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Novelty Edition

Vasy's method

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Learn to:

- meromorphically continue resolvents on asymptotically hyperbolic manifolds
- avoid 0- and b-calculi
- avoid semiclassical methods!

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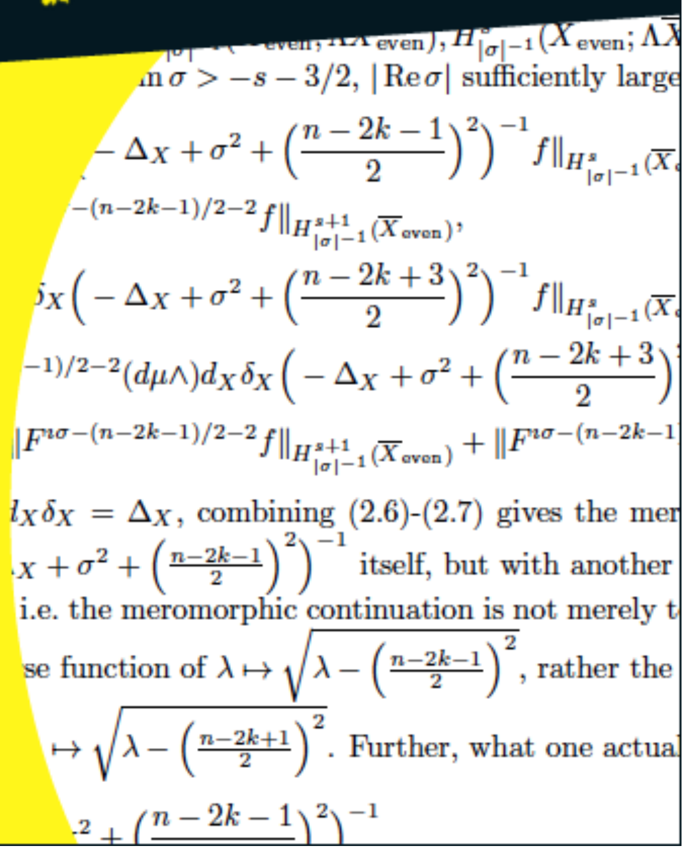

1. Introduction

We present the method introduced by András Vasy [V1],[V2] to prove meromorphic continuations of resolvents of Laplacians on asymptotically hyperbolic spaces in a simple model case. In particular, we prove Melrose's *radial estimates* [M] indicating the idea behind the general case. As explained in §9 the general case can be treated by the same methods without additional difficulties. We present the details in the simplest case for pedagogical reasons only.

We consider the Laplacian on (M, g) where

$$M = M_0 \cup M_1, \quad M_0 = (0,1]_{y_1} \times \mathbb{S}^1_{y_2}, \quad \partial M_1 = \{y_1 = 1\}, \quad \partial M = \{y_1 = 0\},$$
$$g|_{M_0} = \frac{dy_1^2 + dy_2^2}{y_1^2}.$$

Geometrically this corresponds to "half" of a hyperbolic funnel "cupped" at $\{y_1 = 1\}$. We have

$$-\Delta_g = y_1^2(D_{y_1}^2 + D_{y_2}^2), \quad D_{y_j} = \frac{1}{i}\partial_{y_j}, \quad d\mathrm{vol}_g = \frac{dy_1 dy_2}{y_1^2}$$

and

$$(-\Delta_g - \tfrac{1}{4} - \lambda^2)^{-1} : L^2(M, d\mathrm{vol}_g) \to L^2(M, d\mathrm{vol}_g), \quad \operatorname{Im}\lambda > 0. \tag{1.1}$$

We want to continue this operator meromorphically to $\mathbb{C}$:

$$(-\Delta_g - \tfrac{1}{4} - \lambda^2)^{-1} : L^2_{\mathrm{comp}}(M, d\mathrm{vol}_g) \to L^2_{\mathrm{loc}}(M, d\mathrm{vol}_g), \quad \lambda \in \mathbb{C}.$$

Moreover, we want to achieve it so that

$$(-\Delta_g - \tfrac{1}{4} - \lambda^2)^{-1} = U(\lambda)P(\lambda)^{-1}V(\lambda), \tag{1.2}$$

where U, V are holomorphic families of operators

$$V(\lambda) : L^2_{\mathrm{comp}}(M, d\mathrm{vol}_g) \to \mathscr{Y}_s, \quad U(\lambda) : \mathscr{X}_s \to L^2_{\mathrm{loc}}(M, d\mathrm{vol}_g), \quad \operatorname{Im}\lambda > -s - \frac{1}{2},$$

and

$$P(\lambda) : \mathscr{X}_s \to \mathscr{Y}_s, \quad \operatorname{Im}\lambda > -s - \frac{1}{2},$$

is of a holomorphic family of Fredholm operators, $\mathscr{Y}_s$ and $\mathscr{X}_s \supset \mathscr{Y}_s$ are suitable Hilbert spaces, and $P(\lambda)$ is invertible for $\operatorname{Im}\lambda \gg 1$.

The operators $U(\lambda)$ and $V(\lambda)$ will be sufficiently explicit so that (1.2) implies that (for $\lambda \notin -i\mathbb{N}$),

$$\operatorname{rank} \oint (-\Delta_g - \tfrac{1}{4} - \lambda^2)^{-1} d\lambda = \frac{1}{2\pi i} \operatorname{tr}_{\mathscr{Y}_s} \oint \partial_\lambda P(\lambda) P(\lambda)^{-1} d\lambda, \tag{1.3}$$

where the integration is over any closed curve in $\operatorname{Im}\lambda > -s - \frac{1}{2}$.

For interesting applications it is crucial to consider the semiclassical case, that is, uniform analysis as $\operatorname{Re}\lambda \to \infty$ but to indicate the basic mechanism behind the meromorphic continuation we only present the Fredholm property and invertibility in the upper half-plane.

2. Review of propagation of singularities

We recall the standard result about propagation of singularities due to Duistermaat–Hörmander. In view of the applications here we present it in the case of second order pseudodifferential opeerators. We use the notation of [H3, Chapter 18]

Suppose that X is a compact manifold and $P \in \Psi^m(X)$ and that the symbol of P, $\sigma(P) \in S^m(T^*X \setminus 0)/S^{m-1}(T^*X \setminus 0)$ has a representative

$$p - iq \in S^m(T^*X \setminus 0)$$

which is homogeneous of degree m.

For any operator $A \in \Psi^m(X)$ we can define $\operatorname{WF}(A) \subset T^*X \setminus 0$ (the smallest subset outside of which A has order $-\infty$ – see [H3, (18.1.34)]). We also define $\operatorname{Char}(A)$ the smallest conic closed set outside of which A is *elliptic* – see [H3, Definition 18.1.25].

The first estimate we recall is an *elliptic* estimate: suppose that $A, B \in \Psi^0(X)$ satisfy

$$\operatorname{WF}(A) \cap (\operatorname{Char}(B) \cup \operatorname{Char}(P)) = \emptyset, \tag{2.1}$$

Then for any N there exists a constant K such that

$$\|Au\|_{H^{s+m}} \leq K\|BPu\|_{H^s} + K\|u\|_{H^{-N}}. \tag{2.2}$$

We now move to *propagation* estimates. For that we assume that in an open conic subset of $U \subset T^*X$,

$$q(x,\xi) \geq 0, \quad (x,\xi) \in U, \tag{2.3}$$

and

$$p(x,\xi) = 0 \implies H_p \text{ and } \xi\partial_\xi \text{ are linearly independent at } (x,\xi). \tag{2.4}$$

The *radial* vector field $\xi\partial_\xi$ is invariantly defined as the generator of the $\mathbb{R}_+$ action on $T^*X \setminus 0$ (multiplication of one forms by positive scalars).

The basic propagation estimate is given as follows: suppose that $A, B, B_1 \in \Psi^0(X)$ have wave front sets contained in U.

In addition we assume that

$$(\operatorname{WF}(A) \cup \operatorname{WF}(B)) \cap \operatorname{WF}(I - B_1) = \emptyset,$$

and that $\operatorname{WF}(A)$ is *forward controlled* by $\complement \operatorname{Char}(B)$ in the following sense:

$$\forall\, (x,\xi) \in \operatorname{WF}(A)\ \exists T > 0 \quad \exp(-TH_p)(x,\xi) \notin \operatorname{Char}(B). \tag{2.5}$$

The crucial estimate is then given by

$$\|Au\|_{H^{s+m-1}} \leq K\|B_1 Pu\|_{H^s} + K\|Bu\|_{H^{s+m-1}} + K\|u\|_{H^{-N}}, \tag{2.6}$$

where N is arbitrary and K is a constant depending on N. A direct proof can be found in [H]. The estimate is valid with $u \in \mathcal{D}'(X)$ for which the right hand side is finite – see [DZ2, Exercise E.27]. For our purposes it is enough to prove it for $u \in H^{s+m-1}$ with $Pu \in H^s$.

When the condition (2.3) is change to $q \leq 0$ we need to change the sign in (2.5):

$$\forall\,(x,\xi) \in \mathrm{WF}(A)\ \exists\, T > 0 \quad \exp(+TH_p)(x,\xi) \notin \mathrm{Char}(B)$$

and (2.6) still holds. In that case we say that $\mathrm{WF}(A)$ is *backward controlled* by $\complement\,\mathrm{Char}(B)$.

We outline the standard proof so that the argument for radial estimates (that is, estimates when (2.4) is violated) in §4 is clear.

We note that to establish (2.6) we need to show that

$$\|Au\|_{H^{s+m-1}} \leq K\|B_1 Pu\|_{H^s} + K\|Bu\|_{H^{s+m-1}} + K\|B_1 u\|_{H^{s+m-\frac{3}{2}}} + K\|u\|_{H^{-N}}, \tag{2.7}$$

as we can then proceed by induction by using a "nested" sequence of operators, replacing B_1 and A at each step. We first consider the case of $m = 1$ and $s = 0$. It is enough to prove (2.7) for $u \in C^\infty(X)$ as to obtain the case of $u \in H^{s+m-1}$, $Pu \in H^s$, one can then proceed by approximation – see [DZ2, Lemma E.41].

The key step in proving (2.7) is the construction of an *escape function* $f \in S^0(T^*X)$, homogeneous of degree 0 outside a compact set, with the following properties:

$$\begin{gathered} f(x,\xi) \geq 0,\ (x,\xi) \in T^*X, \quad f(x,\xi) > 0,\ (x,\xi) \in \mathrm{WF}(A), \\ f(x,\xi) H_p f(x,\xi) \leq -\beta f(x,\xi)^2 + C_0 b(x,\xi)^2, \quad b \equiv \sigma(B), \\ \operatorname{supp} f \cap \mathrm{WF}(I - B_1) = \emptyset. \end{gathered} \tag{2.8}$$

Such f can be constructed for any $\beta > 0$ – see Fig. 1 and [DZ2, Lemma E.43].

We choose

$$F \in \Psi^0(X), \quad \sigma(F) \equiv f, \quad \mathrm{WF}(F) \cap \mathrm{WF}(I - B_1) = \emptyset. \tag{2.9}$$

We write

$$\operatorname{Re} P = (P + P^*)/2, \quad \operatorname{Im} P = (P - P^*)/2i,$$

so that $\sigma(\operatorname{Re} P) \equiv p$ and $\sigma(\operatorname{Im} P) = q$. With this notation we have, for $u \in C^\infty(X)$,

$$\operatorname{Im}\langle Pu, F^*Fu\rangle = \langle \tfrac{i}{2}[\operatorname{Re} P, F^*F]u, u\rangle + \operatorname{Re}\langle \operatorname{Im} P\, u, FF^*u\rangle. \tag{2.10}$$

We now put

$$H := \tfrac{i}{2}[\operatorname{Re} P, F^*F] \in \Psi^0(X), \quad \sigma(H) \equiv f H_p f,$$

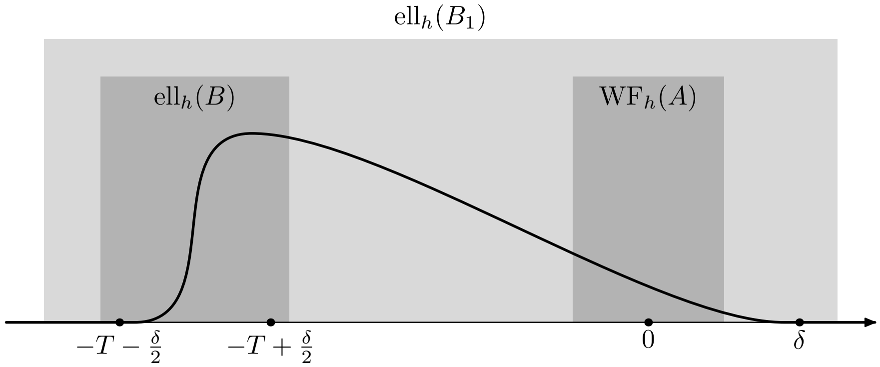

FIGURE 1. The escape function g along one flow line and the restrictions of $\mathrm{WF}_h(A), \mathrm{ell}_h(B), \mathrm{ell}_h(B_1)$ to this line. This figure is borrowed from [DZ2, Fig. E.2] where the more general semiclassical case is considered; we refer to [DZ2, Appendix E] for further details.

and note that in expressions involving Fu, u can be replaced by $B_1 u$ at the expense of errors controlled by $\|u\|_{H^{-N}}$ (that follows from the the condition in (2.9)). Hence, the sharp Gårding inequality [H3, §18.1] and (2.8) imply that

$$\langle Hu, u\rangle \leq -\beta\|Fu\|_{L^2}^2 + C_0\|Bu\|_{L^2}^2 + C\|B_1 u\|_{H^{-1/2}}^2 + C\|u\|_{H^{-N}}^2. \tag{2.11}$$

On the other hand, the sharp Gårding inequality applied to $\sigma(\mathrm{Im}\, P) \equiv q \leq 0$ and the fact that $\mathrm{Re}(F^*[F, \mathrm{Im}\, P]) \in \Psi^{-1}(X)$ give

$$\begin{aligned}\mathrm{Re}\langle(\mathrm{Im}\, P)u, FF^*\rangle &= \mathrm{Re}\langle(\mathrm{Im}\, P)Fu, Fu\rangle + \mathrm{Re}\langle F^*[F, \mathrm{Im}\, P]u, u\rangle \\ &\leq C_1\|Fu\|_{L^2}^2 + C\|B_1 u\|_{H^{-1/2}}^2 + C\|u\|_{H^{-N}}^2,\end{aligned}$$

where the constant C_1 is (obviously) *independent* of F.

Returning to (2.10) we see that for $u \in C^\infty(X)$,

$$\mathrm{Im}\langle Pu, F^*Fu\rangle \leq -(\beta - C_1)\|Fu\|_{L^2}^2 + C_0\|Bu\|_{L^2}^2 + C'\|B_1 u\|_{H^{-1/2}}^2 + C'\|u\|_{H^{-N}}^2,$$

By choosing $\beta > C_1 + 1$ we conclude that

$$\|Fu\|^2 \leq \|Pu\|_{L^2}\|Fu\|_{L^2} + C\|Bu\|_{L^2}^2 + C\|B_1 u\|_{H^{-1/2}}^2 + C\|u\|_{H^{-N}}^2.$$

Since F is elliptic on $\mathrm{WF}(A)$ the estimate (2.7) (with $m = 1$ and $s = 0$) follows.

To obtain the estimate for arbitrary m and s we change P to $\langle D\rangle^{s-m+1}P\langle D\rangle^{-s}$ and u to $\langle D\rangle^s u$ and apply the same argument.

Eventually, it is much better to formulate this results in terms of the flow on the compactified cotangent bundle, especially with semiclassical extensions in mind – see [DZ1, Appendix C] and [DZ2, Appendix E]. In this presentation we refer only to the classical version of the result.

3. Review of hyperbolic estimates

Consider

$$P = D_t^2 - D_x^2, \quad (t,x) \in [0,1] \times \mathbb{S}^1. \tag{3.1}$$

On what space does this operator become invertible?

To answer we recall the notions of Sobolev spaces on manifolds with boundary from [H3, §B.2]. Let $\mathbb{R}^n_+ := \{x_n > 0\}$ and let $H^s(\mathbb{R}^n)$ denote the usual Sobolev space. Then

$$\begin{gathered} \bar{H}^s(\mathbb{R}^n_+) := \{u \in \mathcal{D}'(\mathbb{R}^n_+) : \exists\, U \in H^s(\mathbb{R}^n), \ u = U|_{\mathbb{R}^n_+}\}, \\ \|u\|_{\bar{H}^s(\mathbb{R}^n_+)} = \min\{\|U\|_{H^s(\mathbb{R}^n)} : u = U|_{\mathbb{R}^n_+}\}, \\ \dot{H}^s(\bar{\mathbb{R}}^n_+) := \{u \in H^s(\mathbb{R}^n) : \operatorname{supp} u \subset \bar{\mathbb{R}}^n_+\}, \quad \|u\|_{\dot{H}^s(\bar{\mathbb{R}}^n_+)} = \|u\|_{H^s(\mathbb{R}^n)}. \end{gathered} \tag{3.2}$$

For $s \in \mathbb{R}$ the spaces $\dot{H}^{-s}(\bar{\mathbb{R}}^n_+)$ and $\bar{H}^s(\mathbb{R}^n_+)$ are dual with respect to the the L^2-inner product (extended from $C^\infty_{\rm c}(\mathbb{R}^n_+) \times \bar{C}^\infty_{\rm c}(\mathbb{R}^n_+)$ – see [H3, Theorem B.2.1]). These notions extend to manifolds with smooth boundaries and if the boundaries has multiple components we can have different Sobolev spaces near each of them. For instance we can define

$$\begin{gathered} \widetilde{H}^s([0,1) \times \mathbb{S}^1) := \{u : \exists\, U \in \dot{H}^s([0,\infty) \times \mathbb{S}^1) : u = U|_{[0,1)\times\mathbb{S}^1}\} \\ \widetilde{H}^s((0,1] \times \mathbb{S}^1) := \{u : \exists\, U \in \dot{H}^s((-\infty,1]) \times \mathbb{S}^1) : u = U|_{(0,1]\times\mathbb{S}^1}\}. \end{gathered} \tag{3.3}$$

We now put

$$\mathscr{X}_s = \{u \in \widetilde{H}^{s+1}([0,1) \times \mathbb{S}^1) : Pu \in \widetilde{H}^s([0,1) \times \mathbb{S}^1)\}, \quad \mathscr{Y}_s = \widetilde{H}^s([0,1) \times \mathbb{S}^1). \tag{3.4}$$

with obvious inner products. What is less obvious is that $\widetilde{C}^\infty$ (restrictions to $[0,1) \times \mathbb{S}^1$ oif smooth functions supported in $t > 0$) is dense in $\mathscr{X}_s$ (see for instance [DZ2, Lemma E.41]).

Then in fact

$$P : \mathscr{X}_s \to \mathscr{Y}_s \tag{3.5}$$

is an *invertible* operator (and hence a Fredholm operator).

This in fact follows from general results about strictly hyperbolic operators – see [H3, §23.2]. We recall the case which is important to us.

Suppose that $P = D_t^2 - R(t,x,D_x)$, $x \in N$, where N is a compact manifold and $R \in C^\infty(\mathbb{R}_t; \mathrm{Diff}^2(N))$. Assume that P is strictly hyperbolic with respect t. For any $T > 0$ and $s \in \mathbb{R}$, let $\widetilde{H}^s([0,T) \times N)$ be defined by (3.3) (with $\mathbb{S}^1$ replaced by N and 1 by T). Then

$$\forall\, f \in \widetilde{H}^s([0,T) \times N) \ \exists!\, u \in \widetilde{H}^{s+1}([0,T) \times N), \quad Pu = f. \tag{3.6}$$

In particular, if we define the spaces $\mathscr{X}_s$ and $\mathscr{Y}_s$ as in (3.4) then the operator (3.5) is invertible.

If P^* is the (formal) adjoint of P with respect to an L^2 inner product then

$$P^* : \widetilde{\mathscr{Y}}_{-s} \to \widetilde{\mathscr{X}}_{-s} \tag{3.7}$$

is invertible, where

$$\begin{aligned} \widetilde{\mathscr{X}}_r &= \{u \in \widetilde{H}^r((0,T] \times N) : P^* u \in \widetilde{H}^{r-1}([0,T) \times N)\}, \\ \widetilde{\mathscr{Y}}_r &= \widetilde{H}^{r-1}((0,T] \times N). \end{aligned} \tag{3.8}$$

These are not the dual space but they can be used to prove Fredholm properties of operators similar to P.

In our application we will need the following estimates which can be concluded from the invertibility of P and P^*: if $u \in \bar{H}^s((0,T) \times N)$ then

$$\|u\|_{\bar{H}^{s+1}((T/2,T)\times N)} \le C\|Pu\|_{\bar{H}^s((0,T)\times N)} + C\|u\|_{\bar{H}^{s+1}((0,T/2))\times N)}. \tag{3.9}$$

If $v \in \widetilde{H}^{-s}((0,T] \times N)$ and $P^* v \in H^{-s-1}$ then

$$\|v\|_{\widetilde{H}^{-s}((0,T]\times N)} \le C\|P^* v\|_{\widetilde{H}^{-s-1}((0,T]\times N)}. \tag{3.10}$$

We stress that the support condition in the definition of $\widetilde{H}^{-s}$ effectively fixes zero "initial" (final) condition at $t = T$.

4. Radial estimates in the model case

To obtain meromorphic continuation of the resolvent (1.1) we need propagation estimates at *radial* points in addition to the standard propagation estimates reviewed in §2. These estimates were developed by Melrose [M] in the context of scattering theory on asymptotically Euclidean spaces and are crucial in the Vasy approach [V1]. A semiclassical version valid for very general radial sets was given in Dyatlov–Zworski [DZ1] (see also [DZ2, Appendix E]).

Suppose that $X = \mathbb{R} \times \mathbb{S}^1$ and $P \in \Psi^2(X)$ with $\sigma(P)$ represented by $p \in S^2(T^*X)$ where

$$p = x_1 \xi_1^2 + \xi_2^2. \tag{4.1}$$

The Hamilton vector field is given by

$$H_p = \xi_1(2x_1\partial_{x_1} - \xi_1\partial_{\xi_1}) + 2\xi_2\partial_{x_2}. \tag{4.2}$$

We see that the condition (2.4) is violated at

$$\Gamma := \{(0, x_2, \xi_1, 0) : x_2 \in \mathbb{S}^1, \xi_1 \in \mathbb{R} \setminus 0\} \subset T^*X \setminus 0. \tag{4.3}$$

Nevertheless we have the following propagation estimates valid in spaces with restricted regularity.

There exists $s_0 \ge 0$ (depending on lower order terms in P) such that for $s \ge s_0$

$$\|Au\|_{H^{s+1}} \le K\|B_1 Pu\|_{H^s} + K\|B_1 u\|_{H^{s_0}} + K\|u\|_{H^{-N}}, \tag{4.4}$$

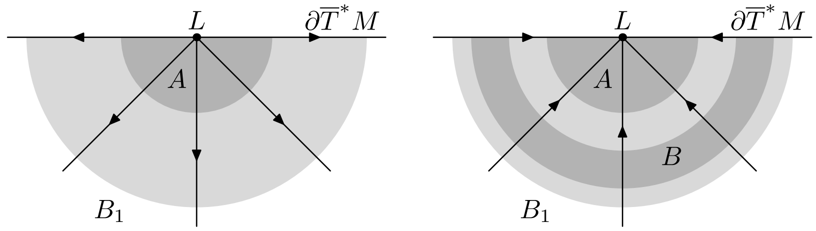

FIGURE 2. An illustration of the behaviour of the Hamilton flows for radial sources (left, $L = \partial\overline{\Gamma}_+$, boundary of the compactification of the conic set Γ_+ – see below) and for radial sinks (right, $L = \partial\overline{\Gamma}_-$) and of the localization of operators in the estimates (4.4) and (4.5) respectively. The horizontal line on the top denotes the boundary, $\partial\overline{T}^*X$, of the *compactified* cotangent bundle $\overline{T}^*X$. The shaded half-discs then correspond to conic neighbourhoods in T^*X. Near $\Gamma_\pm$ an explicit (projective) compactification is given by $r = 1/|\xi_1|$, (so that $\partial\overline{T}^*X = \{r = 0\}$), $\theta = \xi_2/|\xi_1|$, with x (the base variable) unchanged. In this variables, near $\partial\overline{\Gamma}_\pm$ (boundaries of compactifications of $\Gamma_\pm$ we check that $r\partial_r = -\xi_1\partial_{\xi_1} - \xi_2\partial_{\xi_2}$ and $\theta\partial_\theta = \xi_2\partial_{\xi_2}$. Hence near $\Gamma_\pm$, $H_p = \pm r(\theta\partial_\theta + r\partial_r + 2x_1\partial_{x_1} + 2\theta\partial_{x_2})$ and (after rescaling) we see a source and a sink.

for any $A, B_1 \in \Psi^0$ such that

$$\mathrm{WF}(A) \cap \mathrm{WF}(I - B_1) = \emptyset, \quad \mathrm{WF}(B_1) \subset U,$$

where U is a (small) conic neighbourhood of

$$\Gamma_+ := \Gamma \cap \{\xi_1 > 0\},$$

with Γ given by (4.3). That is a *radial source estimate.*

Remark. We prove (4.4) for $u \in C^\infty$ but the approximation argument [DZ2, Lemma E.41] shows that it is then valid for $u \in H^{s+1}$ such that $Pu \in H^s$. However using a standard regularization argument [DZ2, Exercise E.27] (4.4) holds for all $u \in \mathcal{D}'(X)$ such that the right hand side is finite. That will be used in §8 where we need the fact that $Pu = 0$ and $u \in H^{s_0}$ implies $u \in C^\infty$.

Now suppose that A and B satisfy the same conditions but with U a (small) conic neighbourhood of

$$\Gamma_- := \Gamma \cap \{\xi_1 < 0\}.$$

(The smallness of U means that U stays away from $\Gamma_+ := \Gamma \cap \{\xi_1 > 0\}$.) Assume in addition $B \in \Psi^0(X)$, $\mathrm{WF}(B) \subset \mathrm{WF}(B_1)$ and that $\complement\,\mathrm{Char}(B)$ forward controls $\mathrm{WF}(A)$ in the sense of (2.5).

Then, with the same s_0 as in (4.4), for $s \geq s_0$,

$$\|Au\|_{H^{-s+1}} \leq K\|B_1P^*u\|_{H^{-s}} + K\|Bu\|_{H^{-s}} + K\|u\|_{H^{-N}}. \tag{4.5}$$

This is a *radial sink estimate*. The principal symbols of P and P^* are the same but s_0 depends on lower order terms – see (4.9).

Proof of (4.4). We would like to mimic the proof (2.7) but with F producing a global good sign for the commutator (since in (2.7) we do not have the control term Bu). We take $\psi_1 \in C_c^\infty((-2\delta, 2\delta); [0,1])$, $\psi_1(t) = 1$, for $|t| < \delta$, $t\psi_1'(t) \le 0$, and $\psi_2 \in C^\infty(\mathbb{R})$, $\psi_2(t) = 0$ for $t \le 1$, $\psi_2(t) = 1$, $t \ge 2$, and propose

$$F_s := \psi_1(x_1)\psi_1(D_{x_2}/D_{x_1})\psi_2(D_{x_1})D_{x_1}^{s-\frac{1}{2}} \in \Psi^{s-\frac{1}{2}}(X),$$
$$\sigma(F_s) \equiv f_s(x,\xi) = \psi_1(x_1)\psi_1(\xi_2/\xi_1)\psi_2(\xi_1)\xi_1^{s-\frac{1}{2}}.$$

We note that because of the cut-off ψ_2, $D_{x_1}^s$ and D_{x_2}/D_{x_1} are well defined.

For $|\xi| > 3$ (which implies that $\xi_1 > 2$ on the support of f_s if δ is small enough),

$$\begin{aligned} H_p f_s(x,\xi) = \xi_1^{s+\frac{1}{2}} & \big(2x_1\psi_1'(x_1)\psi_1(\xi_2/\xi_1) + \psi_1(x_1)(\xi_2/\xi_1)\psi_1'(\xi_2/\xi_1) \\ & -(s-\tfrac{1}{2})\psi_1(x_1)\psi_1(\xi_2/\xi_1)\big)\,\psi_2(\xi_1) \le -(s-\tfrac{1}{2})\xi_1 f_s. \end{aligned}$$

In particular,

$$f_s H_p f_s + (s-\tfrac{1}{2})\xi_1 f_s^2 \le 0, \qquad |\xi| > 3. \tag{4.6}$$

We now repeat the argument reviewed in §2 paying more attention to lower order terms. We write

$$\begin{aligned} & P = P_0 + R + iQ, \quad P_0 := x_1 D_{x_1}^2 + D_{x_2}^2 - iD_{x_1}, \\ & R, Q \in \Psi^1(X), \quad P_0 = P_0^*, \quad R^* = R, \quad Q = Q^*, \end{aligned} \tag{4.7}$$

where we take formal adjoints with respect to the measure $dx_1 dx_2$. Hence, arguing as in the proof of (2.7),

$$\operatorname{Im}\langle Pu, F_s^* F_s\rangle \le \langle Hu, u\rangle + \operatorname{Re}\langle QFu, Fu\rangle + C\|B_1 u\|^2_{H^{s-\frac{1}{2}}} + C\|u\|^2_{H^{-N}},$$

where now $H = \frac{i}{2}[P_0, F_s^* F_s]$.

From this and (4.6) we see that if

$$((s - \frac{1}{2})\xi_1 - \sigma(Q))f_s^2 \ge c\xi_1 f_s^2, \quad c > 0, \tag{4.8}$$

then

$$\|F_s u\|^2_{H^{\frac{1}{2}}} \le C\|B_1 Pu\|_{H^{s-1}}\|F_s u\|_{H^{\frac{1}{2}}} + C\|B_1 u\|^2_{H^{s-\frac{1}{2}}} + C\|u\|^2_{H^{-N}}.$$

Ellipticity of $F_s \in \Psi^{s-\frac{1}{2}}$ on $\mathrm{WF}(A)$ (assumed to lie in a small neighbourhood of Γ_+) shows that

$$\|Au\|_{H^s} \le C\|B_1 Pu\|_{H^{s-1}} + C\|B_1 u\|_{H^{s-\frac{1}{2}}} + C\|u\|_{H^{-N}}.$$

We need $s - \frac{1}{2} > s_0$ where (4.8) gives us the natural condition on s_0:

$$-s_0 + \xi_1^{-1}\sigma(Q) \le 0 \text{ in a conic neighbourhood of } \Gamma_+. \tag{4.9}$$

□

Remark. The estimate (4.4) holds also for sinks but the condition (4.9) changes. We apply the argument above to $-P$ (so that the direction of the flow changes) and then we need

$$-s_0 - \xi_1^{-1}\sigma(Q) \leq 0 \text{ in a conic neighbourhood of } \Gamma_-. \tag{4.10}$$

The proof of (4.5) is similar with a different sign and a need for a control operator B (since propagation is now towards the region where A lives and the sign of the terms coming from $\psi(x_1)$ and $\psi(\xi_2/\xi_1)$ is "wrong").

5. Fredholm property for the extended operator

Returning to (1.1) we consider solving

$$(-\Delta_g - \lambda^2 - \tfrac{1}{4})u_\pm = f \in C_{\rm c}^\infty(M), \quad M = (0,1)_{y_1} \times \mathbb{S}^1 \cup M_1, \quad \partial M_1 = \{y_1 = 1\} \subset M_0,$$

demanding that

$$u_\pm(x) = y_1^{\mp i\lambda + \frac{1}{2}} F_\pm(x), \quad F_\pm|_{M_0} \in C^\infty([0,1] \times \mathbb{S}^1),$$

see for instance [GZ, (3.6)] for a general discussion. These are the *outgoing* (+) and *incoming* (-) solutions.

In particular, for $\pm \operatorname{Im} \lambda > 0$, $\lambda^2 + \frac{1}{4} \notin \operatorname{Spec}(-\Delta_g)$, a unique solution exists and it lies in $L^2(M, d\mathrm{vol}_g)$. In view of (1.1) we see that

$$u_+ = (-\Delta_g - \lambda^2 - \tfrac{1}{2})^{-1} f, \quad y_1^{i\lambda - \frac{1}{2}} u_+|_{M_0} \in C^\infty([0,1] \times \mathbb{S}^1), \tag{5.1}$$

see §6 for a direct proof of a stronger statement.

In this terminology, we want to construct the meromorphic continuation of the outgoing resolvent.

Of course (except in special cases), $y_1^{i\lambda - \frac{1}{2}} u_- \notin C^\infty([0,1] \times \mathbb{S}^1)$. In view of (5.1) the outgoing condition can be formulated by requiring *regularity* of solutions to an equation conjugated by $y_1^{-i\lambda - \frac{1}{2}}$ and solved by F_+. Here we extend y_1 to be a constant outside of a neighbourhood of 0 – there are some *subtleties* in the semiclassical case when we need invertibility of the extended operator.

Hence we consider

$$y_1^{i\lambda - \frac{1}{2}}(-\Delta_g - \lambda^2 - \tfrac{1}{4}) y_1^{-i\lambda + \frac{1}{2}} = y_1^2 D_{y_1}^2 - (2\lambda + i) y_1 D_{y_1} + y_1^2 D_{y_2}^2. \tag{5.2}$$

Then we perform the following change of variables

$$x_1 = y_1^2, \quad x_2 = y_2$$

so that $D_{y_1} = 2y_1 D_{x_1}$, $D_{y_1}^2 = 4y_1^2 D_{x_1}^2 - 2iD_{x_1}$, and

$$y_1^{i\lambda - \frac{1}{2}}(-\Delta_g - \lambda^2 - \tfrac{1}{4}) y_1^{-i\lambda + \frac{1}{2}} = x_1 \left(4x_1 D_{x_1}^2 - 4(\lambda + i) D_{x_1} + D_{x_2}^2\right). \tag{5.3}$$

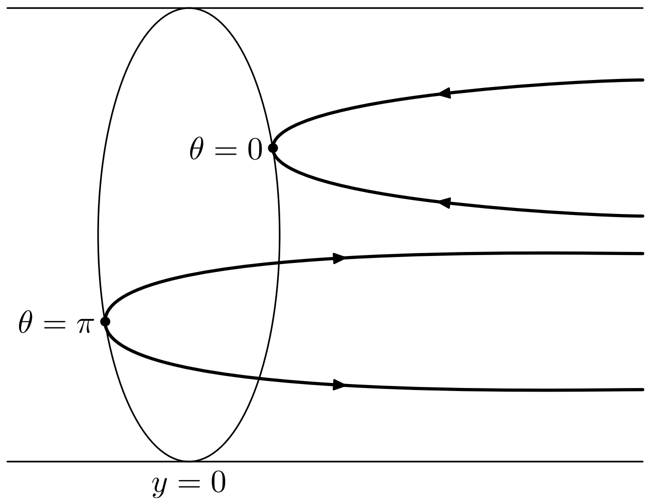

FIGURE 3. The radial sets and the direction of the flow: $y = -x_1$, $(\xi_1, \xi_2) = -|\xi|(\cos\theta, \sin\theta)$. The radial *source*, Γ_+, corresponds to $\theta = \pi$ and the radial sink, Γ_-, to $\theta = 0$.

We now define

$$P_0(\lambda) := 4x_1 D_{x_1}^2 - 4(\lambda + i)D_{x_1} + D_{x_2}^2, \quad (x_1, x_2) \in X_0 := (0,1) \times \mathbb{S}^1. \tag{5.4}$$

For $x_1 < 0$ the operator $P_0(\lambda)$ in (5.4) is hyperbolic and we can simply extend it by imposing *no* boundary conditions at $x_1 = -1$.

Remark. For $\lambda \in \mathbb{R}$ the operator $-\Delta_g$ is self–adjoint with respect to the measure

$$\frac{dy_1}{y_1^2}\, dy_2.$$

Hence the conjugated operator is self-adjoint with respect to the measure

$$\frac{dy_1}{y_1}\, dy_2 = \frac{1}{2}\frac{dx_1}{x_1}\, dx_2.$$

Since that operator is equal to $x_1 P(\lambda)$ we see that $P(\lambda)$ is formally self-adjoint with respect to $dx_1 dx_2$. All the adjoints will now be taken with respect to this measure (what really matters to us the behaviour at $x_1 = 0$). With this convention we see that, for all $\lambda \in \mathbb{C}$,

$$P(\lambda)^* = P(\bar{\lambda}). \tag{5.5}$$

The spaces are now defined as

$$\mathscr{Y}_s := \bar{H}^s(X), \qquad \mathscr{X}_s := \{u \in \bar{H}^{s+1}(X) : P(\lambda)u \in \bar{H}^s(X)\}, \tag{5.6}$$

where

$$X = (-1,1) \times \mathbb{S}^1 \cup M_1, \quad \partial X = \{-1\} \times \mathbb{S}^1,$$

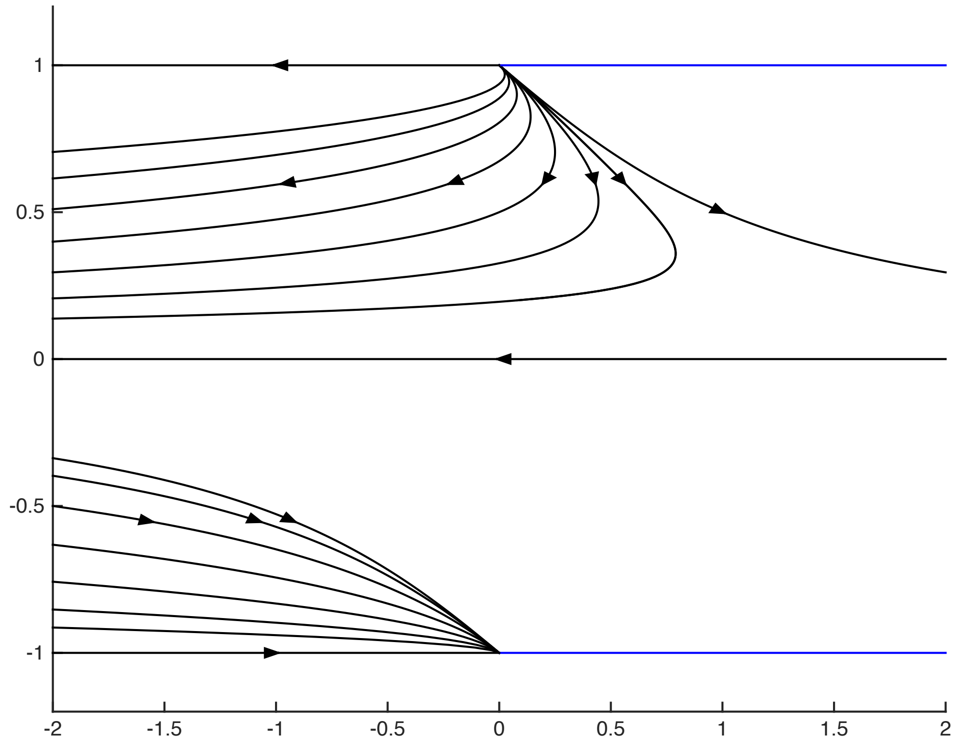

Figure 4. Plots of the characteristic sets of the semiclassical principal symbol, $p(x,\xi) = x_1\xi_1^2 - z\xi_1 + \xi_2^2$, for $z = 1$ and different values of ξ_2. The x-axis corresponds to x_1 and the y-axis to $\rho := 2\arctan(\xi_1)/\pi$. The source $x_1 = 0$, $\rho = 1$ and the sink $x_1 = 0$, $\rho = -1$ are shown with the flow lines of H_p. The key fact is the separation of the characteristic set into two disjoint components corresponding to $\rho \geq 0$ and $\rho < 0$. Understanding of this flow is important when considering the high energy problem. See also Fig. 5 where the characteristic sets for more values of ξ_2 are shown and the separation is even more dramatic.

and the Sobolev spaces $\bar{H}^s$ are defined in (3.2). We note that there is no dependence on lower order terms in the definition so effectively we could demand that $(x_1 D_{x_1}^2 + D_{x_2}^2)u \in \bar{H}^s(X)$.

We now have the following crucial result:

$$P(\lambda) : \mathscr{X}_s \longrightarrow \mathscr{Y}_s \quad \text{is a Fredholm operator for } s + frac12 > -\operatorname{Im}\lambda. \tag{5.7}$$

Proof of (5.7). If $\chi_+ \in C_c^\infty$, $\operatorname{supp}\chi_+ \subset \{x_1 > 0\}$ then elliptic estimates show that

$$\|\chi_+ u\|_{H^{s+1}} \leq \|\chi_+ u\|_{H^{s+2}} \leq C\|Pu\|_{H^s} + C\|u\|_{H^{-N}}.$$

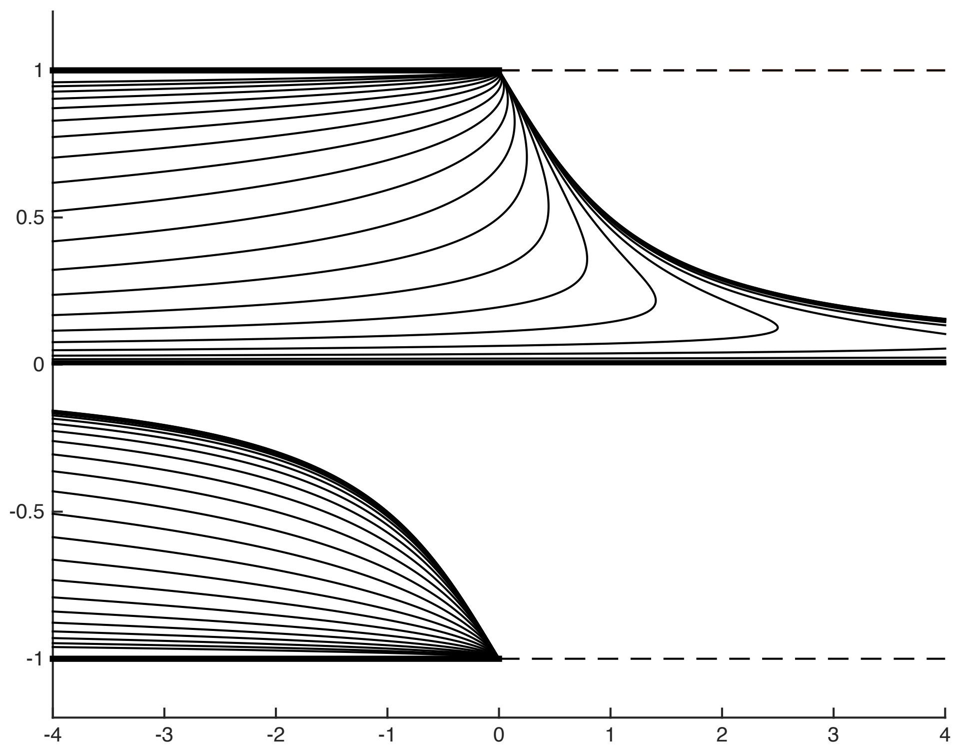

FIGURE 5. Same plot as in Fig. 4 but with more values of ξ_2. The separation of the two components of the characteristic set is visible more dramatically. What should be noted that the only way for a trajectory to go from $x_1 = 0$ into $x_1 > 0$ and back is by starting at the radial source. Trajectories which enter $x_1 < 0$ from $x_1 > 0$ never come back – that is crucial for the non-trapping estimates.

Near $x_1 = 0$ we use the estimates (4.4) which give for, for $\chi_0 \in C_c^\infty$, $\operatorname{supp} \chi_0 \subset \{|x_1| < 1/2\}$

$$\|\chi_0 u\|_{H^{s+1}} \leq C\|Pu\|_{H^s} + C\|u\|_{H^{s_0}}, \quad s > s_0$$

(We need to microlocalize to neighbourhoods of $\{\pm\xi_1 > |\xi|/C\}$, use (4.4) for P and $-P$ respectively – see the remark preceding (4.10). Elsewhere the operator is elliptic.) To calculate the value of s_0 we return to (4.9) which we need to rescale:

$$-4s_0 + \xi_1^{-1}\sigma(Q) \leq 0,$$

where in the notation of (4.7)

$$P = 4x_1 D_{x_1}^2 - 4(\lambda + i) D_{x_1} + D_{x_2}^2, \quad P_0 = 4x_1 D_{x_1}^2 + D_{x_2}^2 - 4iD_{x_1},$$
$$Q = -4 \operatorname{Im} \lambda D_{x_1}, \quad R = -4 \operatorname{Re} \lambda D_{x_1}.$$

Hence we need,

$$s_0 \geq - \operatorname{Im} \lambda \tag{5.8}$$

Finally if χ_- is supported in $\{x_1 < -1/3\}$ then the hyperbolic estimate (3.9) shows that

$$\|\chi_- u\|_{\bar{H}^{s+1}(X)} \leq C \|Pu\|_{\bar{H}^s(X)} + C \|u\|_{\bar{H}^{s+1}(\{x_1 > 1/3\}))}.$$

Putting these estimates together gives

$$\|u\|_{\mathscr{X}^s} \leq \|Pu\|_{\mathscr{Y}^s} + \|u\|_{H^{s_0}}.$$

If $s > s_0$ this immediately shows that the kernel of P is finite dimensional.

To show that the range of P on $\mathscr{X}_s$ is of finite codimension we note that if $v \in \dot{H}^{-s}$ is orthogonal to Pu for all $u \in \bar{C}^\infty(X)$ (we recall that $\bar{C}^\infty(X)$ is dense in $\mathscr{X}^s$ – see the remark after (3.4)), then $P^* v = 0$. We can then use (3.10) to see that $v \equiv 0$ in $\{x_1 < 0\}$. For $s > s_0$ the estimates (4.5) show that

$$\|\chi_0 v\|_{\dot{H}^{-s}(X)} \leq C \|P^* v\|_{\dot{H}^{-s-1}(X)} + C \|v\|_{\dot{H}^{-N}(X)} = C \|v\|_{\dot{H}^{-N}(X)}.$$

(The control term is negligible as $v = 0$ for $x_1 < 0$ and all the bicharacteristics propagate into that region – see Fig. 3.) Finally we have the elliptic estimate

$$\|\chi_+ v\|_{\dot{H}^{-s}} \leq C \|v\|_{\dot{H}^{-N}},$$

and hence

$$P^* v = 0, \ v \in \dot{H}^{-s}(X) \implies \|v\|_{\dot{H}^{-s}} \leq C \|v\|_{\dot{H}^{-N}},$$

and hence the space of $v \in \dot{H}^{-s}$ such that $\langle v, Pu \rangle = 0$, for all $u \in \mathscr{X}_s$, is finite dimensional. (To show that this implies that the image of $\mathscr{X}_s$ has finite codimension requires a standard functional analysis argument – see [H4, Proof of Theorem 26.1.7].) □

6. Asymptotic expansions

Let $X_1 = (0,1) \times \mathbb{S}^1 \cup M_1$ and consider $P_0(\lambda)$ given by (5.4) in $x_1 > 0$. Under the changes of variables $y_1^2 = x_1$ we have

$$-\Delta_g - \lambda^2 - \tfrac{1}{4} = y_1^{-i\lambda + \frac{1}{2}} y_1^2 P_0(\lambda) y_1^{i\lambda - \frac{1}{2}},$$

and

$$(-\Delta_g - \lambda^2 - \tfrac{1}{4})^{-1} : L^2(M, d\mathrm{vol}_g) \longrightarrow L^2(M, d\mathrm{vol}_g), \quad \operatorname{Im} \lambda > 0$$

(we always assume that $\lambda^2 + \frac{1}{4} \notin \operatorname{Spec}(\Delta_g)$). Hence we have a formal inverse,

$$P_0(\lambda)^{-1} = x_1^{\frac{i\lambda}{2}-\frac{1}{4}}(-\Delta_g - \lambda^2 - \tfrac{1}{4})^{-1} x_1^{-\frac{i\lambda}{2}+\frac{5}{4}}.$$

with the mapping property

$$P_0(\lambda)^{-1} : x_1^{-\frac{\rho}{2}-\frac{1}{2}} L^2 \to x_1^{-\frac{\rho}{2}+\frac{1}{2}} L^2, \quad \rho := \operatorname{Im}\lambda > 0. \tag{6.1}$$

Here we used the fact that $2dy_1/y_1^2 = dx_1/x_1^{3/2}$ and that

$$L^2(y_1^{-2} dy_1 dy_2) = L^2\left(x_1^{-\frac{3}{2}} dx_1 dx_2\right) = x_1^{\frac{3}{4}} L^2, \quad L^2 := L^2(dx_1 dx_2).$$

In particular,

$$P_0(\lambda)^{-1} : C^\infty_{\rm c}(X_1) \longrightarrow x_1^{-\frac{\rho}{2}+\frac{1}{2}} L^2, \quad \rho = \operatorname{Im}\lambda > 0, \tag{6.2}$$

We claim that in fact we have a stronger mapping property than (6.2):

$$P_0(\lambda)^{-1} : C^\infty_{\rm c}(X_1) \longrightarrow \bar{C}^\infty(X_1), \quad \operatorname{Im}\lambda \geq c_0, \quad i\lambda \notin \mathbb{Z}. \tag{6.3}$$

This implies a stronger version of (5.1) since the smoothness is now in (y_1^2, y_2). Here $c_0 > 0$ is some fixed constant.

To prove (6.3) we will use a classical tool for obtaining asymptotic expansions, the *Mellin transform* – see [Ma, Theorem 7.3] for a general version and references.

Suppose that

$$P_0(\lambda)u = f, \quad f \in C^\infty_{\rm c}(X_1), \quad u \in x_1^{-\frac{\rho}{2}+\frac{1}{2}} L^2, \quad \rho = \operatorname{Im}\lambda \geq c_0, \tag{6.4}$$

for some sufficiently large constant $c_0 > 0$. We want to show that $u \in \bar{C}^\infty((0,1) \times \mathbb{S}^1)$. By replacing u by $\chi(x_1)u$,

$$\chi \in C^\infty_{\rm c}((-1,1);[0,1]), \quad \chi = 1 \text{ near } 0,$$

we can assume that

$$u \in C^\infty((0,1)_{x_1} \times \mathbb{S}^1) \cap x_1^{-\frac{\rho}{2}+\frac{1}{2}} L^2, \tag{6.5}$$

where smoothness for $x_1 > 0$ follows from ellipticity of $P_0(\lambda)$ there.

We need the following result

$$(6.4) \implies (x_1 D_{x_1})^\ell D^k_{x_2} u \in x_1^{-\frac{\rho}{2}+\frac{1}{2}} L^2 \tag{6.6}$$

This implies that for any k

$$x_1^N u \in C^k([0,1] \times \mathbb{S}^1) \tag{6.7}$$

if N is large enough.

The proof of a more general version is presented in the general case in the Appendix to §9 – this is the only place where the model case is significantly simpler as one can separate variables.

Proof of (6.3). We define the Mellin transform (for functions with support in $[0,1)$) as

$$Mu(s,x_2) := \int_0^1 u(x) x_1^s \frac{dx_1}{x_1}.$$

This is well defined for $\operatorname{Re} s > \rho/2$:

$$\begin{aligned} \|Mu(s,x_2)\|^2_{L^2(dx_2)} &= \int_{\mathbb{S}^1} \left| \int_0^1 x_1^{s+i\lambda/2-1/2} (x_1^{-i\lambda/2-1/2} u(x_1,x_2)) dx_1 \right|^2 dx_2 \\ &\leq \left(\int_0^1 t^{-\rho+2\operatorname{Re} s-1} dt \right) \|x_1^{\rho/2-1/2} u\|_{L^2} \\ &= (2\operatorname{Re} s - \rho)^{-1} \|x_1^{\rho/2-1/2} u\|_{L^2}. \end{aligned}$$

In view of (6.6)

$$s \longmapsto Mu(s,x_2).$$

is a holomorphic family of *smooth* functions in $\operatorname{Re} s > \operatorname{Im} \lambda/2$. We claim now that $Mu(s,x_2)$ continues meromorphically to all of $\mathbb{C}$. In fact,

$$M(x_1 f)(s,x_2) = M(x_1 P(\lambda) u)(s,x_2) = -s(s+i\lambda) Mu(s,x_2) + D^2_{x_2} Mu(s+1,x_2),$$

where $s \mapsto M(x_1 f)(s,x_2)$ is entire as f vanishes near $x_1 = 0$.

Hence,

$$\begin{aligned} Mu(s,x_2) =& \frac{D^{2k}_{x_2} Mu(s+k+1)}{s(s+i\lambda)\cdots(s+k)(s+k+i\lambda)} \\ & - \sum_{j=0}^k \frac{D^{2j}_{x_2} M(x_1 f)(s+j,x_2)}{s(s+i\lambda)\cdots(s+j)(s+j+i\lambda)}, \end{aligned}$$

and that provides a meromorphic continuations with possible poles at $-i\lambda - k$, $k \in \mathbb{N}$.

The Mellin transform inversion formula, a contour deformation and the residue theorem (applied to simple poles thanks to our assumption that $i\lambda \notin \mathbb{Z}$) then give

$$u(x) = x_1^{i\lambda}(b_0(x_2) + x_1 b_1(x_2) + \cdots) + a_0(x_2) + x_1 a_1(x_2) + \cdots,$$

where the regularity of remainders comes from (6.7). (The basic point is that

$$M(x_1^a \chi(x_1))(s) = (s+a)^{-1} F(s), \quad F(s) = -\int x_1^{a+s} \chi'(x_1) dx_1,$$

so that $F(s)$ is an entire function with $F(-a) = 1$.)

Since $Pu(x) = 0$ for $0 < x_1 < \varepsilon$ the equation shows that b_k is determined by $b_0, \cdots b_{k-1}$. We claim that $b_k \equiv 0$: if $b_0 \neq 0$ then

$$v(y) = y_1^{-i\lambda-\frac{1}{2}} u(y_1^2, y_2) = y_1^{i\lambda+\frac{1}{2}} b_0(y_2) + \mathcal{O}(y^{i\lambda+\frac{3}{2}}) + \mathcal{O}(1),$$

would not be in $L^2(M, d\mathrm{vol}_g)$. This completes the proof of (6.3). $\square$

7. Additional hyperbolic estimates

Here we will show that if

$$u \in C^\infty([-1,1]\times \mathbb{S}^1), \quad u|_{x_1\geq 0} \equiv 0, \quad P(\lambda)u = 0 \implies u \equiv 0. \tag{7.1}$$

As pointed out by András Vasy this follows from general properties of the de Sitter wave equation [V3, Proposition 5.3] but we provide a simple direct proof.

Proof of (7.1)*:* We note that if $u|_{x_1\geq -\varepsilon} = 0$ for some $\varepsilon > 0$ then $u \equiv 0$ by standard energy estimates. We want to make that argument quantitative. Hence, for $x_1 < 0$, we write

$$\begin{aligned}\operatorname{div}\left(\frac{1}{2}|x_1|^{-N}(-x_1|u_{x_1}|^2 + |u_{x_2}|^2), -|x_1|^{-N}\operatorname{Re}\bar{u}_{x_1}u_{x_2}\right) &= -|x_1|^{-N}\bar{u}_{x_1}P(\lambda)u \\ - N|x_1|^{-N-1}(-x_1|u_{x_1}|^2 + |u_{x_2}|^2) + \tfrac{1}{2}|x_1|^{-N}|u_{x_1}|^2 &+ (i\lambda - 1)|x_1|^{-N}|u_{x_1}|^2.\end{aligned}$$

Fix $\delta > 0$. Applying the divergence theorem we see that for N large enough (depending on λ) we have

$$\begin{aligned}\int_{\mathbb{S}^1}(|u_{x_1}|^2 + |u_{x_2}|^2)|_{x_1=\delta}\,dx_2 &\leq C\varepsilon^{-N}\int_{\mathbb{S}^1}(|u_{x_1}|^2 + |u_{x_2}|^2)|_{x_1=\varepsilon}\,dx_2 \\ &\leq C_K\varepsilon^{-N+K},\end{aligned}$$

for any K, as $\varepsilon \to 0+$ (since u vanishes to infinite order at $x_1 = 0$). By choosing $K > N$ we see that the left hand side is 0 and that implies that u is zero. □

8. Meromorphic continuation

To show that $P(\lambda)^{-1} : \mathscr{Y}_s \to \mathscr{X}_s$ is a *meromorphic* family of operators for

$$\operatorname{Im}\lambda > -s - \frac{1}{2}$$

we need to find λ_0 such that $P(\lambda_0) : \mathscr{X}_s \to \mathscr{Y}_s$ is invertible. In fact we claim that

$$\operatorname{Im}\lambda_0 > 0, \ \lambda_0^2 + \tfrac{1}{4} \notin \operatorname{Spec}(-\Delta_g), \ s > 0 \implies P(\lambda_0) : \mathscr{X}_s \to \mathscr{Y}_s \text{ is invertible.} \tag{8.1}$$

To prove it we start with

$$\operatorname{Im}\lambda_0 > 0, \ \lambda_0^2 + \tfrac{1}{4} \notin \operatorname{Spec}(-\Delta_g), \ P(\lambda_0)u = 0, \ \ u \in \mathscr{X}_s, \ s > 0 \implies u \equiv 0. \tag{8.2}$$

Proof of (8.2). We note that if $u \in \mathscr{X}_s$ satisfied $P(\lambda_0)u = 0$ then

$$(-\Delta_g - \lambda_0^2 - \tfrac{1}{4})v = 0, \quad v(y) := y_1^{-i\lambda_0 + \frac{1}{2}} u(y_1^2, y_2).$$

For $\operatorname{Im}\lambda_0 > 0$, $v \in L^2(M, d\mathrm{vol}_g)$ that contradicts (1.1) once we make sure that $\lambda_0^2 + \frac{1}{4}$ cannot be an eigevalues of $-\Delta_g$.

Hence $u|_{x_1>0} \equiv 0$, $u \in C^\infty([-1,1]\times\mathbb{S}^1)$, and $P(\lambda) \equiv 0$. From (7.1) we conclude that $u \equiv 0$. □

Proof of (8.1). In view of (8.2) we need to show that $P(\lambda_0)^* w = 0$, $w \in \dot{H}^{-s}(X)$, implies that $w \equiv 0$. It is enough to do this for $\lambda_0 \notin i\mathbb{N}$ since invertibility at one point shows that the index of $P(\lambda)$ is zero and invertibility for all $\operatorname{Im}\lambda > 0$ (except when $\lambda^2 + \frac{1}{4} \in \operatorname{Spec}(-\Delta_g)$) follows from (8.2).

Arguing as in the proof of (5.7) we see that

$$\operatorname{supp} w \subset \overline{X}_1, \quad X_1 := (0,1]\times\mathbb{S}^1 \cup M_1,$$

that is $w \in \dot{H}^{-s}(X_1)$.

We now show that $\operatorname{supp} w \cap X_1 \neq \emptyset$ (that is there is some support in $x_1 > 0$; in fact by unique continuation results for second order elliptic operators (see for instance [H3, §17.2]) this shows that $\operatorname{supp} w = \overline{X}_1$). In other words we we need to show that we cannot have $\operatorname{supp} w \subset \{x_1 = 0\}$. Since $\operatorname{WF}(w) \subset N^*\partial X_1$ we can restrict w to $x_2 = \text{const}$ and it is then a linear combination of $\delta^{(k)}(x_1)$. But $P(\bar{\lambda}_0)(\delta^{(k)}(x_1)) = (k+1-\bar{\lambda}_0/i)\delta^{(k+1)}$ and that does not vanish for $\operatorname{Im}\lambda_0 > 0$.

Mapping property (6.3) and the definition of $P(\lambda)$ show that for any $f \in C^\infty_{\mathrm{c}}(X_1)$ (that is f supported in $x_1 > 0$) there exists $u \in \bar{C}^\infty(X_1)$ such that $P(\lambda_0)u = f$ in X_1. Then (with L^2 inner products meant as distributional pairings),

$$\langle f, w\rangle = \langle P(\lambda_0)u, w\rangle = \langle u, P(\lambda_0)^* w\rangle = 0.$$

Since $w \in \dot{\mathcal{D}}(X_1)$ and $u \in \bar{C}^\infty(X_1)$ the pairing is justified. In view of support properties of w, we can find f such that the left hand side does not vanish. This gives a contradiction. □

Remark. A different proof of the existence of λ_0 with $P(\lambda_0)$ invertible is obtained using semiclassical versions of the propagation estimates of §§2 and 4 after an additional conjugation by $(1+x_1/2)^{i\lambda/4}$ (to guarantee semiclassical ellipticity for $\operatorname{Im}\lambda \gg 1$) providing invertibility of $P(\lambda_0)$ for $\operatorname{Im}\lambda_0 \gg 1$.

Existence of the inverse at some point λ_0 (guaranteed by (8.1)) and the standard Fredholm analytic theory [DZ2, Theorem C.5] give that

$$P(\lambda)^{-1} : \mathscr{Y}_s \to \mathscr{X}_s \ \text{ is a meromorphic family of operators in } \operatorname{Im}\lambda > -s - \frac{1}{2}. \tag{8.3}$$

We can now define the meromorphic family

$$R(\lambda) : C^\infty_{\mathrm{c}}(M) \to C^\infty(M), \quad \lambda \in \mathbb{C}$$

such that

$$R(\lambda) = (-\Delta - \lambda^2 - \tfrac{1}{4})^{-1}, \quad \operatorname{Im}\lambda > 0.$$

First we put

$$V(\lambda) : C^\infty_{\rm c}(M) \to C^\infty_{\rm c}(X), \quad f(y) \longmapsto Tf(x) := \begin{cases} x_1^{\frac{i}{2}\lambda-\frac{5}{4}} f(\sqrt{x}_1, x_2), & x_1 > 0, \\ \quad 0, & x_1 \le 0, \end{cases}$$

$$U(\lambda) : C^\infty(X) \to C^\infty(M), \quad u(x) \longmapsto y_1^{-i\lambda+\frac{1}{2}} u(y_1^2, y_2),$$

and then

$$R(\lambda) = U(\lambda)P(\lambda)^{-1}V(\lambda). \tag{8.4}$$

9. The general case

The general case is analyzed by the same type of arguments once we assume that the metric on M, near ∂M, is given by

$$g = \frac{dy_1^2 + h(y_1^2, y', dy')}{y_1^2}, \quad y_1|_{\partial M} = 0, \quad dy_1|_{\partial M} \neq 0, \quad y' \in \partial M. \tag{9.1}$$

The class of *even* asymptotically hyperbolic metrics can be put into this normal form – see [V2] and references given there.

Near ∂M, the Laplacian has the form ($\dim M = n$)

$$-\Delta_g = (y_1 D_{y_1})^2 - i(n - 1 + y_1^2 \gamma(y_1^2, y'))y_1 D_{y_1} - y_1^2 \Delta_h,$$

where Δ_h is the Laplacian for the family of metrics depending on y_1^2 and $\gamma \in C^\infty$.

We now have

$$y_1^{i\lambda-\frac{n-1}{2}}(-\Delta_g - \lambda^2 - \tfrac{(n-1)^2}{4})y_1^{-i\lambda+\frac{n-1}{2}} = x_1 P(\lambda), \quad x_1 = y_1^2, \quad x' = y',$$

where

$$P(\lambda) = 4(x_1 D_{x_1}^2 - (\lambda + i)D_{x_1}) + \Delta_h - i\gamma(x)\left(2x_1 D_{x_1} - \lambda - i\tfrac{n-1}{4}\right). \tag{9.2}$$

There is no conceptual (and very little technical – we address what seems to be the only point below) difference between the analysis we presented (with $\gamma \equiv 0$ and $\Delta_h = D_{x_2}^2$) and the case of $P(\lambda)$ above.

Hence we obtain the meromorphic continuation of

$$P(\lambda)^{-1} : C^\infty_{\rm c}(X_1) \longrightarrow \bar{C}^\infty(X_1)$$

which gives the meromorphic continuation of

$$(-\Delta_g - \lambda^2 - \tfrac{(n-1)^2}{4})^{-1} : C^\infty_{\rm c}(M) \longrightarrow C^\infty(M). \tag{9.3}$$

In fact we get all the precise statements derived in the model case. As first shown by Vasy in [V1],[V2] this recovers and strengthens the results of Mazzeo–Melrose [MaM] and Guillarmou [Gu].

Appendix. The special structure of $P(\lambda)$ in the model case was used in the proof of (6.6). The analogue of (6.6) for operators of the form (9.2) is given as follows: for $\operatorname{Im}\lambda \geq c_0 > 0$, $\lambda \notin i\mathbb{N}$, and

$$L^2 = L^2((0,1)_{x_1} \times \partial M_{x'}; dx_1 dx'),$$

$$P(\lambda)u = f \in C^\infty_{\rm c}((0,1)\times\partial M), \quad u \in x_1^{-\frac{\varrho}{2}+\frac12} L^2 \implies (x_1 D_{x_1})^\ell D^\alpha_{x'} u \in x_1^{-\frac{\varrho}{2}+\frac12} L^2, \quad (9.4)$$

for any $\ell \in \mathbb{N}$ and $\alpha \in \mathbb{N}^{n-1}$.

Proof of (9.4). We will prove (9.4) working in the original coordinates y. Writing

$$Q(\lambda^2) := -\Delta_g - \lambda^2 - \tfrac{(n-1)^2}{4},$$

(9.4) is equivalent to

$$Q(\lambda^2)u \in C^\infty_{\rm c}((0,1)\times\partial M), \quad u \in L^2 \implies (y_1 D_{y_1})^\ell D^\alpha_{y'} u \in L^2, \quad (9.5)$$

$L^2 := L^2(M, d\mathrm{vol}_g)$.

From invertibility of $Q(\lambda^2)$ away from the spectrum we see that[†]

$$\begin{gathered} Q(\lambda^2)^{-1} : H^k(M, d\mathrm{vol}_g) \to H^{k+2}(M, d\mathrm{vol}_g), \\ H^k(M, d\mathrm{vol}_g) := \{u : (y_1 D_{y_1})^\ell y_1^{|\alpha|} D^\alpha_{y'} u \in L^2(M, d\mathrm{vol}_g),\ \ell + |\alpha| \leq k\}, \quad \ell \in \mathbb{N}. \end{gathered} \quad (9.6)$$

From the spectral theorem we also have, for $\operatorname{Im}\lambda > 0$,

$$\begin{aligned} \|Q(\lambda^2)^{-1}\|_{L^2\to L^2} &= \frac{1}{d(\lambda^2, \mathrm{Spec}(-\Delta_g - \frac{(n-1)^2}{4}))}, \\ \|Q(\lambda^2)^{-1}\|_{L^2\to H^2} &\leq \frac{(1 + C|\lambda|)}{d(\lambda^2, \mathrm{Spec}(-\Delta_g - \frac{(n-1)^2}{4}))} \end{aligned} \quad (9.7)$$

We now consider weighted estimates and, for $\lambda^2 + \alpha^2 \notin \mathrm{Spec}(-\Delta_g - \frac{(n-1)^2}{4})$, we write

$$\begin{aligned} y_1^\alpha Q(\lambda^2) y_1^{-\alpha} &= Q(\lambda^2+\alpha^2) + \alpha(2iy_1 D_{y_1} - n + 1 y_1^2 \gamma(y_1^2, y')) \\ &= \left(I + \alpha(2iy_1 D_{y_1} - n + y_1^2\gamma(y_1^2, y'))Q(\lambda^2+\alpha^2)^{-1}\right) Q(\lambda^2+\alpha^2). \end{aligned}$$

Estimates (9.7) show that for $\operatorname{Im}\lambda > c(\alpha)$

$$y_1^\alpha Q(\lambda^2)^{-1} y_1^{-\alpha} : L^2(M, d\mathrm{vol}_g) \to H^2(M, d\mathrm{vol}_g).$$

In particular, for $\operatorname{Im}\lambda > c_0$,

$$Q(\lambda^2) y_1^{-1} v = 0, \quad v \in L^2(M, d\mathrm{vol}_g) \implies v = 0. \quad (9.8)$$

[†]This follows from standard integration by parts arguments: considering $\langle Q(\lambda^2)v, v\rangle$ we first show that for $u \in C^\infty([0,1)\times\partial M))$, $\|y_1 D_{y_k}\|_{L^2} \leq C\|Q(\lambda^2)v\|_{L^2} + C\|u\|_{L^2}$. We then consider $\|Q(\lambda^2)v\|^2_{L^2}$ and the form of Δ_g in (9.3) shows that we control the H^2 norm. Commuting $y_1 D_{y_k}$ with $Q(\lambda^2)$ gives the general estimate.

We now prove (9.5) and for that we first show that $D_{y_k} u \in L^2$, $k > 1$. In fact,

$$Q(\lambda^2)(D_{y_k} u) = F_k := D_{y_k} f - y_1^2 \partial_{y_k} \gamma y_1 D_{y_1} u - y_1^2 [\Delta_h, D_{y_k}] u \in L^2,$$

where we used (9.6) which shows that $u \in H^N(M, d\mathrm{vol}_g)$ for any N. Now,

$$y_1 D_{y_k} - y_1 Q(\lambda^2)^{-1} F_k \in L^2, \quad Q(\lambda^2) y_1^{-1} (y_1 D_{y_k} - y_1 Q(\lambda^2)^{-1} F_k) = 0.$$

Hence (9.8) shows that $D_{y_k} u = Q(\lambda^2)^{-1} F_k \in L^2$. This argument can be iterated showing (9.5). $\square$

Acknowledgements. I should like to thank Semyon Dyatlov, Peter Hintz and András Vasy for helpful comments on the first version of this note and the National Science Foundation for partial support under the grant DMS-1500852.

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