVISITED

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Dedicated to the memory of Yuri Safarov

Abstract. We revisit Vasy's method [Va1],[Va2] for showing meromorphy of the resolvent for (even) asymptotically hyperbolic manifolds. It provides an effective definition of resonances in that setting by identifying them with poles of inverses of a family of Fredholm differential operators. In the Euclidean case the method of complex scaling made this available since the 70's but in the hyperbolic case an effective definition was not known till [Va1],[Va2]. Here we present a simplified version which relies only on standard pseudodifferential techniques and estimates for hyperbolic operators. As a byproduct we obtain more natural invertibility properties of the Fredholm family.

1. Introduction

We present a version of the method introduced by András Vasy [Va1],[Va2] to prove meromorphic continuations of resolvents of Laplacians on even asymptotically hyperbolic spaces – see (1.2). That meromorphy was first established for any asymptotically hyperbolic metric by Mazzeo–Melrose [MazMe]. Other early contributions were made by Agmon [Ag], Fay [Fa], Guillopé–Zworski [GuZw], Lax–Phillips [LaPh], Mandouvalos [Man], Patterson [Pa] and Perry [Pe]. Guillarmou [Gu] showed that the evenness condition was needed for a global meromorphic continuation and clarified the construction given in [MazMe].

Vasy's method is dramatically different from earlier approaches and is related to the study of stationary wave equations for Kerr–de Sitter black holes – see [Va1] and [DyZw2, §5.7]. Its advantage lies in relating the resolvent to the inverse of a family of *Fredholm differential operators*. Hence, microlocal methods can be used to prove results which have not been available before, for instance existence of resonance free strips for non-trapping metrics [Va2]. Another application is the work of Datchev–Dyatlov [DaDy] on the fractal upper bounds on the number of resonances for (even) asymptotically hyperbolic manifolds and in particular for convex co-compact quotients of $\mathbb{H}^n$. Previously only the case of convex co-compact Schottky quotients was known [GuLiZw] and that was established using transfer operators and zeta function methods. In the context of black holes the construction has been used to obtain a quantitative version of Hawking radiation [Dr], exponential decay of waves in the Kerr–de Sitter case [Dy1], the description of quasi-normal modes for perturbations of Kerr–de Sitter black holes [Dy2] and rigorous definition of quasi-normal modes for Kerr–Anti de Sitter black holes [Ga]. The construction of the Fredholm family also plays a role in the study of linear and non-linear scattering problems – see [BaVaWu], [HiVa1], [HiVa2] and references given there.

A related approach to meromorphic continuation, motivated by the study of Anti-de Sitter black holes, was independently developed by Warnick [Wa]. It is based on physical space techniques for hyperbolic equations and it also provides meromorphic continuation of resolvents for even asymptotically hyperbolic metrics [Wa, §7.5].

We should point out that for a large class of asymptotically Euclidean manifolds an effective characterization of resonances has been known since the introduction of the method of complex scaling by Aguilar–Combes, Balslev–Combes and Simon in the 1970s – see [DyZw2, §4.5] for an elementary introduction and references and [WuZw] for a class asymptotically Euclidean manifolds to which the method applies.

In this note we present a direct proof of meromorphic continuation based on standard pseudodifferential techniques and estimates for hyperbolic equations which can found, for instance, in [H3, §18.1] and [H3, §23.2] respectively. In particular, we prove Melrose's *radial estimates* [Me] which are crucial for establishing the Fredholm property. A semiclassical version of the approach presented here can be found in [DyZw2, Chapter 5] – it is needed for the high energy results [DaDy], [Va2] mentioned above.

We now define even asymptotically hyperbolic manifolds. Suppose that $\overline{M}$ is a compact manifold with boundary $\partial M \neq \emptyset$ of dimension $n+1$. We denote by M the interior of $\overline{M}$. The Riemannian manifold (M, g) is even asymptotically hyperbolic if there exist functions $y' \in \bar{\mathcal{C}}^\infty(M; \partial M)$ and $y_1 \in \bar{\mathcal{C}}^\infty(M; (0,2))$[†], $y_1|_{\partial M} = 0$, $dy_1|_{\partial M} \neq 0$, such that

$$\overline{M} \supset y_1^{-1}([0,1]) \ni m \mapsto (y_1(m), y'(m)) \in [0,1] \times \partial M \tag{1.1}$$

is a diffeomorphism, and near ∂M the metric has the form,

$$g|_{y_1 \leq 1} = \frac{dy_1^2 + h(y_1^2)}{y_1^2}, \tag{1.2}$$

where $[0,1] \ni t \mapsto h(t)$, is a smooth family of Riemannian metrics on ∂M. For the discussion of invariance of this definition and of its geometric meaning we refer to [Gu, §2].

Let $-\Delta_g \geq 0$ be the Laplace–Beltrami operator for the metric g. Since the spectrum is contained in $[0, \infty)$ the operator $-\Delta_g - \zeta(n-\zeta)$ is invertible on $H^2(M, d\,\mathrm{vol}_g)$ for

[†]We cannot write a paper about Vasy's method without some footnotes: we follow the notation of [H3, Appendix B] where $\bar{\mathcal{C}}^\infty(M)$ denotes functions which are smoothly extendable across ∂M and $\dot{\mathcal{C}}^\infty(\overline{M})$ functions which are extendable to smooth functions *supported* in $\overline{M}$ – see §3.

$\operatorname{Re}\zeta > n$. Hence we can define

$$R(\zeta) := (-\Delta_g - \zeta(n-\zeta))^{-1} : L^2(M, d\mathrm{vol}_g) \to H^2(M, d\mathrm{vol}_g), \quad \operatorname{Re}\zeta > n. \tag{1.3}$$

We note that elliptic regularity shows that $R(\zeta) : \dot{\mathcal{C}}^\infty(M) \to \mathcal{C}^\infty(M)$, $\operatorname{Re}\zeta > n$. We also remark that as a byproduct of the construction we will show the well known fact that $R(\lambda) : L^2 \to H^2$ is meromorphic for $\operatorname{Re}\zeta > n/2$: the poles correspond to L^2 eigenvalues of $-\Delta_g$ and hence lie in $(n/2, n)$.

We will prove the result of Mazzeo–Melrose [MazMe] and Guillarmou [Gu]:

Theorem 1. *Suppose that (M, g) is an even asymptotically hyperbolic manifold and that $R(\zeta)$ is defined by (1.3). Then*

$$R(\zeta) : \dot{\mathcal{C}}^\infty(M) \to \mathcal{C}^\infty(M),$$

continues meromorphically from $\operatorname{Re}\zeta > n$ to $\mathbb{C}$ with poles of finite rank.

The key point however is the fact that $R(\zeta)$ can be related to $P(i(\zeta - n/2))^{-1}$ where

$$\zeta \longmapsto P(i(\zeta - n/2))$$

is a family of Fredholm differential operators – see §2 and Theorem 2. That family will be shown to be invertible for $\operatorname{Re}\zeta > n$ which proves the meromorphy of $P(i(\zeta - n/2))^{-1}$ – see Theorem 3. We remark that for $\operatorname{Re}\zeta > \frac{n}{2}$, $R(\zeta)$ is meromorphic as an operator $L^2(M) \to L^2(M)$ with poles corresponding to eigenvalues of $-\Delta_g$.

The paper is organized as follows. In §2 we define the family $P(\lambda)$ and the spaces on which it has the Fredholm property. That section contains the main results of the paper: Theorems 2 and 3. In §3 we recall the notation from the theory of pseudodifferential operators and provide detailed references. We also recall estimates for hyperbolic operators needed here. In §4 we prove Melrose's propagation estimates at radial points and in §5 we use them to show the Fredholm property. §6 gives some precise estimates valid for $\operatorname{Im}\lambda \gg 1$. Finally §7 we present invertibility of $P(\lambda)$ for $\operatorname{Im}\lambda \gg 1$ and that proves the meromorphic continuation. Except for references to [H3, 18.1] and [H3, 23.2] and some references to standard approximation arguments [DyZw2, Appendix E] (with material readily available in many other places) the paper is self-contained.

Acknowledgements. I would like to thank Semyon Dyatlov and András Vasy for helpful comments on the first version of this note. I am particularly grateful to Peter Hintz for many suggestions and for his help with the proof of Proposition 8 and to the anonymous referee whose careful reading lead to many improvements. Partial support by the National Science Foundation under the grant DMS-1500852 is also gratefully acknowledged.

2. The Fredholm family of differential operators

Let $y' \in \partial M$ denote the variable on ∂M. Then (1.2) implies that near ∂M, the Laplacian has the form

$$\begin{aligned} -\Delta_g &= (y_1 D_{y_1})^2 + i(n + y_1^2\gamma(y_1^2, y'))y_1 D_{y_1} - y_1^2 \Delta_{h(y_1^2)}, \\ \gamma(t,y') &:= -\partial_t \bar h(t)/\bar h(t), \quad \bar h(t) := \det h(t), \quad D := \tfrac{1}{i}\partial. \end{aligned} \tag{2.1}$$

Here $\Delta_{h(y_1^2)}$ is the Laplacian for the family of metrics on ∂M depending smoothly on y_1^2 and $\gamma \in \mathcal{C}^\infty([0,1]\times\partial M)$. (The logarithmic derivative defining γ is independent of of the density on ∂M needed to define the determinant $\bar h$.)

In §6 we will show that the unique L^2 solutions to

$$(-\Delta_g - \zeta(n-\zeta))u = f \in \dot{\mathcal{C}}^\infty(M), \quad \operatorname{Re}\zeta > n,,$$

satisfy

$$u = y_1^\zeta \bar{\mathcal{C}}^\infty(M) \quad \text{ and } \quad y_1^{-\zeta} u|_{y_1<1} = F(y_1^2, y'), \quad F \in \bar{\mathcal{C}}^\infty([0,1]\times\partial M).$$

Eventually we will show that the meromorphic continuation of the resolvent provides solutions of this form for all $\zeta \in \mathbb{C}$ that are not poles of the resolvent.

This suggests two things:

- To reduce the investigation to the study of smooth solutions we should conjugate $-\Delta_g - \zeta(n-\zeta)$ by the weight y_1^ζ.
- The desired smoothness properties should be stronger in the sense that the functions should be smooth in (y_1^2, y').

Motivated by this we calculate,

$$y_1^{-\zeta}(-\Delta_g - \zeta(n-\zeta))y_1^\zeta = x_1 P(\lambda), \quad x_1 = y_1^2, \quad x' = y', \ \lambda = i(\zeta - \tfrac{n}{2}), \tag{2.2}$$

where, near ∂M,

$$P(\lambda) = 4(x_1 D_{x_1}^2 - (\lambda + i)D_{x_1}) - \Delta_h + i\gamma(x)\left(2x_1 D_{x_1} - \lambda - i\tfrac{n-1}{2}\right). \tag{2.3}$$

The switch to λ is motivated by the fact that numerology is slightly lighter on the ζ-side for $-\Delta_g$ and on the λ-side for $P(\lambda)$.

To define the operator $P(\lambda)$ geometrically we introduce a new manifold using coordinates (1.1) and $x_1 = y_1^2$ for $y_1 > 0$:

$$X = [-1,1]_{x_1} \times \partial M \sqcup \left(M \setminus y^{-1}((0,1))\right). \tag{2.4}$$

We note that $X_1 := X \cap \{x_1 > 0\}$ is diffeomorphic to M but $\overline{X}_1$ and $\overline{M}$ have different $\mathcal{C}^\infty$-structures[‡].

[‡]This construction appeared already in [GuZw, §2] and $P(\lambda) = Q(n/4 - i\lambda/2)$ where $Q(\zeta)$ was defined in [GuZw, (2.6),(3.12)]. However the significance of $Q(\zeta)$ did not become clear until [Va1].

We can extend $x_1 \to h(x_1)$ to a family of smooth non-degenerate metrics on ∂M on $[-1, 1]$. Using (2.1) that provides a natural extension of the function γ appearing (2.2).

The Laplacian $-\Delta_g$ is a self-adjoint operator on $L^2(M, d\,\mathrm{vol}_g)$, where near ∂M and in the notation of (2.1),

$$d\,\mathrm{vol}_g = y_1^{-n-1}\bar{h}(y_1^2, y')dy_1 dy',$$

where dy' in a density on ∂M used to define the determinant $\bar{h} = \det h$. The conjugation (2.2) shows that for $\lambda \in \mathbb{R}$ ($\zeta \in \frac{n}{2} + i\mathbb{R}$) $x_1 P(\lambda)$ is formally self-adjoint with respect to $x_1^{-1}\bar{h}(x)dx_1 dx'$ and consequently $P(\lambda)$ is formally self-adjoint for

$$d\mu_g = \bar{h}(x)dx. \tag{2.5}$$

This will be the measure used for defining $L^2(X)$ in what follows. In particular we see that the formal adjoint with respect to $d\mu_g$ satisfies

$$P(\lambda)^* = P(\bar{\lambda}). \tag{2.6}$$

We can now define spaces on which $P(\lambda)$ is a Fredholm operator. For that we denote by $\bar{H}^s(X^\circ)$ the space of restrictions of elements of H^s on an extension of X across the boundary to the interior of X – see [H3, §B.2] and §3.2 – and put

$$\mathscr{Y}_s := \bar{H}^s(X^\circ), \quad \mathscr{X}_s := \{u \in \mathscr{Y}_{s+1} : P(0)u \in \mathscr{Y}_s\}. \tag{2.7}$$

Since the dependence on λ in $P(\lambda)$ occurs only in lower order terms we can replace $P(0)$ by $P(\lambda)$ in the definition of $\mathscr{X}$.

Motivation: Since for $x_1 < 0$ the operator $P(\lambda)$ is hyperbolic with respect to the surfaces $x_1 = a < 0$ the following elementary example motivates the definition (2.7). Consider $P = D_{x_1}^2 - D_{x_2}^2$ on $[-1, 0] \times \mathbb{S}^1$ and define

$$Y_s := \{u \in \bar{H}^s([-1, \infty) \times \mathbb{S}^1) : \mathrm{supp}\, u \subset [-1, 0] \times \mathbb{S}^1\}, \quad X_s := \{u \in Y_{s+1} : Pu \in Y_s\}.$$

Then standard hyperbolic estimates – see for instance [H3, Theorem 23.2.4] – show that for any $s \in \mathbb{R}$, the operator $P : X_s \to Y_s$ is invertible. Roughly, the support condition gives 0 initial values at $x_1 = 0$ and hence $Pu = f$ can be uniquely solved for $x_1 < 0$.

We can now state the main theorems of this note:

Theorem 2. *Let $\mathscr{X}_s, \mathscr{Y}_s$ be defined in (2.7). Then for $\mathrm{Im}\,\lambda > -s - \frac{1}{2}$ the operator*

$$P(\lambda) : \mathscr{X}_s \to \mathscr{Y}_s,$$

has the Fredholm property, that is

$$\dim\{u \in \mathscr{X}_s : P(\lambda)u = 0\} < \infty, \quad \dim \mathscr{Y}_s / P(\lambda)\mathscr{X}_s < \infty,$$

and $P(\lambda)\mathscr{X}_s$ is closed.

The next theorem provides invertibility of $P(\lambda)$ for $\operatorname{Im}\lambda > 0$ and that shows the meromorphy of $P(\lambda)^{-1}$ – see [DyZw2, Theorem C.4]. We will use that in Proposition 8 to show the well known fact that in addition to Theorem 1 $R(\frac{n}{2} - i\lambda)$ is meromorphic on $L^2(M, d\mathrm{vol}_g)$ for $\operatorname{Im}\lambda > 0$.

Theorem 3. *For* $\operatorname{Im}\lambda > 0$, $\lambda^2 + (\frac{n}{2})^2 \notin \operatorname{Spec}(-\Delta_g)$ *and* $s > -\operatorname{Im}\lambda - \frac{1}{2}$,

$$P(\lambda) : \mathscr{X}_s \to \mathscr{Y}_s$$

is invertible. Hence, for $s \in \mathbb{R}$ *and* $\operatorname{Im}\lambda > -s - \frac{1}{2}$, $\lambda \mapsto P(\lambda)^{-1} : \mathscr{Y}_s \to \mathscr{X}_s$, *is a meromorphic family of operators with poles of finite rank.*

For interesting applications it is crucial to consider the semiclassical case, that is, uniform analysis as $\operatorname{Re}\lambda \to \infty$ – see [DyZw2, Chapter 5] – but to indicate the basic mechanism behind the meromorphic continuation we only present the Fredholm property and invertibility in the upper half-plane.

3. Preliminaries

Here we review the notation and basic facts need in the proofs of Theorems 2 and 3.

3.1. **Pseudodifferential operators.** We use the notation of [H3, §18.1] and for X, an open $\mathcal{C}^\infty$-manifold we denote by $\Psi^m(X)$ the space of *properly supported* pseudodifferential operators of order m. (The operator $A : \mathcal{C}^\infty_c(X) \to \mathcal{D}'(X)$ is properly supported if the projections from support of the Schwartz kernel of A in $X \times X$ to each factor are proper maps, that is inverse images of compact sets are compact. The support of the Schwartz kernel of any differential operator is contained in the diagonal in $X \times X$ and clearly has that property.)

For $A \in \Psi^m(X)$ we denote by $\sigma(A) \in S^m(T^*X \setminus 0)/S^{m-1}(T^*X \setminus 0)$ the symbol of A, sometimes writing $\sigma(A) = a \in S^m(T^*X \setminus 0)$ with an understanding that a is a *representative* from the equivalence class in the quotient.

We will use the following basic properties of the symbol map: if $A \in \Psi^m(X)$ and $B \in \Psi^k(X)$ then

$$\sigma(AB) = \sigma(A)\sigma(B) \in S^{m+k}/S^{m+k-1},$$
$$\sigma(i[A, B]) = H_{\sigma(A)}\sigma(B) \in S^{m+k-1}/S^{m+k-2},$$

where for $a \in S^m$, H_a is the *Hamiton vector field* of a.

For any operator $P \in \Psi^m(X)$ we can define $\mathrm{WF}(P) \subset T^*X \setminus 0$ (the smallest subset outside of which A has order $-\infty$ – see [H3, (18.1.34)]). We also define $\operatorname{Char}(P)$ the smallest conic closed set outside of which P is *elliptic* – see [H3, Definition 18.1.25]. A typical application of the symbolic calculus and of this notation is the following statement [H3, Theorem 18.1.24′]: if $P \in \Psi^m(X)$ and V is an open conic set such that $\overline{V} \cap \mathrm{Char}(P) = \emptyset$ then there exists $Q \in \Psi^{-m}(X)$ such that

$$\mathrm{WF}(I - PQ) \cap V = \mathrm{WF}(I - QP) \cap V = \emptyset. \tag{3.1}$$

This means that Q is a *microlocal* inverse of P in V.

We also recall that the operators in $A \in \Psi^m(X)$ have mapping properties

$$A : H^s_{\mathrm{loc}}(X) \to H^{s-m}_{\mathrm{loc}}(X), \quad A : H^s_{\mathrm{comp}}(X) \to H^{s-m}_{\mathrm{comp}}(X), \quad s \in \mathbb{R}.$$

Combined with (3.1) we obtain the following *elliptic* estimate: if $A, B \in \Psi^0(X)$ have *compactly supported* Schwartz kernels, $P \in \Psi^m(X)$ and

$$\mathrm{WF}(A) \cap (\mathrm{Char}(B) \cup \mathrm{Char}(P)) = \emptyset,$$

then for any N there exists C such that

$$\|Au\|_{H^{s+m}} \leq C\|BPu\|_{H^s} + C\|u\|_{H^{-N}}. \tag{3.2}$$

3.2. **Hyperbolic estimates.** If X is a smooth compact manifold with boundary we follow [H3, §B.2] and define Sobolev spaces of extendible distributions, $\bar{H}^s(X^\circ)$ and of supported distributions $\dot{H}^s(X)$. Here $X = X^\circ \sqcup \partial X$ and X° is the interior of X. These are modeled on the case of $X = \overline{\mathbb{R}}^n_+$, $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_1 > 0\}$ in which case

$$\bar{H}^s(\mathbb{R}^n_+) = \{u : \exists U \in H^s(\mathbb{R}^n),\ u = U|_{x_1>0}\},$$
$$\dot{H}^s(\overline{\mathbb{R}}^n_+) := \{u \in H^s(\mathbb{R}^n) : \mathrm{supp}\, u \subset \overline{\mathbb{R}}^n_+\}.$$

The key fact is that the L^2 pairing (defined using a smooth density on X)

$$\dot{\mathcal{C}}^\infty(X) \times \bar{\mathcal{C}}^\infty(X^\circ) \ni (u, v) \mapsto \int_X u(x)\bar{v}(x)dx,$$

extends by density to $(u, v) \in \dot{H}^{-s}(X) \times \bar{H}(X^\circ)$ and provides the identification of dual spaces,

$$(\bar{H}^s(X^\circ))^* \simeq \dot{H}^{-s}(X), \quad s \in \mathbb{R}. \tag{3.3}$$

Suppose that $P = D_t^2 + P_1(t, x, D_x)D_t + P_0(t, x, D_x)$, $x \in N$, where N is a compact manifold and $P_j \in \mathcal{C}^\infty(\mathbb{R}_t; \Psi^{2-j}(N))$, is strictly hyperbolic with respect to the level surfaces $t = \mathrm{const}$ – see [H3, §23.2]. For any $T > 0$ and $s \in \mathbb{R}$, we define

$$\widetilde{H}^s([0,T) \times N) = \left\{u : u = U|_{[0,T)\times N}, \quad U \in H^s(\mathbb{R} \times N),\ \mathrm{supp}\, U \subset [0, \infty) \times N\right\},$$

with the norm defined as infimum of H^s norms over all $U \in H^s$ with $u_{[0,T)} = U$. (These spaces combines the $\dot{H}^s$ space at the $t = 0$ with $\bar{H}^s$ at $t = T$.)

Then

$$\forall\, f \in \widetilde{H}^s([0,T) \times N)\ \ \exists!\, u \in \widetilde{H}^{s+1}([0,T) \times N), \quad Pu = f, \tag{3.4}$$

and

$$\|u\|_{\widetilde{H}^{s+1}([0,T)\times N)} \leq C\|f\|_{\widetilde{H}^s([0,T)\times N)}, \tag{3.5}$$

see [H3, Theorem 23.2.4].

If we define

$$Y_s := \widetilde{H}^s([0,T)\times N),\quad X_s := \{u\in Y_{s+1} : Pu \in Y_s\}$$

then $P : X_s \to Y_s$ is *invertible*. In our application we will need the following estimate which follows from the invertibility of P: if $u \in \bar{H}^s((0,T)\times N)$ then for any $\delta > 0$,

$$\|u\|_{\bar{H}^{s+1}((0,T)\times N)} \leq C\|Pu\|_{\bar{H}^s((0,T)\times N)} + C\|u\|_{\bar{H}^{s+1}((0,\delta))\times N)}. \tag{3.6}$$

The operator $P(\lambda)$ defined in (2.3) is of the form $x_1(D_{x_1}^2 - P_1(x)D_{x_1} + P_0(x, D_{x'}))$ where $P_1 \in \mathcal{C}^\infty$ and P_0 is elliptic with a negative principal symbol for $-1 \leq x_1 < -\varepsilon < 0$, for any fixed ε. That means that for $t = 1 + x_1$ and $T = 1 - \varepsilon$ or $t = -\varepsilon - x_1$, $T = 1 - \varepsilon$, the operator is (up to the non-zero smooth factor x_1) is of the form to which estimates (3.5) and (3.6) apply.

We will also need an estimate valid all the way to $x_1 = 0$:

Lemma 1. *Suppose that* $u \in \dot{\mathcal{C}}^\infty(X \cap \{x_1 \leq 0\})$ *and* $P(\lambda)u = 0$. *Then* $u \equiv 0$.

As pointed out by András Vasy this follows from general properties of the de Sitter wave equation [Va3, Proposition 5.3] but we provide a simple direct proof.

Proof. We note that if $u|_{x_1 \geq -\varepsilon} = 0$ for some $\varepsilon > 0$ then $u \equiv 0$ by (3.5). That follows from energy estimates. We want to make that argument quantitative. We will work in $[-1,-\varepsilon]\times\partial M$ and define $d : \mathcal{C}^\infty(\partial M) \to \mathcal{C}^\infty(\partial M; T^*\partial M)$ to be the differential. We denote by d^* its Hodge adjoint with with respect to the (x_1-dependent) metrics h, $d_h^* : \mathcal{C}^\infty(\partial M; T^*\partial M) \to \mathcal{C}^\infty(\partial M)$. Then

$$P(\lambda) = 4x_1 D_{x_1}^2 + d_h^* d - 4(\lambda + i)D_{x_1} - i\gamma(x)(2x_1 D_{x_1} - \lambda - i\tfrac{n-1}{2}).$$

Since for $f \in \mathcal{C}^\infty(\partial M)$ and any fixed x_1, $h = h(x_1)$,

$$\begin{aligned}\int_{\partial M} d_h^*(v du)\bar{f}\, d\mathrm{vol}_h &= \int_{\partial M} \langle v du, df\rangle_h\, d\mathrm{vol}_h = \int_{\partial M} \left(\langle du, d(\bar{v}f)\rangle_h - \langle du, d\bar{v}\rangle_h \bar{f}\,\right) d\mathrm{vol}_h \\ &= \int_{\partial M} \left(v d_h^* du - \langle du, d\bar{v}\rangle\right)\bar{f}\, d\mathrm{vol}_h,\end{aligned}$$

we conclude that $d_h^*(v du) = v d_h^* du - \langle du, d\bar{v}\rangle_h$. From this we derive the following form of the energy identity valid for $x_1 < 0$:

$$\begin{aligned}&\partial_{x_1}\left(|x_1|^{-N}(-x_1|\partial_{x_1}u|^2 + |du|_h^2 + |u|^2)\right) + |x_1|^{-N} d_h^*\left(\mathrm{Re}(\bar{u}_{x_1} du)\right) = \\ &\quad 2\,\mathrm{Re}\,|x_1|^{-N}\bar{u}_{x_1}P(\lambda)u + N|x_1|^{-N-1}\left(-x_1|u_{x_1}|^2 + |du|_h^2 + |u|^2\right) + |x_1|^{-N}R(\lambda, u),\end{aligned}$$

where $R(\lambda, u)$ is a quadratic form in u and du, independent of N. We now fix $\delta > 0$ and apply Stokes's theorem in $[-\delta, -\varepsilon]\times M$. For N large enough (depending on λ)

that gives

$$\begin{aligned}\int_{\partial M} (|u_{x_1}|^2 + |du|_h^2)|_{x_1=-\delta} \, d\,\mathrm{vol}_h &\leq C\varepsilon^{-N} \int_{\partial M} (|u_{x_1}|^2 + |du|_h^2)|_{x_1=-\varepsilon} \, d\,\mathrm{vol}_h \\ &\leq C_K \varepsilon^{-N+K},\end{aligned}$$

for any K, as $\varepsilon \to 0+$ (since u vanishes to infinite order at $x_1 = 0$). By choosing $K > N$ we see that the left hand side is 0 and that implies that u is zero. $\square$

4. Propagation of singularities at radial points

To obtain meromorphic continuation of the resolvent (1.3) we need propagation estimates at *radial* points. These estimates were developed by Melrose [Me] in the context of scattering theory on asymptotically Euclidean spaces and are crucial in the Vasy approach [Va1]. A semiclassical version valid for very general sinks and sources was given in Dyatlov–Zworski [DyZw1] (see also [DyZw2, Appendix E]).

To explain this estimates we first review the now standard results on propagation of singularities due to Hörmander [H]. Thus let $P \in \Psi^m(X)$, with a real valued symbol $p := \sigma(P)$. Suppose that in an open conic subset of $U \subset T^*X \setminus 0$, $\pi(U) \Subset X$ ($\pi : T^*X \to X$),

$$p(x,\xi) = 0, \ (x,\xi) \in U \implies H_p \text{ and } \xi\partial_\xi \text{ are linearly independent at } (x,\xi). \tag{4.1}$$

Here H_p is the Hamilton vector field of p and $\xi\partial_\xi$ is the *radial* vector field. The latter is invariantly defined as the generator of the $\mathbb{R}_+$ action on $T^*X \setminus 0$ (multiplication of one forms by positive scalars).

The basic propagation estimate is given as follows: suppose that $A, B, B_1 \in \Psi^0(X)$ and $\mathrm{WF}(A) \cup \mathrm{WF}(B) \subset U$, $\mathrm{WF}(I - B_1) \cap U = \emptyset$.

We also assume that that $\mathrm{WF}(A)$ is *forward controlled* by $\complement\,\mathrm{Char}(B)$ in the following sense: for any $(x,\xi) \in \mathrm{WF}(A)$ there exists $T > 0$ such that

$$\exp(-TH_p)(x,\xi) \notin \mathrm{Char}(B), \quad \exp([-T,0]H_p)(x,\xi) \subset U. \tag{4.2}$$

The forward control can be replaced by backward control, that is we can demand existence of $T < 0$. That is allowed since the symbol is real.

The crucial estimate is then given by

$$\|Au\|_{H^{s+m-1}} \leq C\|B_1 Pu\|_{H^s} + C\|Bu\|_{H^{s+m-1}} + C\|u\|_{H^{-N}}, \tag{4.3}$$

where N is arbitrary and C is a constant depending on N. A direct proof can be found in [H]. The estimate is valid with $u \in \mathcal{D}'(X)$ for which the right hand side is finite – see [DyZw2, Exercise E.28].

We will consider a situation in which the condition (4.1) is violated. We will work on the manifold X given by (2.4), near $x_1 = 0$. In the notation of (4.1) we assume

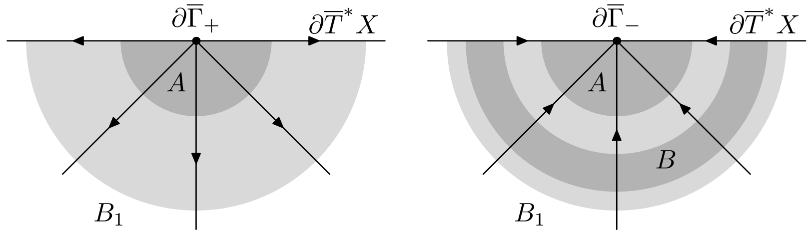

FIGURE 1. An illustration of the behaviour of the Hamilton flows for radial sources and for radial sinks and of the localization of operators in the estimates (4.10) and (4.13) respectively. The horizontal line on the top denotes the boundary, $\partial\overline{T}^*X$, of the *fiber–compactified* cotangent bundle $\overline{T}^*X$. The shaded half-discs then correspond to conic neighbourhoods in T^*X. In the simplest example of $X = (-1,1) \times \mathbb{R}/\mathbb{Z}$, and $p = x_1\xi_1^2 + \xi_2^2$, $H_p = \xi_1(2x_1\partial_{x_1} - \xi_1\partial_{\xi_1}) + 2\xi_2\partial_{x_2}$, $x_2 \in \mathbb{R}/\mathbb{Z}$. Near $\partial\overline{\Gamma}_\pm$ explicit (projective) compactifications is given by $r = 1/|\xi_1|$, (so that $\partial\overline{T}^*X = \{r = 0\}$), $\theta = \xi_2/|\xi_1|$, with x (the base variable) unchanged. In this variables, near $\partial\overline{\Gamma}_\pm$ (boundaries of compactifications of $\Gamma_\pm$ we check that $r\partial_r = -\xi_1\partial_{\xi_1} - \xi_2\partial_{\xi_2}$ and $\theta\partial_\theta = \xi'\partial_{\xi'}$. Hence near $\Gamma_\pm$, $H_p = \pm r(\theta\partial_\theta + r\partial_r + 2x_1\partial_{x_1} + 2\theta\partial_{x_2})$ and (after rescaling) we see a source and a sink.

that, near $x_1 = 0$,

$$P \in \mathrm{Diff}^2(X), \quad p = \sigma(P) = x_1\xi_1^2 + q(x,\xi'), \quad q(x_1,x',\xi') := |\xi'|^2_{h(x_1,x')}, \tag{4.4}$$

$(x',\xi') \in T^*\partial M$, $(x,\xi) \in T^*X \setminus 0$. The Hamilton vector field is given by

$$H_p = \xi_1(2x_1\partial_{x_1} - \xi_1\partial_{\xi_1}) + \partial_{x_1}q(x,\xi')\partial_{\xi_1} + H_{q(x_1)}, \tag{4.5}$$

where $H_{q(x_1)}$ is the Hamilton vectorfield of $(x',\xi') \mapsto q(x_1,x',\xi')$ on $T^*\partial M$.

We see that the condition (4.1) is violated at

$$\begin{gathered} \Gamma = \{(0,x',\xi_1,0) : x' \in \partial M, \xi_1 \in \mathbb{R}\setminus 0\} \subset T^*X \setminus 0, \\ \Gamma = N^*Y \setminus 0, \quad Y := \{x_1 = 0\}. \end{gathered} \tag{4.6}$$

In fact, $H_p|_{N^*Y} = -\xi_1(\xi\partial_\xi|_{N^*Y})$. Nevertheless Propositions 2 and 3 below provide propagation estimates valid in spaces with restricted regularity.

We note that $\Gamma = p^{-1}(0) \cap \pi^{-1}(Y)$ and that near $\pi^{-1}(Y)$, $\Sigma =: p^{-1}(0)$ has two *disjoint* connected components:

$$\begin{gathered} \Sigma = \Sigma_+ \sqcup \Sigma_-, \qquad \Gamma_\pm := \Sigma_\pm \cap \Gamma, \\ \Sigma_\pm \cap \{|x_1| < 1\} := \{(-q(x,\xi')/\rho^2, x', \rho, \xi') : \pm\rho > 0, \ |x_1| < 1\}. \end{gathered} \tag{4.7}$$

The set Γ_+ is a source and Γ_- is a sink for the flow projected to the sphere at infinity – see Fig. 1.

We now write P as follows:

$$P = P_0 + iQ, \quad P_0 = P_0^*, \quad Q = Q^*, \tag{4.8}$$

where the formal L^2-adjoints are taken with respect to the density $dx_1 d\,\mathrm{vol}_h$.

We can now formulate the following propagation result at the source. We should stress that changing P to $-P$ changes a source into a sink and the relevant thing is the sign of $\sigma(Q) \in S^1/S^0$ which then changes – see (4.9) below.

We first state a *radial source estimate*:

Proposition 2. *In the notation of* (4.7) *and* (4.8) *put*

$$s_+ = \sup_{\Gamma_+} |\xi_1|^{-1}\sigma(Q) - \tfrac{1}{2}, \tag{4.9}$$

and take $s > s_+$. For any $B_1 \in \Psi^0(X)$ satisfying $\mathrm{WF}(I - B_1) \cap \Gamma_+ = \emptyset$ there exists $A \in \Psi^0(X)$ with $\mathrm{Char}(A) \cap \Gamma_+ = \emptyset$ such that for $u \in \mathcal{C}_\mathrm{c}^\infty(X)$

$$\|Au\|_{H^{s+1}} \leq C\|B_1 Pu\|_{H^s} + C\|u\|_{H^{-N}}, \tag{4.10}$$

for any N.

Remarks. 1. The supremum in (4.9) should be understood as being taken at the ξ-infinity or as $s_+ = \sup_{x' \in \partial M} \lim_{\xi_1 \to \infty} |\xi_1|^{-1}\sigma(Q)(0, x', \xi_1, 0) - \frac{1}{2}$.

2. An approximation argument – see [DyZw2, Lemma E.42] for a textbook presentation and also [HaVa],[Va1],[Me] – shows that (4.10) is valid for $u \in H^{-N}$, $\operatorname{supp} u \cap \partial X = \emptyset$, such that $B_1 u \in H^{s+1}$, $B_1 Pu \in H^s$.

3. Using a regularization argument – see for instance [H, §3.5] or [DyZw2, Exercises E.28, E.33] – (4.10) holds for all $u \in \mathcal{D}'(X)$, $\operatorname{supp} u \subset K$ where K is a fixed compact subset of X°, such that $B_1 u \in H^r$ for some $r > s_+ + 1$. In particular, when combined with the hyperbolic estimate (3.6), that gives

$$Pu \in \bar{\mathcal{C}}^\infty(X), \quad u \in \bar{H}^r(X), \quad r > s_+ + 1 \implies u \in \bar{\mathcal{C}}^\infty(X). \tag{4.11}$$

In fact, the smoothness near $x_1 = 0$ is obtained from the estimate (4.10) and elliptic estimates applied to χu, $\chi \in \mathcal{C}_\mathrm{c}^\infty(X)$ and then the hyperbolic estimates show smoothness for $x_1 < -\varepsilon$.

4. To see that the threshold (4.9) is essentially optimal for (4.11) we consider $X = (-1, 1) \times \mathbb{R}/\mathbb{Z}$ and $P = x_1 D_{x_1}^2 - i(\rho + 1)D_{x_1} - D_{x_2}^2$, $x_2 \in \mathbb{R}/\mathbb{Z}$, $\rho \in \mathbb{R}$. In this case $s_+ = -\rho - \frac{1}{2}$. Put $u(x) := \chi(x_1)(x_1)_+^{-\rho}$, $\rho \notin -\mathbb{N}$, and and note that

$$(x_1 D_{x_1}^2 - i(\rho + 1)D_{x_1})(x_1)_+^{-\rho} = 0.$$

Hence $Pu \in \mathcal{C}_\mathrm{c}^\infty(X)$ and $u \in H^{-\rho + \frac{1}{2}-} \setminus H^{-\rho+\frac{1}{2}}$.

The *radial sink estimate* requires a control condition similar to that in (4.2). There is also a change in the regularity condition.

Proposition 3. *In the notation of* (4.7) *and* (4.8) *put*

$$s_- = \sup_{\Gamma_-} |\xi_1|^{-1}\sigma(Q) - \tfrac{1}{2}, \tag{4.12}$$

and take $s > s_-$. *For any* $B_1 \in \Psi^0(X)$ *satisfying* $\mathrm{WF}(I - B_1) \cap \Gamma_- = \emptyset$ *there exist* $A, B \in \Psi^0(X)$ *such that*

$$\mathrm{Char}(A) \cap \Gamma_- = \emptyset, \quad \mathrm{WF}(B) \cap \Gamma_- = \emptyset$$

and for $u \in \mathcal{C}^\infty_{\rm c}(X)$,

$$\|Au\|_{H^{-s}} \le C\|B_1 Pu\|_{H^{-s-1}} + C\|Bu\|_{H^{-s}} + C\|u\|_{H^{-N}}, \tag{4.13}$$

for any N.

Remark. A regularization method – see [DyZw2, Exercise 34] – shows that (4.13) is valid for $u \in \mathcal{D}'(X^\circ)$, $\operatorname{supp} u \subset K$ where $K \Subset X^\circ$ is a fixed set, and for which the right hand side of (4.13) is finite.

Proof of Proposition 2. The basic idea is to produce an operator $F_s \in \Psi^{s+\frac12}(X)$, elliptic on $\mathrm{WF}(A)$ such that for $s > s_+$ and $u \in \mathcal{C}^\infty_{\rm c}(X)$, we have

$$\|F_s u\|^2_{H^{\frac12}} \le C\|B_1 Pu\|_{H^s}\|F_s u\|_{H^{\frac12}} + C\|B_1 u\|^2_{H^{s+\frac12}} + C\|u\|^2_{H^{-N}}. \tag{4.14}$$

This is achieved by writing, in the notation of (4.8),

$$\operatorname{Im}\langle Pu, F_s^* F_s u\rangle = \langle \tfrac{i}{2}[P_0, F_s^* F_s]u, u\rangle + \operatorname{Re}\langle Qu, F_s^* F_s u\rangle, \tag{4.15}$$

and using the first term on the right hand side to control the left hand side of (4.14). We note here that since $\mathrm{WF}(F_s) \cap \mathrm{WF}(I - B_1) = \emptyset$, then in any expression involving F_s we can replace u and Pu by $B_1 u$ and $B_1 Pu$ respectively by introducing errors $\mathcal{O}(\|u\|_{H^{-N}})$ for any N. Hence from now on we will consider estimates with u only.

To construct a suitable F_s we take $\psi_1 \in \mathcal{C}^\infty_{\rm c}((-2\delta, 2\delta); [0,1])$, $\psi_1(t) = 1$, for $|t| < \delta$, $t\psi_1'(t) \le 0$, and $\psi_2 \in \mathcal{C}^\infty(\mathbb{R})$, $\psi_2(t) = 0$ for $t \le 1$, $\psi_2(t) = 1$, $t \ge 2$, and propose

$$\begin{gathered} F_s := \psi_1(x_1)\psi_1(-\Delta_h/D^2_{x_1})\psi_2(D_{x_1})D_{x_1}^{s+\frac12} \in \Psi^{s+\frac12}(X), \\ \sigma(F_s) =: f_s(x,\xi) = \psi_1(x_1)\psi_1(q(x,\xi')/\xi_1^2)\psi_2(\xi_1)\xi_1^{s+\frac12}. \end{gathered}$$

We note that because of the cut-off ψ_2, $D_{x_1}^{s+\frac12}$ and $-\Delta_h/D^2_{x_1}$ are well defined.

For $|\xi|$ large enough (which implies that $\xi_1 > |\xi|/C$ on the support of f_s if δ is small enough) we use (4.5) to obtain

$$\begin{aligned} H_p f_s(x,\xi) = \xi_1^{s+\frac32} \big(& 2x_1\psi_1'(x_1)\psi_1(\xi_2/\xi_1) + 2\psi_1(x_1)(q(x,\xi')/\xi_1^2)\psi_1'(q(x,\xi')/\xi_1^2) \\ & -(s+\tfrac12)\psi_1(x_1)\psi_1(q(x,\xi')/\xi_1^2)\big)\,\psi_2(\xi_1) \le -(s+\tfrac12)\xi_1 f_s. \end{aligned} \tag{4.16}$$

In particular,

$$f_s H_p f_s + (s+\tfrac12)\xi_1 f_s^2 \le 0, \qquad |\xi| > C_0. \tag{4.17}$$

The inequality (4.17) is important since $\sigma(\frac{i}{2}[P_0, F_s^* F_s]) = f_s H_p f_s$. Hence returning to (4.15), using (4.17), the sharp Gårding inequality [H3, Theorem 18.1.14] and the fact that $F_s^*[Q, F_s] \in \Psi^{2s+1}(X)$, we see that

$$\begin{aligned}\operatorname{Im}\langle Pu, F_s^* F_s\rangle &= \langle \tfrac{i}{2}[P_0, F_s^* F_s]u, u\rangle + \langle QF_s u, F_s u\rangle + \langle F_s^*[Q, F_s]u, u\rangle \\ &\leq \langle \tfrac{i}{2}[P_0, F_s^* F_s]u, u\rangle + \langle QF_s u, F_s u\rangle + C\|u\|^2_{H^{s+\frac{1}{2}}} \\ &\leq \langle (-(s+\tfrac{1}{2})D_{x_1} + Q)F_s u, F_s u\rangle + C\|u\|^2_{H^{s+\frac{1}{2}}}.\end{aligned}$$

Since D_{x_1} is elliptic (and positive) on $\mathrm{WF}(F_s)$ we can use (3.1) to see that if $s > s_+$ (where s_+ is given in (4.9)) then

$$\begin{aligned}\|F_s u\|^2_{H^{\frac{1}{2}}} &\leq -\operatorname{Im}\langle Pu, F_s^* F_s u\rangle + C\|u\|^2_{H^{s+\frac{1}{2}}} \leq \|Pu\|_{H_s}\|F_s^* F_s u\|_{H^{-s}} + C\|u\|^2_{H^{s+\frac{1}{2}}} \\ &\leq 2\|Pu\|^2_{H_s} + \tfrac{1}{2}\|F_s u\|^2_{H^{\frac{1}{2}}} + C\|u\|^2_{H^{s+\frac{1}{2}}}.\end{aligned}$$

Recalling the remark made after (4.15) this gives (4.14). Choosing A so that $F_s \in \Psi^{s+\frac{1}{2}}$ is elliptic on $\mathrm{WF}(A)$ we obtain

$$\|Au\|_{H^{s+1}} \leq C\|B_1 Pu\|_{H^s} + C\|B_1 u\|_{H^{s+\frac{1}{2}}} C\|u\|_{H^{-N}}. \tag{4.18}$$

It remains to eliminate the second term on the right hand side. We note that $\mathrm{WF}(B_1) \cap \mathrm{Char}(A)$ forward controlled by $\complement\,\mathrm{Char}(A)$ in the sense of (4.2). Since (4.1) is satisfied on $\mathrm{WF}(B_1) \cap \mathrm{Char}(A)$ we apply (4.3) to obtain

$$\begin{aligned}\|B_1 u\|_{H^{s+\frac{1}{2}}} &\leq C\|B_2 Pu\|_{H^{s-\frac{1}{2}}} + C\|Au\|_{H^{s+\frac{1}{2}}} + C\|u\|_{H^{-N}} \\ &\leq C\|B_2 Pu\|_{H^s} + \tfrac{1}{2}\|Au\|_{H^s} + C'\|u\|_{H^{-N}}, \quad s + \tfrac{1}{2} > -N,\end{aligned} \tag{4.19}$$

where B_2 has the same propeties as B_1 but a larger microsupport. (Here we used an interpolation estimate for Sobolev spaces based on $t^{s+\frac{1}{2}} \leq \gamma t^s + \gamma^{-2N-2s-1} t^{-N}$, $t \geq 0$ – that follows from rescaling $\tau^{s+\frac{1}{2}} \leq \tau^s + \tau^{-N}$, $\tau \geq 0$.)

Combining (4.18) and (4.19) gives (4.10) with B_1 replaced by B_2. Relabeling the operators concludes the proof. □

Proof of Proposition 3. The proof of (4.13) is similar to the proof of Proposition 2. We now use $G_s \in \Psi^{-s-\frac{1}{2}}(X)$ given by the same formula:

$$\begin{gathered}G_s := \psi_1(x_1)\psi_1(-\Delta_h/D_{x_1}^2)\psi_2(D_{x_1})D_{x_1}^{-s-\frac{1}{2}} \in \Psi^{-s-\frac{1}{2}}(X), \\ \sigma(G_s) =: g_s(x,\xi) = \psi_1(x_1)\psi_1(q(x,\xi')/\xi_1^2)\psi_2(\xi_1)\xi_1^{-s-\frac{1}{2}}.\end{gathered}$$

However now,

$$\begin{aligned}g_s H_g g_s(x,\xi) = \xi_1^{-s+\frac{1}{2}} g_s(x,\xi) &\left(2x_1\psi_1'(x_1)\psi_1(\xi_2/\xi_1) + 2\psi_1(x_1)(q(x,\xi')/\xi_1^2)\psi_1'(q(x,\xi')/\xi_1^2)\right. \\ &\left. -(s+\tfrac{1}{2})\psi_1(x_1)\psi_1(q(x,\xi')/\xi_1^2)\right)\psi_2(\xi_1) \\ \leq -(s+\tfrac{1}{2})&|\xi_1| g_s^2 + C_0|\xi_1|^{-2s} b(x,\xi)^2,\end{aligned}$$

where $b = \sigma(B)$ is chosen to control the terms involving $t\psi_1'(t)$ (which now have the "wrong" sign compared to (4.16)). The proof now proceeds in the same way as the proof of (4.10) but we have to carry over the $\|Bu\|_{H^s}$ terms. □

5. Proof of Theorem 1

We first show that $\ker_{\mathscr{X}_s} P(\lambda)$ is finite dimensional when $\operatorname{Im}\lambda > -s - \frac{1}{2}$. Using standard arguments this follows from the definition (2.7) and the estimate (5.1) below. To formulate it suppose that

$$\chi \in \mathcal{C}^\infty_{\rm c}(X), \quad \chi|_{x_1 < -2\delta} \equiv 0, \quad \chi|_{x_1 > -\delta} \equiv 1,$$

where $\delta > 0$ is a fixed (small) constant. Then for $u \in \mathscr{X}_s$ and $s > -\operatorname{Im}\lambda - \frac{1}{2}$,

$$\|u\|_{\bar H^{s+1}(X^\circ)} \leq C\|P(\lambda)u\|_{\bar H^s(X^\circ)} + \|\chi u\|_{H^{-N}(X)}. \tag{5.1}$$

Proof of (5.1). If $\chi_+ \in \mathcal{C}^\infty_{\rm c}$, $\operatorname{supp}\chi_+ \subset \{x_1 > 0\}$ then elliptic estimates show that

$$\|\chi_+ u\|_{H^{s+1}} \leq \|\chi_+ u\|_{H^{s+2}} \leq C\|Pu\|_{H^s} + C\|\chi u\|_{H^{-N}}.$$

Near $x_1 = 0$ we use the estimates (4.10) (valid for $u \in \mathscr{X}_s$) – see Remark 2 after Proposition 2) which give for, for $\chi_0 \in \mathcal{C}^\infty_{\rm c}$, $\operatorname{supp}\chi_0 \subset \{|x_1| < \delta/2\}$

$$\|\chi_0 u\|_{H^{s+1}(X)} \leq C\|P(\lambda)u\|_{\bar H^s(X)} + C\|\chi u\|_{H^{-N}(X)}. \tag{5.2}$$

To prove (5.2) we microlocalize to neighbourhoods of $\{\pm\xi_1 > |\xi|/C\}$ and use (4.10) for $P(\lambda)$ and $-P(\lambda)$ respectively – from (2.3) we see that $s_+ = -\operatorname{Im}\lambda - \frac{1}{2}$ for $P = P(\lambda)$ and $s_+ = -\operatorname{Im}\lambda - \frac{1}{2}$ for $P = -P(\lambda)$ (a rescaling by a factor of 4 is needed by comparing (2.3) with (4.4)). Elsewhere the operator is elliptic in $|x_1| < \delta$.

Finally if χ_- is supported in $\{x_1 < -\delta/2\}$ then the hyperbolic estimate (3.6) shows that

$$\|\chi_- u\|_{\bar H^{s+1}(X)} \leq C\|P(\lambda)u\|_{\bar H^s(X)} + C\|\chi_0 u\|_{H^{s+1}(X)}.$$

Putting these estimates together gives (5.1). □

To show that the range of P on $\mathscr{X}_s$ is of finite codimension (and hence closed [H3, Lemma 19.1.1]) we need the following

Lemma 4. *The cokernel of $P(\lambda)$ in $\dot H^{-s}(X) \simeq \mathscr{Y}_s^*$ (see (3.3))*

$$\operatorname{coker}_{\mathcal{X}_s} P(\lambda) := \{v \in \dot H^{-s}(X) : \forall\, u \in \mathscr{X}_s,\ \langle P(\lambda)u, v\rangle = 0\},$$

is equal to the kernel of $P(\bar\lambda)$ on $\dot H^{-s}(X)$: $\operatorname{coker}_{\mathcal{X}_s} P(\lambda) = \ker_{\dot H^{-s}(X)} P(\bar\lambda)$.

Proof. In view of (2.6) we have, for $u \in \bar{\mathcal{C}}^\infty(X^\circ)$ and $v \in \dot H^{-s}(X)$,

$$\langle P(\lambda)u, v\rangle = \langle u, P(\bar\lambda)v\rangle.$$

Since $\bar{\mathcal{C}}^\infty(X^\circ)$ is dense in $\mathscr{X}_s$ (see for instance Lemma [DyZw2, Lemma E.42]) it follows that $\langle P(\lambda)u, v\rangle = 0$ for all $u \in \mathscr{X}_s$ if and only if $P(\bar\lambda)v = 0$. □

Hence to show that $\operatorname{coker}_{\mathscr{X}_s}$ is finite dimensional it suffices to prove that the kernel of $P(\bar\lambda)$ is finite dimensional. We claim an estimate from which this follows:

$$u \in \ker_{\dot H^{-s}(X)} P(\bar\lambda) \implies \|u\|_{\dot H^{-s}(X)} \leq C\|\chi u\|_{H^{-N}(X)}, \quad s > -\operatorname{Im}\lambda - \tfrac12 \tag{5.3}$$

where χ is the same as in (5.1).

Proof of (5.3). The hyperbolic estimate (3.5) shows that if $P(\bar\lambda)u = 0$ for $u \in \dot H^{-s}(X)$ (with any $\lambda \in \mathbb{C}$ or $s \in \mathbb{R}$) then $u|_{x_1<0} \equiv 0$. We can now apply (4.13) with $P = P(\lambda)$ near Γ_- and $P = -P(\lambda)$ near Γ_+. We again see that the threshold condition is the same at both places: we require that $s > -\operatorname{Im}\lambda - \frac12$. Since u vanishes in $x_1 < 0$ there $\operatorname{WF}(Bu) \cap \operatorname{Char} P(\lambda) = \emptyset$ and hence (using (3.1)) $\|Bu\|_{\dot H^{-s}(X)} \leq C\|\chi u\|_{-N}$. Hence (4.13) and elliptic estimates give (5.3). $\square$

6. Asymptotic expansions

To prove Theorem 3 we need a regularity result for L^2 solutions of

$$(-\Delta_g - \lambda^2 - (\tfrac{n}{2})^2)^{-1} u = f \in \mathcal{C}^\infty_{\rm c}(M), \quad \operatorname{Im}\lambda > \tfrac{n}{2}. \tag{6.1}$$

To formulate it we recall the definition of X given in (2.4) and of $X_1 := X \cap \{x_1 > 0\}$. We also define $j : M \to X_1$ to be the natural identification, given by $j(y_1, y') = (y_1^2, y')$ near the boundary. Then we have

Proposition 5. *For $\operatorname{Im}\lambda \gg 1$ and $\lambda \notin i\mathbb{N}$, the unique L^2-solution u to (6.1) satisfies*

$$u = y_1^{-i\lambda + \frac{n}{2}} j^* U, \quad U \in \bar{\mathcal{C}}^\infty(X_1). \tag{6.2}$$

In other words, near the boundary, $u(y) = y_1^{-i\lambda+\frac{n}{2}} U(y_1^2, y')$ where U is smoothly extendible.

Remark. Once Theorem 3 is established then the relation between $P(\lambda)^{-1}$ and the meromorphically continued resolvent $R(\frac{n}{2} - i\lambda)$ shows that $y_1^{-s}R(s) : \dot{\mathcal{C}}^\infty(M) \to j^*\bar{\mathcal{C}}^\infty(X_1)$ is meromorphic away from $s \in \mathbb{N}$ – see §7. That means that away from exceptional points (6.2) remains valid for $u = R(\frac{n}{2} - i\lambda)$.

To give a direct proof of Proposition 5 we need a few lemmas. For that we define Sobolev spaces $H^k_g(M, d\operatorname{vol}_g)$ associated to the Laplacian $-\Delta_g$:

$$H^k_g(M) := \{u : y_1^{|\alpha|} D_y^\alpha u \in L^2(M, d\operatorname{vol}_g), \ |\alpha| \leq k\}, \quad \ell \in \mathbb{N}. \tag{6.3}$$

(In invariant formulation can be obtained by taking vector fields vanishing at ∂M – see [MazMe].) Let us also put

$$Q(\lambda^2) := -\Delta_g - \lambda^2 - (\tfrac{n}{2})^2. \tag{6.4}$$

Lemma 6. *With $H^k_g(M)$ defined by (6.3) and $Q(\lambda^2)$ by (6.4) we have for any $k \geq 0$,*

$$Q(\lambda^2)^{-1} : H^k_g(M) \to H^{k+2}_g(M), \quad \operatorname{Im}\lambda > \tfrac{n}{2}. \tag{6.5}$$

Proof. Using the notation from the proof of (2.1) and Lemma 1 we write

$$Q(\lambda^2) = (y_1 D_{y_1})^2 + y_1^2 d_h^* d - i(n + y_1^2 \gamma(y_1^2, y'))y_1 D_{y_1}$$

so that for $u \in \mathcal{C}_c^\infty(M)$ supported near ∂M, and with the inner products in $L_g^2 = L^2(M, d\,\mathrm{vol}_g)$,

$$\langle Q(\lambda^2)u, u\rangle_{L_g^2} = \int_M (|y_1 D_{y_1}|^2 + y_1^2 |du|_h^2) d\,\mathrm{vol}_g \,.$$

Hence, $\|u\|_{H_g^1} \leq C\|Q(\lambda^2)u\|_{L_g^2} + C\|u\|_{L_g^2}$. Using this and expanding $\langle Q(\lambda)u, Q(\lambda)u\rangle_{L_g^2}$ we see that

$$\|u\|_{H_g^2} \leq C\|Q(\lambda^2)u\|_{L_g^2} + C\|u\|_{L_g^2}, \quad u \in \mathcal{C}_c^\infty(M).$$

Since $\mathcal{C}_c^\infty(M)$ is dense in $H_g^2(M)$ it follows that for $\operatorname{Im}\lambda > \frac{n}{2}$, $Q(\lambda)^2 : L_g^2 \to H_g^2$. Commuting $y_1 V$, where $V \in \bar{\mathcal{C}}^\infty(M; TM)$, with $Q(\lambda^2)$ gives the general estimate,

$$\|u\|_{H_g^{k+2}} \leq C\|Q(\lambda^2)u\|_{H_g^k} + C\|u\|_{L_g^2}, \quad u \in \mathcal{C}_c^\infty(M),$$

and that gives (6.5). □

Lemma 7. *For any $\alpha > 0$ there exists $c(\alpha) > 0$ such that for $\operatorname{Im}\lambda > c(\alpha)$,*

$$y_1^\alpha Q(\lambda^2)^{-1} y_1^{-\alpha} : L_g^2(M) \to H_g^2(M). \tag{6.6}$$

Proof. We expand the conjugated operator as follows:

$$\begin{aligned} y_1^\alpha Q(\lambda^2) y_1^{-\alpha} &= Q(\lambda^2 + \alpha^2) - \alpha(2iy_1 D_{y_1} - n - y_1^2\gamma(y_1^2, y')) \\ &= (I + K(\lambda, \alpha))^{-1} Q(\lambda^2 + \alpha^2), \\ K(\lambda, \alpha) &:= \alpha(2iy_1 D_{y_1} - n - y_1^2\gamma(y_1^2, y'))Q(\lambda^2 + \alpha^2)^{-1}. \end{aligned} \tag{6.7}$$

The inverse of $Q(\lambda^2 + \alpha^2)$ exists due to the following bound provided by the spectral theorem (since $\operatorname{Spec}(-\Delta_g) \subset [0, \infty)$) and (6.5) (with $k = 0$):

$$\|Q(\mu^2)^{-1}\|_{L_g^2 \to H_g^k} \leq \frac{(1 + C|\mu|)^{k/2}}{d(\mu^2, [-(\frac{n}{2})^2, \infty))}, \quad k = 0, 2. \tag{6.8}$$

It follows that for $\operatorname{Im}\lambda > c(\alpha)$, $I + K(\lambda, \alpha)$ in (6.7) is invertible on L_g^2. Hence we can invert $y_1^\alpha Q(\lambda^2) y_1^{-\alpha}$ with the mapping property given in (6.6). □

Proof of Proposition 5. The first step of the proof is a strengthening of Lemma 6 for solutions of (6.1). We claim that if u solves (6.1) and $u \in L_g^2$ then, near the boundary ∂M,

$$V_1 \cdots V_N u \in L_g^2, \quad V_j \in \bar{\mathcal{C}}^\infty(M, TM), \quad V_j y_1|_{y_1} = 0, \tag{6.9}$$

for any N. The condition on V_j means that V_j are tangent to the boundary ∂M (for more on spaces defined by such conditions see [H3, §18.3]).

To obtain (6.9) we see that if V is a vector field tangent to the boundary of ∂M then

$$\begin{aligned} Q(\lambda^2)Vu = F &:= Vf + [(y_1 D_{y_1})^2, V]u + y_1^2[\Delta_{h(y_1^2)}, V] - i[(n + y_1^2\gamma(y))y_1 D_{y_1}, V] \\ &= Vf + y_1^2 Q_2 u + y_1 Q_1 u, \end{aligned}$$

where Q_j are differential operators of order j. Lemma 6 shows that $F \in L^2_g$. From Lemma 6 we also know that $y_1 V u \in L^2_g$. Hence,

$$y_1 Vu - y_1 Q(\lambda^2)^{-1} F \in L^2_g, \quad Q(\lambda^2) y_1^{-1}(y_1 Vu - y_1 Q(\lambda^2)^{-1}F) = 0.$$

But for $\operatorname{Im}\lambda > c_0$, Lemma 7 shows that

$$Q(\lambda^2) y_1^{-1} v = 0, \quad v \in L^2(M, d\mathrm{vol}_g) \implies v = 0. \tag{6.10}$$

Hence $Vu = Q(\lambda^2)^{-1} F \in L^2_g$. This argument can be iterated showing (6.9).

We now consider $P(\lambda)$ as an operator on X_1, formally selfadjoint with respect to $d\mu = dx_1 d\,\mathrm{vol}_h$. Since we are on open manifolds the two $\mathcal{C}^\infty$ structures agree and we can consider $P(\lambda)$ as operator on $\mathcal{C}^\infty(M)$. Since

$$Q(\lambda^2) = y_1^{-i\lambda + \frac{n}{2}} y_1^2 P(\lambda) y_1^{i\lambda - \frac{n}{2}} = x_1^{-\frac{i\lambda}{2} + \frac{n}{4}} x_1 P(\lambda) x_1^{\frac{i\lambda}{2} - \frac{n}{4}},$$

we can define

$$T(\lambda) := x_1^{\frac{i\lambda}{2} - \frac{n}{4}} Q(\lambda^2)^{-1} x_1^{-\frac{i\lambda}{2} + \frac{n}{4} + 1}, \quad \operatorname{Im}\lambda > \tfrac{n}{2}, \tag{6.11}$$

which satisfies

$$\begin{gathered} P(\lambda)T(\lambda)f = f, \quad f \in \mathcal{C}^\infty_{\mathrm{c}}(X_1), \\ T(\lambda) : x_1^{-\frac{\varrho}{2} - \frac{1}{2}} L^2 \to x_1^{-\frac{\varrho}{2} + \frac{1}{2}} L^2, \quad \rho := \operatorname{Im}\lambda > \tfrac{n}{2}. \end{gathered} \tag{6.12}$$

Here we used the fact that $2dy_1/y_1 = dx_1/x_1$ and that

$$L^2(y_1^{-n-1} dy_1 d\mathrm{vol}_h) = L^2\left(x_1^{-\frac{n}{2}-1} dx_1 d\mathrm{vol}_h\right) = x_1^{\frac{n}{4}+\frac{1}{2}} L^2, \quad L^2 := L^2(dx_1 d\mathrm{vol}_h).$$

Proposition 5 is equivalent to the following mapping property of $T(\lambda)$:

$$T(\lambda) : \mathcal{C}^\infty_{\mathrm{c}}(X_1) \longrightarrow \bar{\mathcal{C}}^\infty(X_1), \quad \operatorname{Im}\lambda \geq c_0, \quad \lambda \notin i\mathbb{N}. \tag{6.13}$$

To prove (6.13) we will use a classical tool for obtaining asymptotic expansions, the *Mellin transform*. Thus let $u = T(\lambda)f$, $f \in \mathcal{C}^\infty_{\mathrm{c}}(X_1)$. By replacing u by $\chi(x_1)u$, $\chi \in \mathcal{C}^\infty_{\mathrm{c}}((-1,1);[0,1])$, $\chi = 1$ near 0, we can assume that

$$u \in \mathcal{C}^\infty((0,1) \times \partial M) \cap x_1^{-\frac{\varrho}{2}+\frac{1}{2}} L^2, \quad P(\lambda)u = f_1 \in \mathcal{C}^\infty_{\mathrm{c}}((0,1) \times \partial M), \quad \rho > \tfrac{n}{2},$$

where smoothness for $x_1 > 0$ follows from Lemma 6. In addition (6.9) shows that

$$V_1 \cdots V_N u \in x_1^{-\frac{\varrho}{2}+\frac{1}{2}} L^2(dx_1 d\mathrm{vol}_h), \quad V_j \in \bar{\mathcal{C}}^\infty(X_1, TX_1), \quad V_j x_1|_{x_1} = 0. \tag{6.14}$$

In particular, for any k

$$x_1^N u \in C^k([0,1] \times \mathbb{S}^1) \tag{6.15}$$

if N is large enough.

We define the Mellin transform (for functions with support in $[0,1)$) as

$$Mu(s,x') := \int_0^1 u(x) x_1^s \frac{dx_1}{x_1}.$$

This is well defined for $\operatorname{Re} s > \rho/2$:

$$\begin{aligned} \|Mu(s,x')\|^2_{L^2(d\mathrm{vol}_h)} &= \int_{\mathbb{S}^1} \left| \int_0^1 x_1^{s+\frac{i\lambda}{2}-\frac12} (x_1^{-\frac{i\lambda}{2}-\frac12} u(x_1,x')) dx_1 \right|^2 d\mathrm{vol}_h \\ &\leq \left(\int_0^1 t^{-\rho+2\operatorname{Re} s -1} dt \right) \|x_1^{\frac{\rho}{2}-\frac12} u\|_{L^2} = (2\mathrm{Re} s - \rho)^{-1} \|x_1^{\frac{\rho}{2}-\frac12} u\|_{L^2}. \end{aligned}$$

In view of (6.9) $s \longmapsto Mu(s,x_2)$ is a holomorphic family of *smooth* functions in $\operatorname{Re} s > \rho/2$. We claim now that $Mu(s,x')$ continues meromorphically to all of $\mathbb{C}$. In fact, from (2.3) we see that for $f_2 := \frac14 f_1$,

$$M(x_1 f_2)(s,x') = M(\tfrac14 x_1 P(\lambda) u)(s,x') = -s(s+i\lambda) Mu(s,x') + M(Q_2 u)(s+1,x'),$$

where Q_2 is a second order differential operator built out of vector fields tangent to the boundary of X_1. In view of (6.14) $Q_2 u \in x_1^{-\frac{\rho}{2}+\frac12} L^2$ which implies that $M(Q_2 u)(s,x')$ is holomorphic in $\operatorname{Re} s > \rho/2$. Also, $s \mapsto M(x_1 f_2)(s,x')$ is entire as f_1 vanishes near $x_1 = 0$. Hence

$$Mu(s,x') = \frac{1}{s(s+i\lambda)} M(Q_2 u)(s+1,x') - \frac{1}{s(s+i\lambda)} M(x_1 f_2)(s,x'), \tag{6.16}$$

which means that $s \mapsto Mu(s,x')$ is meromorphic in $\operatorname{Re} s > \rho/2 - 1$. Melrose's indicial operator, $I(s)w = x_1^{-s} Q_2(x_1^s w)|_{x_1=0}$, $w \in \mathcal{C}^\infty(\partial M)$, is a differential operator in x' with polynomial coefficients in s and

$$M(Q_2 u)(s+1,x') = I(s) Mu(s+1,x') + M(\widetilde{Q}_2 u)(s+2,x').$$

where $\widetilde{Q}_2$ is a second order operator built from vector fields tangent to ∂M. Hence (6.16) can be iterated and that gives the meromorphic continuation of $Mu(s,x')$ with possible poles at $-i\lambda - k$, $k \in \mathbb{N}$.

The Mellin transform inversion formula, a contour deformation and the residue theorem (applied to simple poles thanks to our assumption that $i\lambda \notin \mathbb{Z}$) then give

$$u(x) \simeq x_1^{i\lambda}(b_0(x') + x_1 b_1(x') + \cdots) + a_0(x') + x_1 a_1(x') + \cdots, \quad a_j, b_j \in \mathcal{C}^\infty(\partial M),$$

where the regularity of remainders comes from (6.15). (The basic point is that

$$M(x_1^a \chi(x_1))(s) = (s+a)^{-1} F(s), \quad F(s) = -\int x_1^{a+s} \chi'(x_1) dx_1,$$

so that $F(s)$ is an entire function with $F(-a) = 1$.)

Since $Pu(x) = 0$ for $0 < x_1 < \varepsilon$ the equation shows that b_k is determined by $b_0, \cdots b_{k-1}$. We claim that $b_k \equiv 0$: if $b_0 \neq 0$ then

$$|x_1^{\frac{\rho}{2}-\frac{1}{2}} u| = x_1^{-\frac{1}{2}} |b_0(x')| + \mathcal{O}(x_1^{\frac{1}{2}}) \notin L^2(dx_1 d\mathrm{vol}_h).$$

contradicting (6.14). It follows that $u \in \bar{\mathcal{C}}^\infty(X_1)$ proving (6.13) and completing the proof of Proposition 5. $\square$

7. Meromorphic continuation

To prove Theorem 3 we recall that $(-\Delta_g - \lambda^2 - (\frac{n}{2})^2)^{-1}$ is a holomorphic family of operators on L^2_g for $\lambda^2 + (\frac{n}{2})^2 \notin \mathrm{Spec}(-\Delta_g)$ and in particular for $\operatorname{Im}\lambda > \frac{n}{2}$.

Proof of Theorem 3. We first show that for $\operatorname{Im}\lambda > 0$, $\lambda^2 + \frac{1}{4} \notin \mathrm{Spec}(-\Delta_g)$,

$$P(\lambda)u = 0, \quad u \in \mathscr{X}_s, \quad s > -\operatorname{Im}\lambda - \tfrac{1}{2} \implies u \equiv 0. \tag{7.1}$$

In fact, from (4.11) we see that $u \in \bar{\mathcal{C}}^\infty(X)$. Then putting $v(y) := y_1^{-i\lambda+\frac{n}{2}} j^*(u|_{X_1})$, $j : M \to X_1$, (2.3) shows that $(-\Delta_g - \lambda^2 - (\frac{n}{2})^2)v = 0$. For $\operatorname{Im}\lambda > 0$ we have $v \in L^2_g$ and hence from our assumptions, $v \equiv 0$. Hence $u|_{X_1} \equiv 0$, and $u \in \bar{\mathcal{C}}^\infty(X)$. Lemma 1 then shows that $u \equiv 0$ proving (7.1).

In view of Lemma 4 we now need to show that $P(\lambda)^* w = 0$, $w \in \dot{H}^{-s}(X)$, implies that $w \equiv 0$. It is enough to do this for $\lambda \notin i\mathbb{N}$ and $\operatorname{Im}\lambda \gg 1$ since invertibility at one point shows that the index of $P(\lambda)$ vanishes. Then (7.1) shows invertibility for all $\operatorname{Im}\lambda > 0$, $\lambda^2 + (\frac{n}{2})^2 \in \mathrm{Spec}(-\Delta_g)$.

Hence suppose that $P(\lambda)^* w = 0$, $w \in \dot{H}^{-s}(X)$. Estimate (3.5) then shows that $\operatorname{supp} w \subset \overline{X}_1$. (For $-1 < x_1 < 0$ we solve a hyperbolic equation with zero initial data and zero right hand side.) We now show that $\operatorname{supp} w \cap X_1 \neq \emptyset$ (that is there is some support in $x_1 > 0$; in fact by unique continuation results for second order elliptic operators, see for instance [H3, §17.2], this shows that $\operatorname{supp} w = \overline{X}_1$). In other words we we need to show that we cannot have $\operatorname{supp} w \subset \{x_1 = 0\}$. Since $\mathrm{WF}(w) \subset N^*\partial X_1$ we can restrict w to fixed values of $x' \in \partial M$ and the restriction and is then a linear combination of $\delta^{(k)}(x_1)$. But

$$P(\bar{\lambda})(\delta^{(k)}(x_1)) = (k + 1 - \bar{\lambda}/i)\delta^{(k+1)}(x_1) - i\gamma(x)(2i(k+1) - \bar{\lambda} - i\tfrac{n-1}{2})\delta^{(k)}(x_1),$$

and that does not vanish for $\operatorname{Im}\lambda > 0$.

Mapping property (6.13) and the definition of $P(\lambda)$ show that for any $f \in \mathcal{C}^\infty_{\mathrm{c}}(X_1)$ (that is f supported in $x_1 > 0$) there exists $u \in \bar{\mathcal{C}}^\infty(X_1)$ such that $P(\lambda)u = f$ in X_1. Then (with L^2 inner products meant as distributional pairings),

$$\langle f, w \rangle = \langle P(\lambda)u, w \rangle = \langle u, P(\lambda)^* w \rangle = 0.$$

Since $w \in \dot{\mathcal{D}}(X_1)$ and $u \in \bar{C}^\infty(X_1)$ the pairing is justified. In view of support properties of w, we can find f such that the left hand side does not vanish. This gives a contradiction. $\square$

Remark. Different proofs of the existence of λ with $P(\lambda)$ invertible can be obtained using semiclassical versions of the propagation estimates of §4. That is done for $\operatorname{Im}\lambda_0 \gg \langle \operatorname{Re}\lambda_0 \rangle$ in [Va2] and for $\operatorname{Im}\lambda_0 \gg 1$ in [DyZw2, §5.5.3].

Theorem 3 guarantees existence of the inverse at many values of λ. Then standard Fredholm analytic theory (see for instance [DyZw2, Theorem C.5]) gives

$$P(\lambda)^{-1} : \mathscr{Y}_s \to \mathscr{X}_s \ \text{ is a meromorphic family of operators in } \operatorname{Im}\lambda > -s - \frac{1}{2}. \quad (7.2)$$

Proof of Theorem 1. We define

$$V(\lambda) : \mathcal{C}^\infty_{\rm c}(M) \to \mathcal{C}^\infty_{\rm c}(X), \quad f(y) \longmapsto Tf(x) := \begin{cases} x_1^{\frac{i\lambda}{2} - \frac{n}{4} - 1} (j^{-1})^* f & x_1 > 0, \\ 0, & x_1 \leq 0, \end{cases}$$

$$U(\lambda) : \bar{\mathcal{C}}^\infty(X) \to \mathcal{C}^\infty(M), \quad u(x) \longmapsto y_1^{-i\lambda + \frac{n}{2}} j^*(u|_{X_1}),$$

where $j : M \to X_1$ is the map defined by $j(y) = (y_1^2, y')$ near ∂M. Then, for $\operatorname{Im}\lambda > \frac{n}{2}$, (2.2) and (2.3) show that

$$R(\tfrac{n}{2} - i\lambda) = U(\lambda) P(\lambda)^{-1} V(\lambda). \quad (7.3)$$

Since $P(\lambda)^{-1} : \bar{\mathcal{C}}^\infty(X) \to \bar{\mathcal{C}}^\infty(X)$ is a meromorphic family of operators in $\mathbb{C}$, Theorem 1 follows. $\square$

Remarks. 1. The structure of the residue of $P(\lambda)^{-1}$ is easiest to describe when the pole at λ_0 is simple and has rank one. In that case,

$$P(\lambda) = \frac{u \otimes v}{\lambda - \lambda_0} + Q(\lambda, \lambda_0), \quad u \in \bar{\mathcal{C}}^\infty(X), \quad v \in \bigcap_{s > -\operatorname{Im}\lambda_0 - \frac{1}{2}} \dot{H}^{-s}(\overline{X}_1)$$

$$P(\lambda_0) u = 0, \quad P(\bar{\lambda}_0) v = 0,$$

and where $Q(\lambda, \lambda_0)$ is holomorphic near λ_0. We note that $u \in \mathcal{C}^\infty(X)$ because of (4.11). The regularity of $v \in \dot{H}^{-s}$, $s > -\operatorname{Im}\lambda_0 - \frac{1}{2}$ just misses the threshold for smoothness – in particular there is no contradiction with Theorem 3!

2. The relation (7.3) between $R(\frac{n}{2} - i\lambda)$ and $P(\lambda)$ shows that unless the elements of the kernel of $P(\bar{\lambda})$ are supported on $\partial X_1 = \{x_1 = 0\}$ then the multiplicities of the poles of $R(\frac{n}{2} - i\lambda)$ agree.

For completeness we conclude with the proof of the following standard fact:

Proposition 8. *If* $R(\zeta) := (-\Delta_g - \zeta(n - \zeta))^{-1}$ *for* $\operatorname{Re}\zeta > n$ *then*

$$R(\zeta) : L^2(M, d\mathrm{vol}_g) \to L^2(M, d\mathrm{vol}_g), \quad (7.4)$$

is meromorphic for $\operatorname{Re}\zeta > \frac{n}{2}$ *with simple poles where* $\zeta(n-\zeta) \in \operatorname{Spec}(-\Delta_g)$*.*

Proof. The spectral theorem implies that $R(\zeta)$ is holomorphic on L^2_g in $\{\operatorname{Re}\zeta > \frac{n}{2}\} \setminus [\frac{n}{2}, n]$. In the λ-plane that corresponds to $\{\operatorname{Im}\lambda > 0\} \setminus i[0, \frac{n}{2}]$.

From (6.11) and (6.12) we see that boundeness of $R(\frac{n}{2} - i\lambda)$ on $L^2_g(M)$ is equivalent to

$$P(\lambda)^{-1} : x_1^{-\frac{\rho}{2}-\frac{1}{2}} L^2(X_1) \to x_1^{-\frac{\rho}{2}+\frac{1}{2}} L^2(X_1), \quad \rho := \operatorname{Im}\lambda. \tag{7.5}$$

We will first prove (7.7) for $0 < \rho \leq 1$. From Theorem 3 we know that except at a discrete set of poles, $P(\lambda)^{-1} : \bar{H}^s(X_1) \to \bar{H}^{s+1}(X_1)$, $s > -\rho - \frac{1}{2}$. We claim that for $-1 \leq s < -\frac{1}{2}$

$$x_1^s L^2(X_1) \hookrightarrow \bar{H}^s(X_1), \quad \bar{H}^{s+1}(X_1) \hookrightarrow x_1^{s+1} L^2(X_1). \tag{7.6}$$

By duality the first inclusion follows from the inclusion

$$\dot{H}^r(X_1) \hookrightarrow x_1^r L^2, \quad 0 \leq r \leq 1. \tag{7.7}$$

Because of interpolation we only need to prove this for $r = 1$ in which case it follows from Hardy's inequality, $\int_0^\infty |x_1^{-1} u(x_1)|^2 dx_1 \leq 4 \int_0^\infty |\partial^2_{x_1} u(x_1)|^2 dx_1$. The second inclusion follows from (7.7) and the fact that $\bar{H}^r(X_1) = \dot{H}^r(X_1)$ for $0 \leq r < \frac{1}{2}$ – see [Ta, Chapter 4, (5.16)]. We can now take $s = -\frac{\rho}{2} - \frac{1}{2}$ in (7.6) which for $0 < \rho \leq 1$ is in the allowed range. That proves (7.5) for $0 < \operatorname{Im}\lambda \leq 1$, except at the poles and consequently establishes (7.4) for $\frac{n}{2} < \operatorname{Re} s \leq \frac{n}{2} + 1$. We can choose a polynomial $p(s)$ such that $p(s)R(s) : \mathcal{C}^\infty_{\mathrm{c}}(M) \to \mathcal{C}^\infty(M)$ is holomorphic near $[\frac{n}{2}, n]$. The maximum principle applied to $\langle p(s)R(s)f, g\rangle$, $f, g \in \mathcal{C}^\infty_{\mathrm{c}}(M)$ now proves that $p(s)R(s)$ is bounded on $L^2_g(M)$ near $[\frac{n}{2}, n]$ concluding the proof. ∎

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