University of Washington

Quantum Chaos in Scattering Theory

Maciej Zworski

UC Berkeley

10 April 2007

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- describe simple models of chaotic scattering in dimension two
- relate the dimension of the trapped set to the density of resonances
- relate the pressure to the quantum decay rate.

Simplest model:

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`www.cims.nyu.edu/~dbindel/resonant1d`

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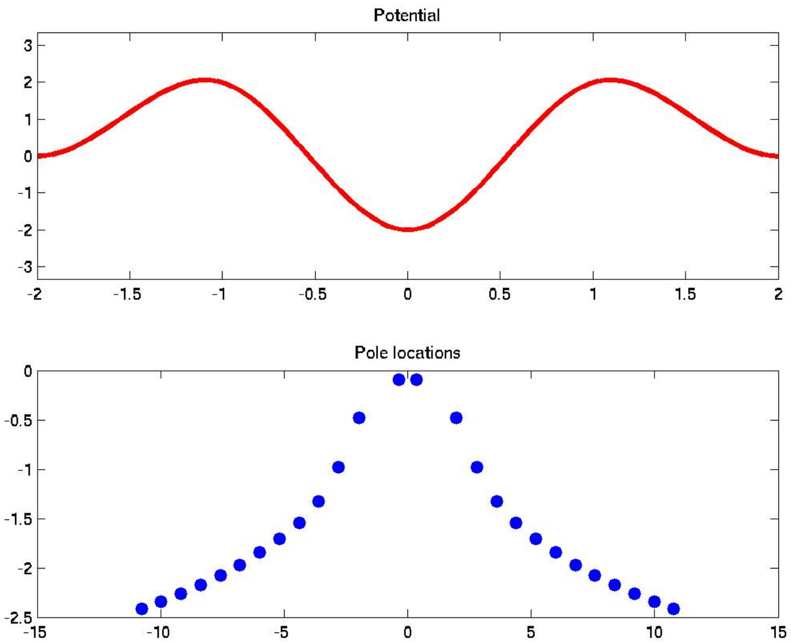

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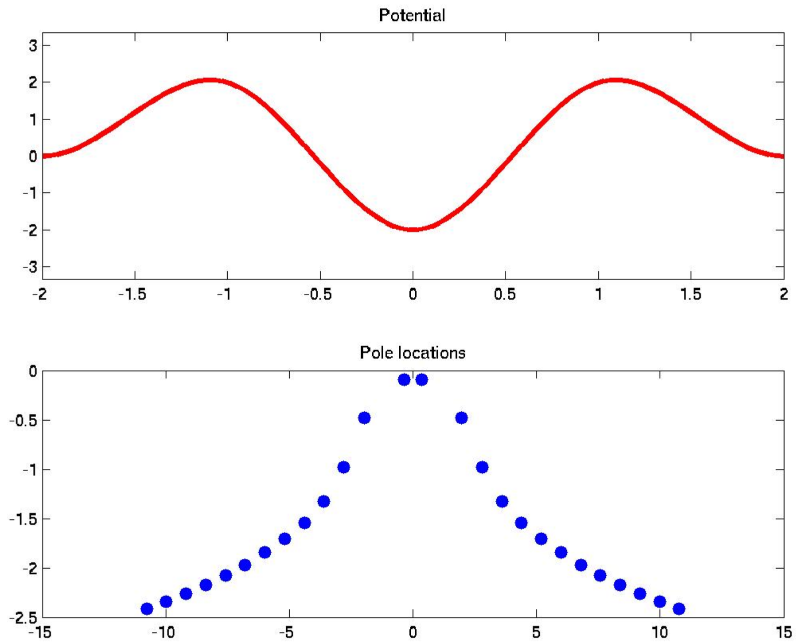

We went from $h = 1$ to $h = 1/7$ in 49 steps.

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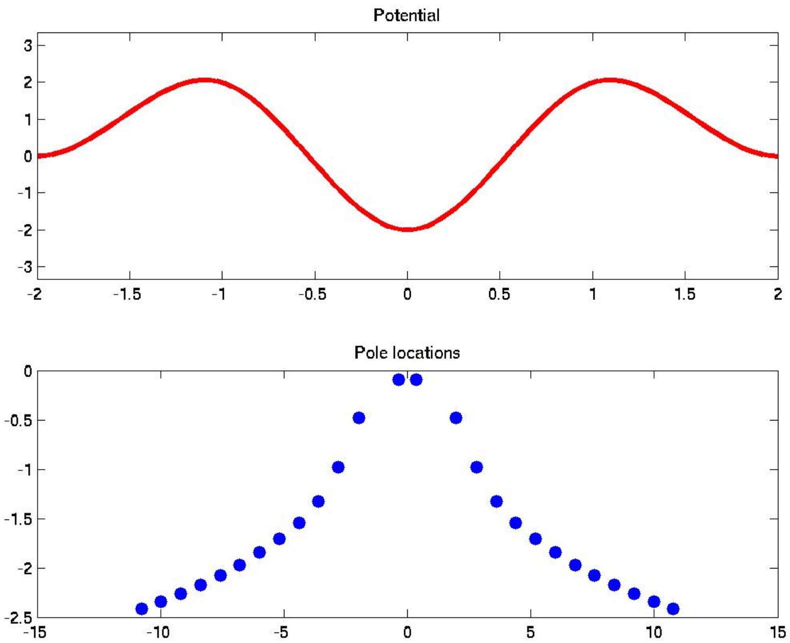

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Please note that as h decreases the density of resonances goes up.

Before explaining how resonances are related to the long time behaviour of scattered waves we discuss the more familiar case of eigenvalues and eigenfuctions.

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A vibrating string, is described using eigenvalues and eigefunctions:

$$\sum_{j=0}^{\infty} \cos(t\lambda_j) c_j u_j(x) + \sum_{j=0}^{\infty} \lambda_j^{-1} \sin(t\lambda_j) d_j u_j(x)$$

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This decomposition in the basis of harmonic analysis, signal processing, and many other things.

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where w_0 and w_1 are the initial conditions of our wave.

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An example from a poem by Li Bai:

> *A thousand valleys’ rustling pines resound.*
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The word 余响 (餘響) appearing in the original led to this modern translation and can be interpreted as an early (8th century) mention of resonances in literature.

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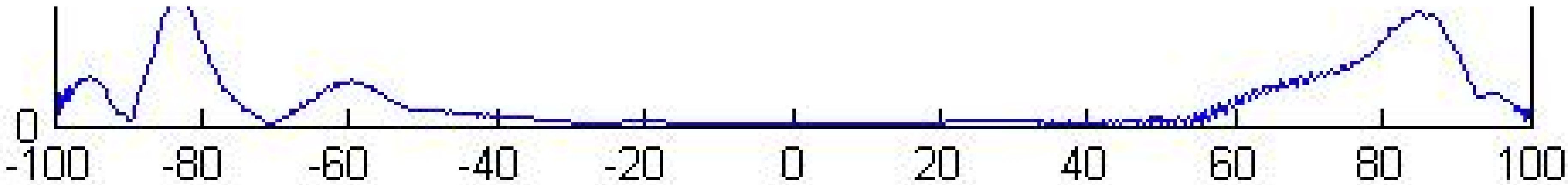

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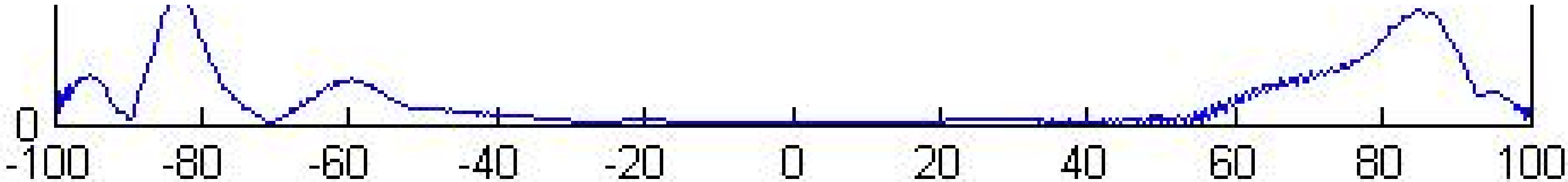

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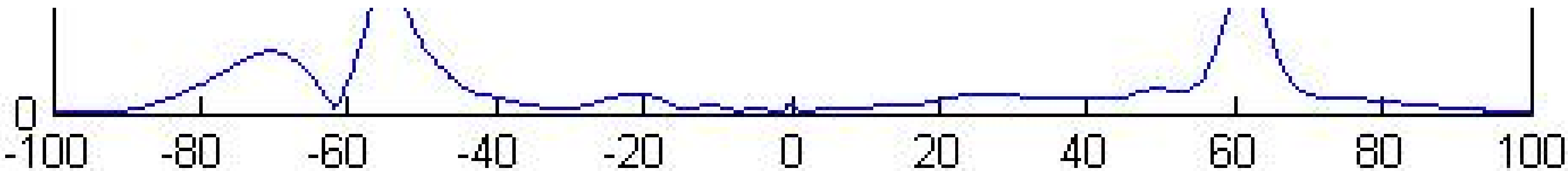

resonant $v = 2.4$

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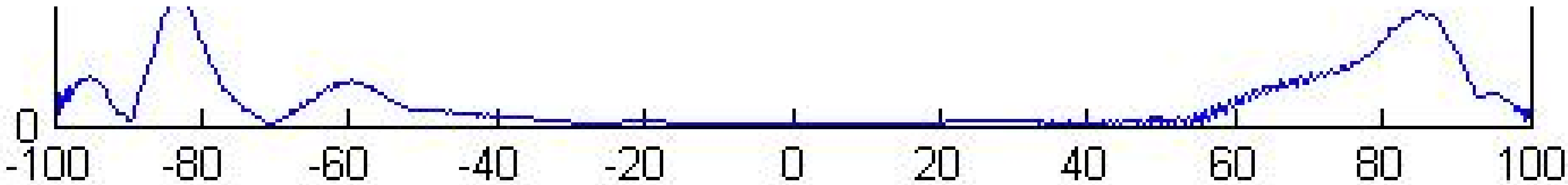

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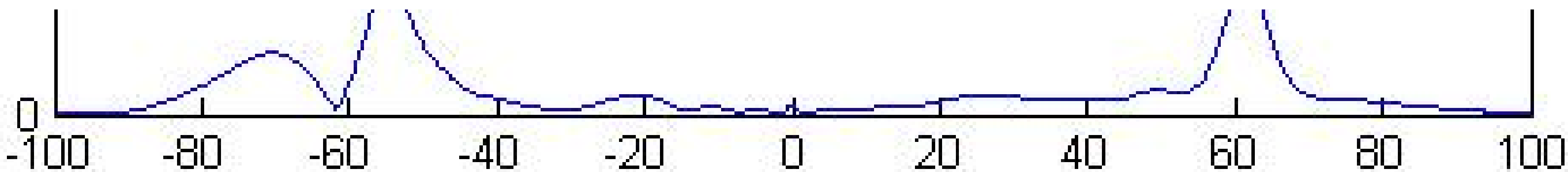

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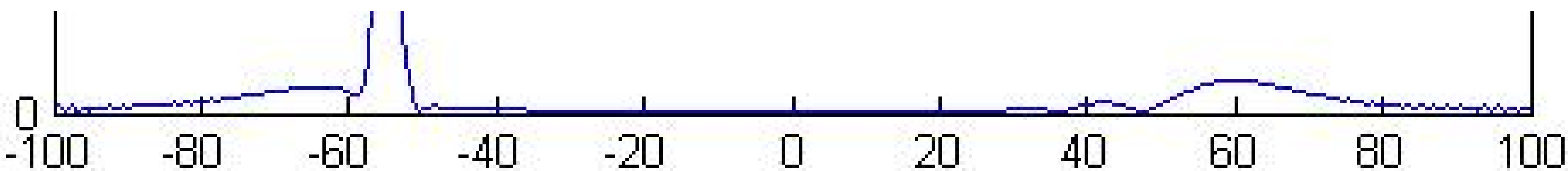

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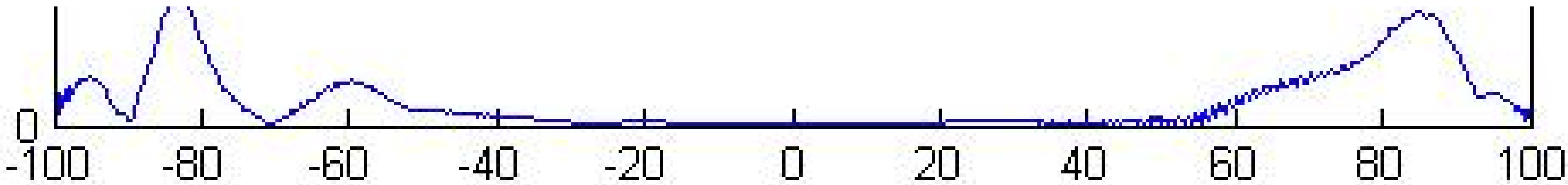

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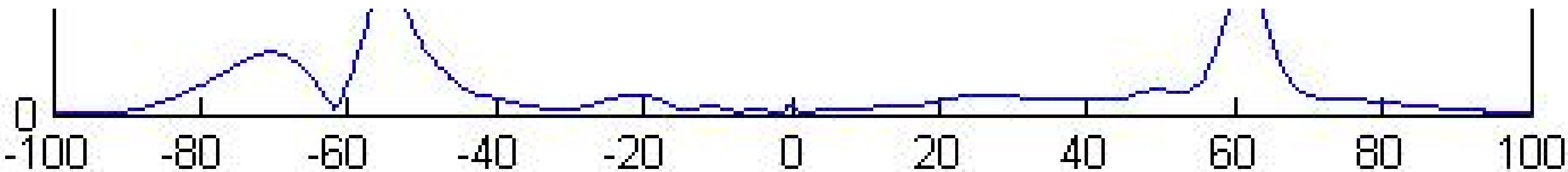

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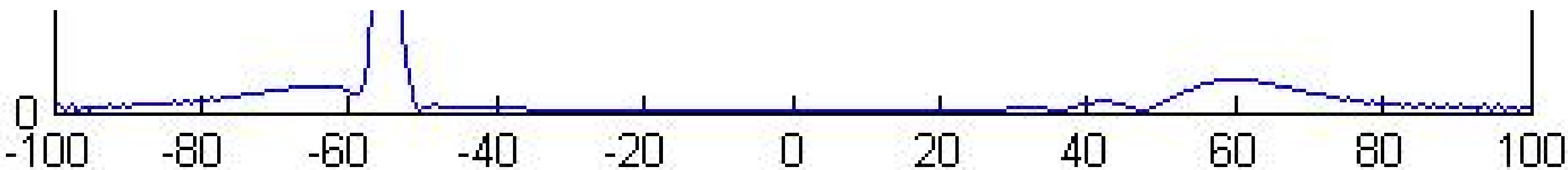
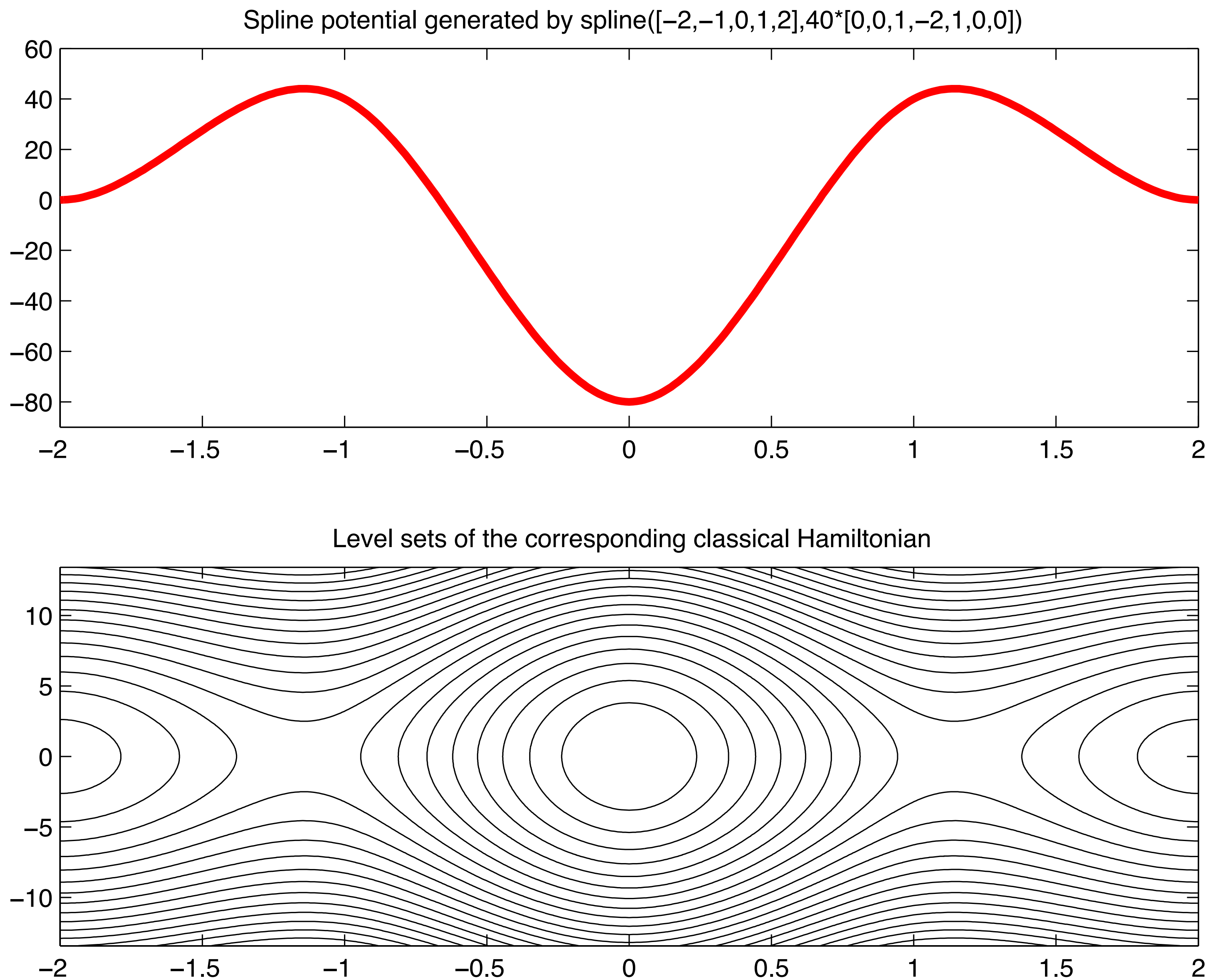
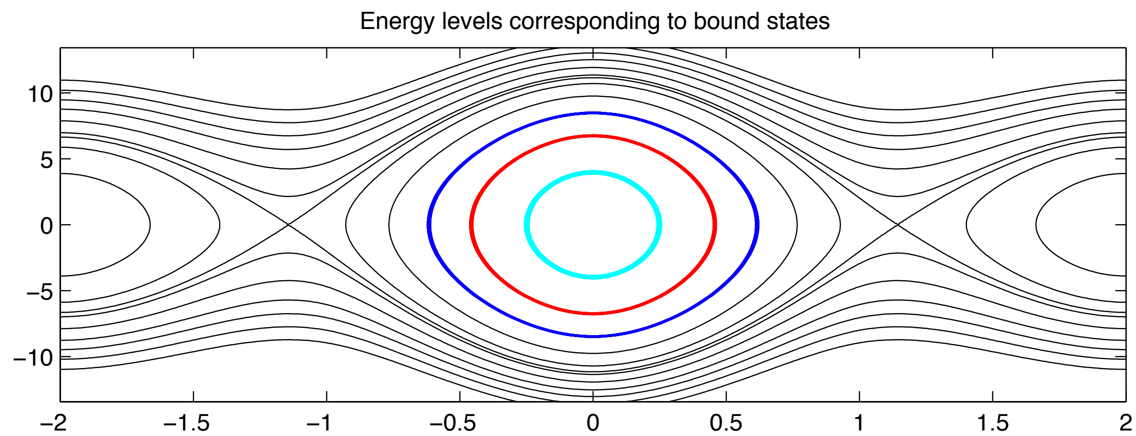


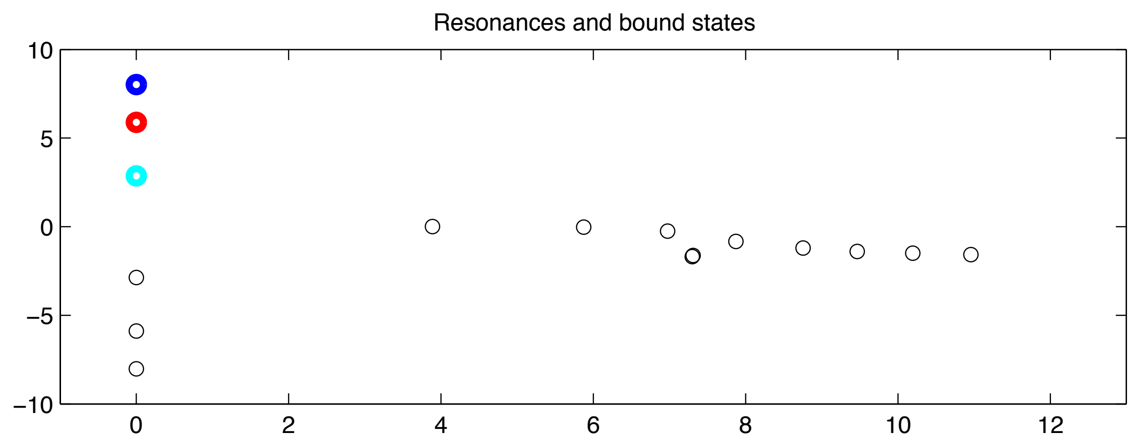
nonresonant $v = 3.4$

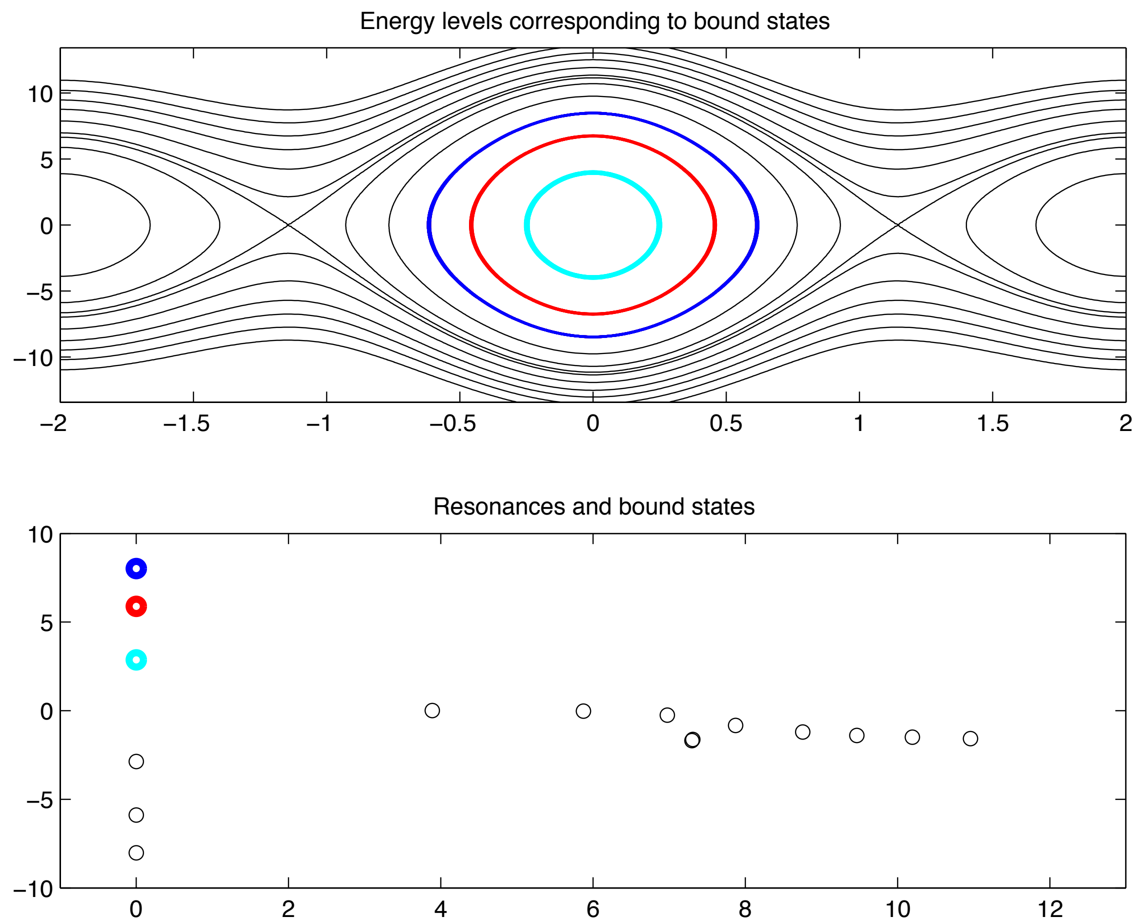
Note the fat, resonant tail in the second figure.



A double barrier potential and the phase portrait for the classical Hamiltonian $\xi^2 + V(x)$.

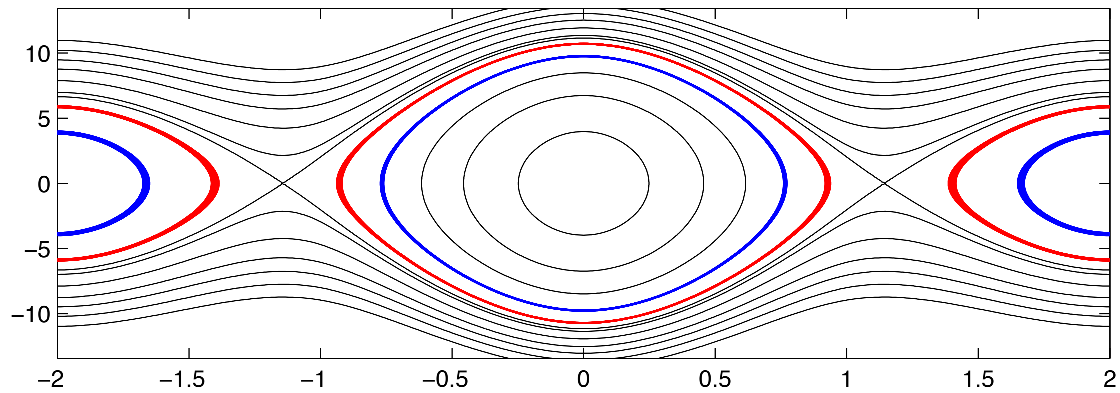

Energy levels corresponding to bound states
10
5
0
−5
−10
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2


Resonances and bound states
10
5
0
−5
−10
0
2
4
6
8
10
12

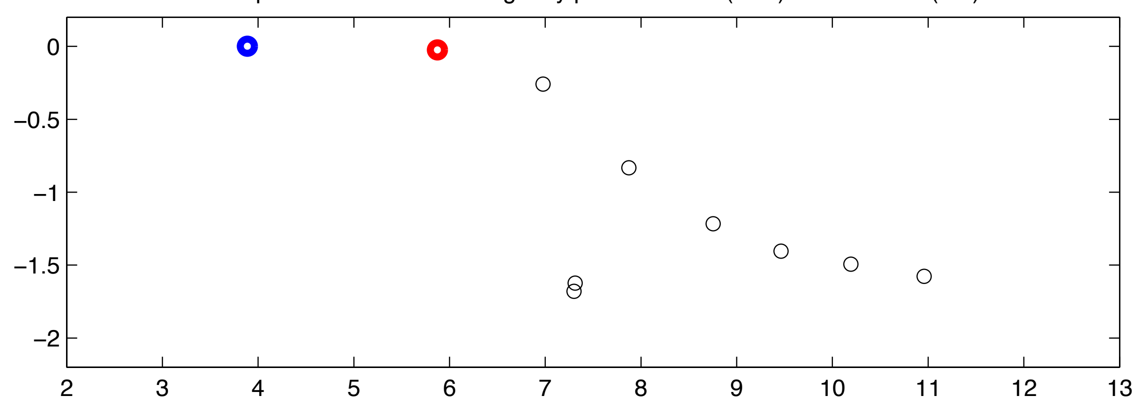


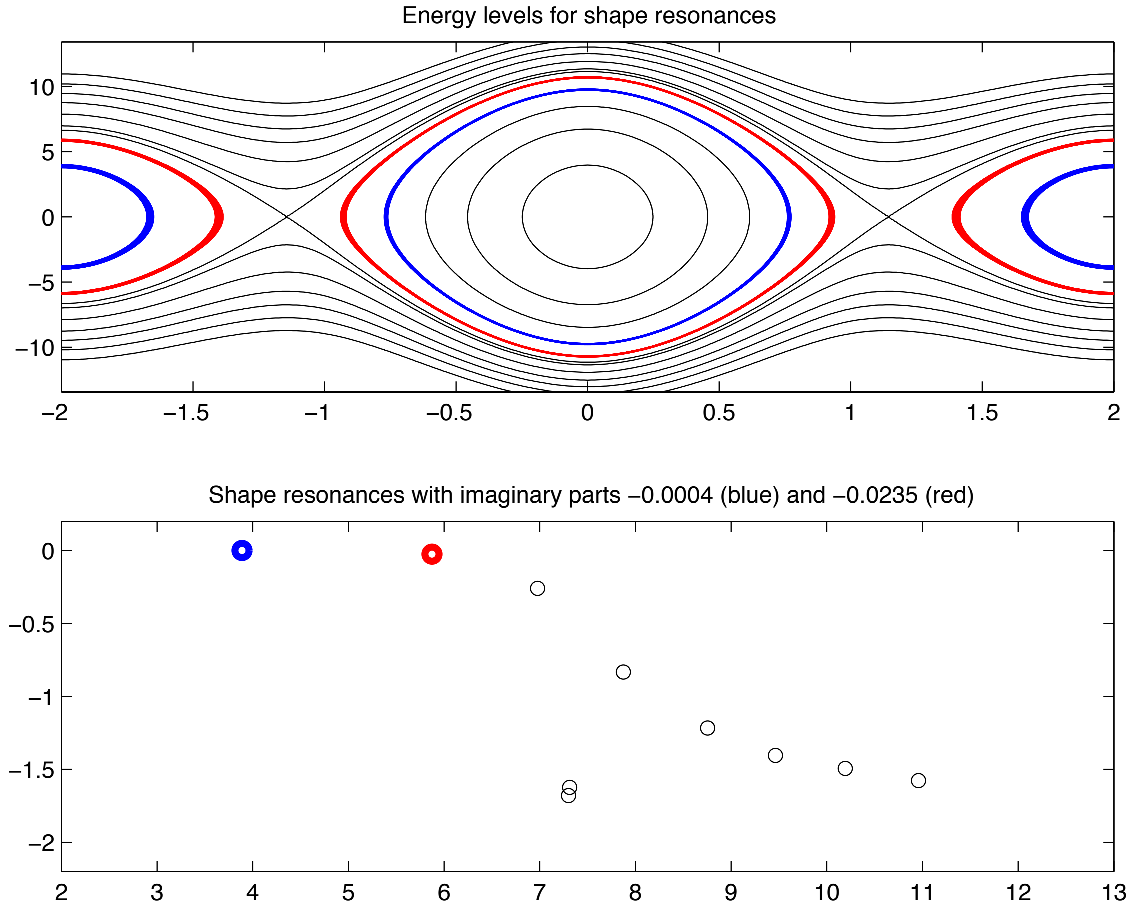
The bound states and resonances for the same potential. The colour coding gives an approximate classical/quantum correspondence between the bound states and energy level satisfying Bohr-Sommerfeld quantization conditions.

Energy levels for shape resonances

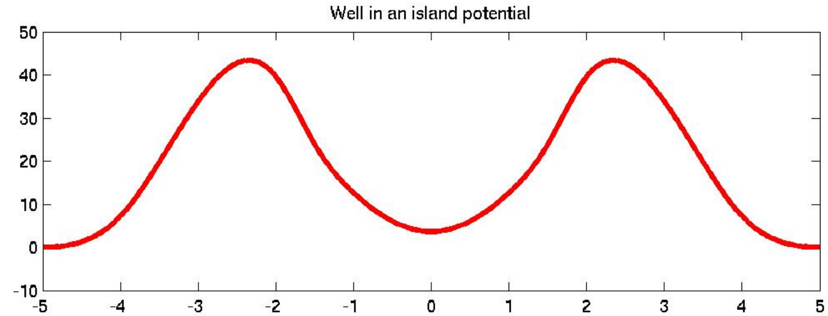


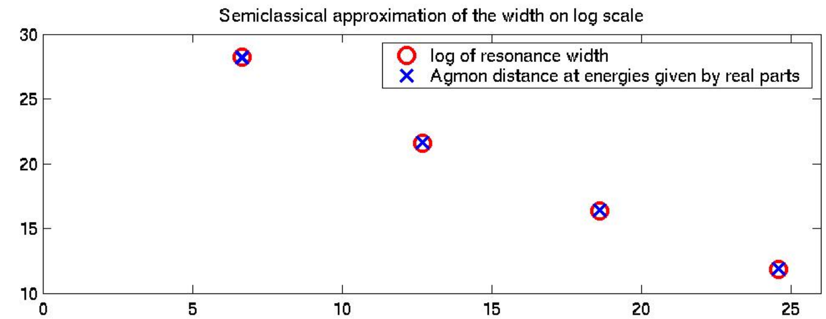
Shape resonances with imaginary parts −0.0004 (blue) and −0.0235 (red)

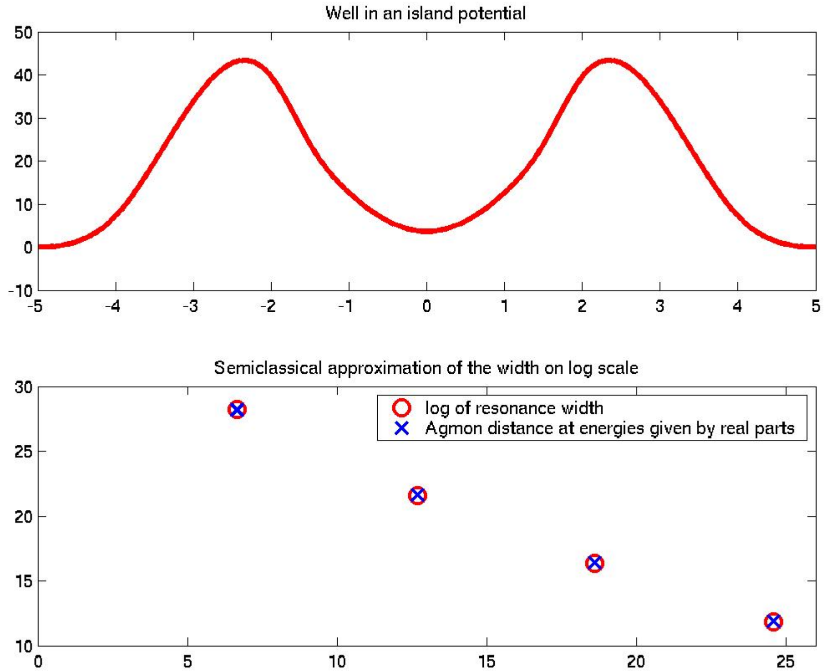




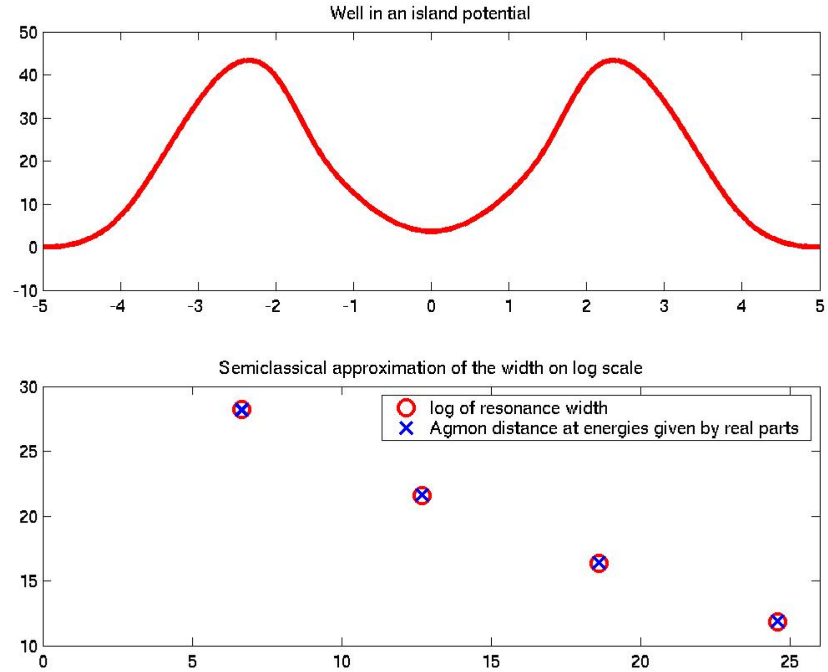
Resonances close to the real axis. The real part is related to the bounded components of the energy levels and the imaginary part is related to tunneling between the bounded and unbounded energy levels.


Well in an island potential
50
40
30
20
10
0
-10
-5
-4
-3
-2
-1
0
1
2
3
4
5


Semiclassical approximation of the width on log scale
log of resonance width
Agmon distance at energies given by real parts
30
25
20
15
10
0
5
10
15
20
25



"Well in an island" potential and the plot of $(\operatorname{Re}\lambda_j^2, -\log(-\operatorname{Im}\lambda_j^2))$ comparing the logarithms of resonance width (imaginary parts) to Agmon/tunneling distances (Helffer-Sjöstrand 1985):



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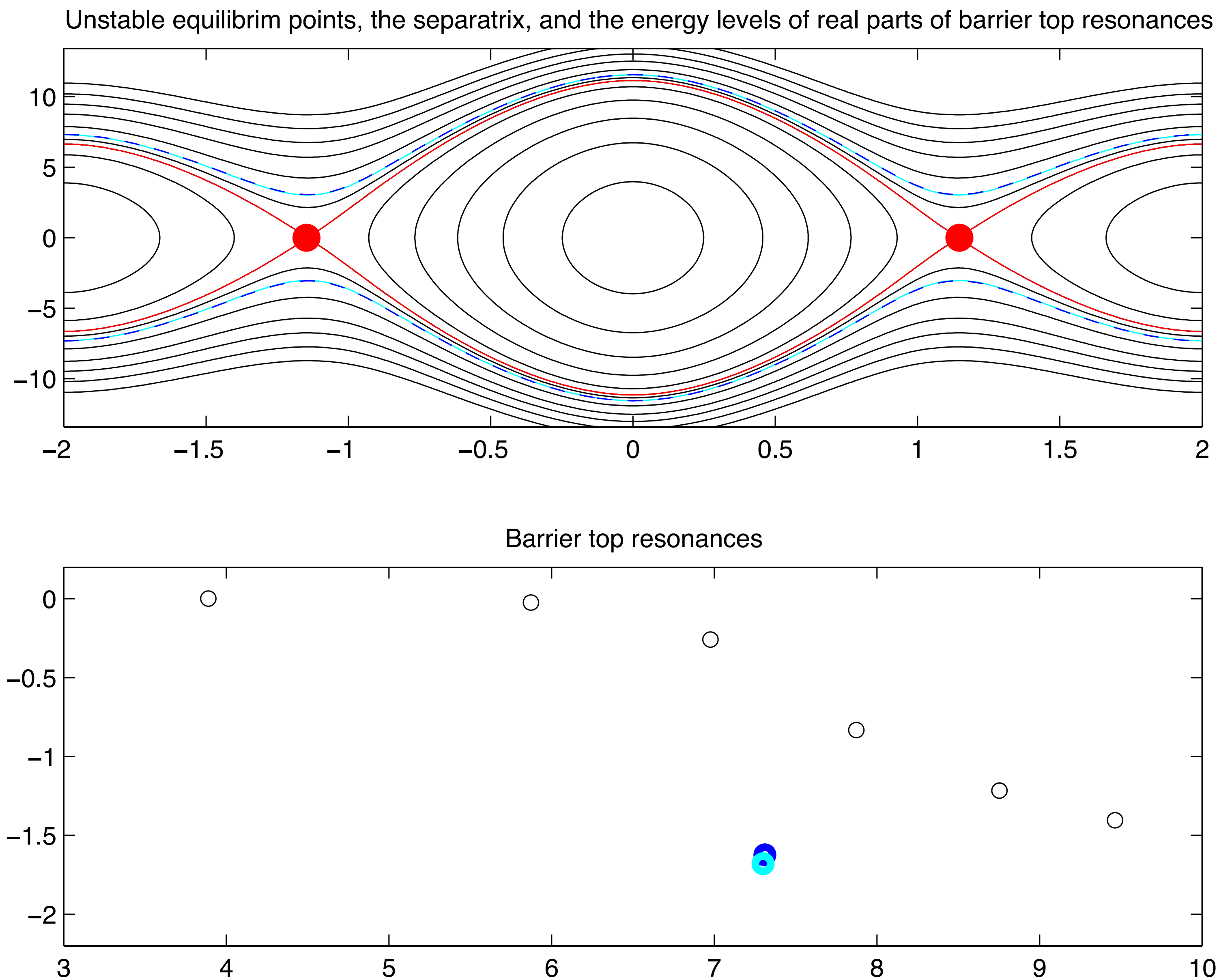
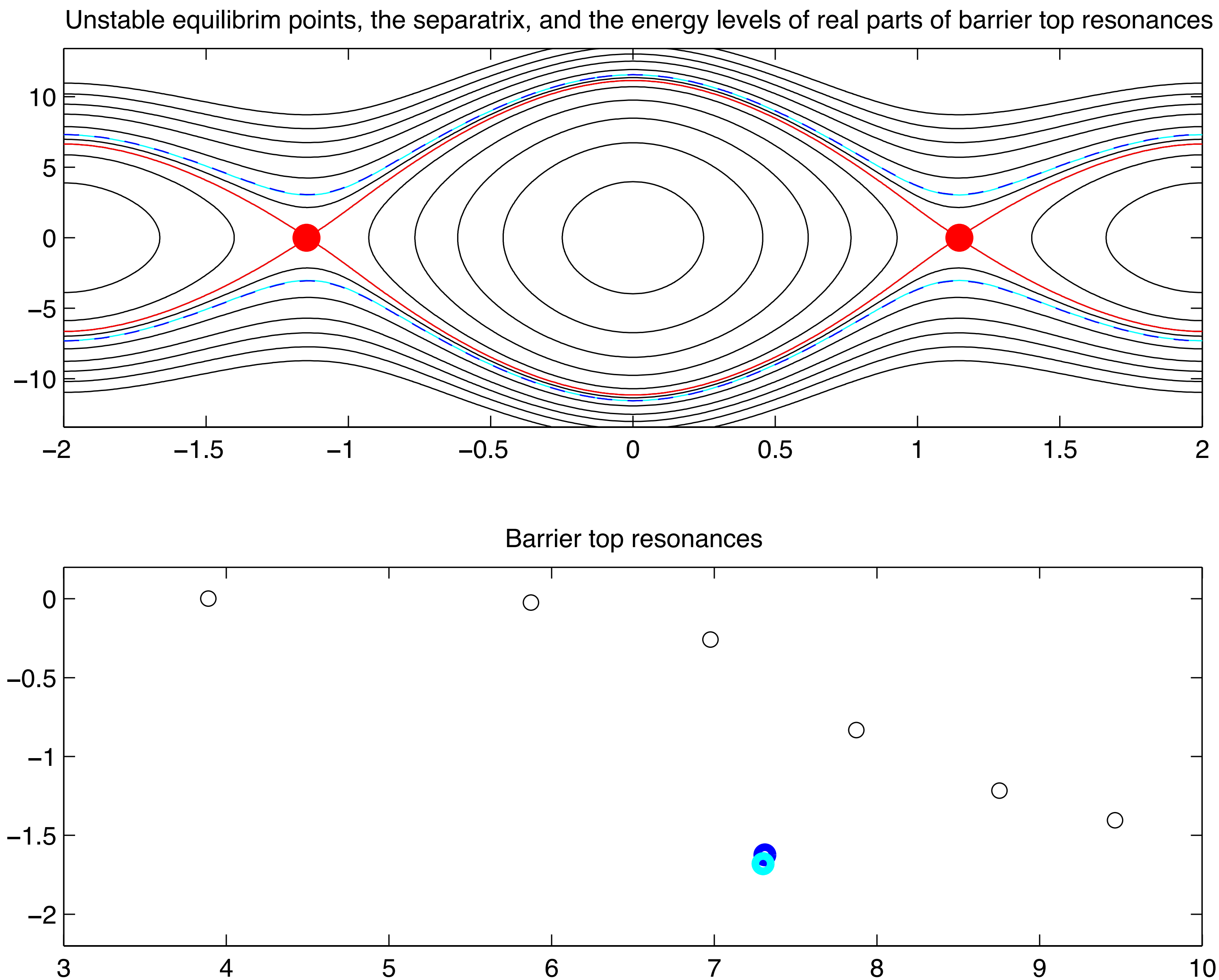
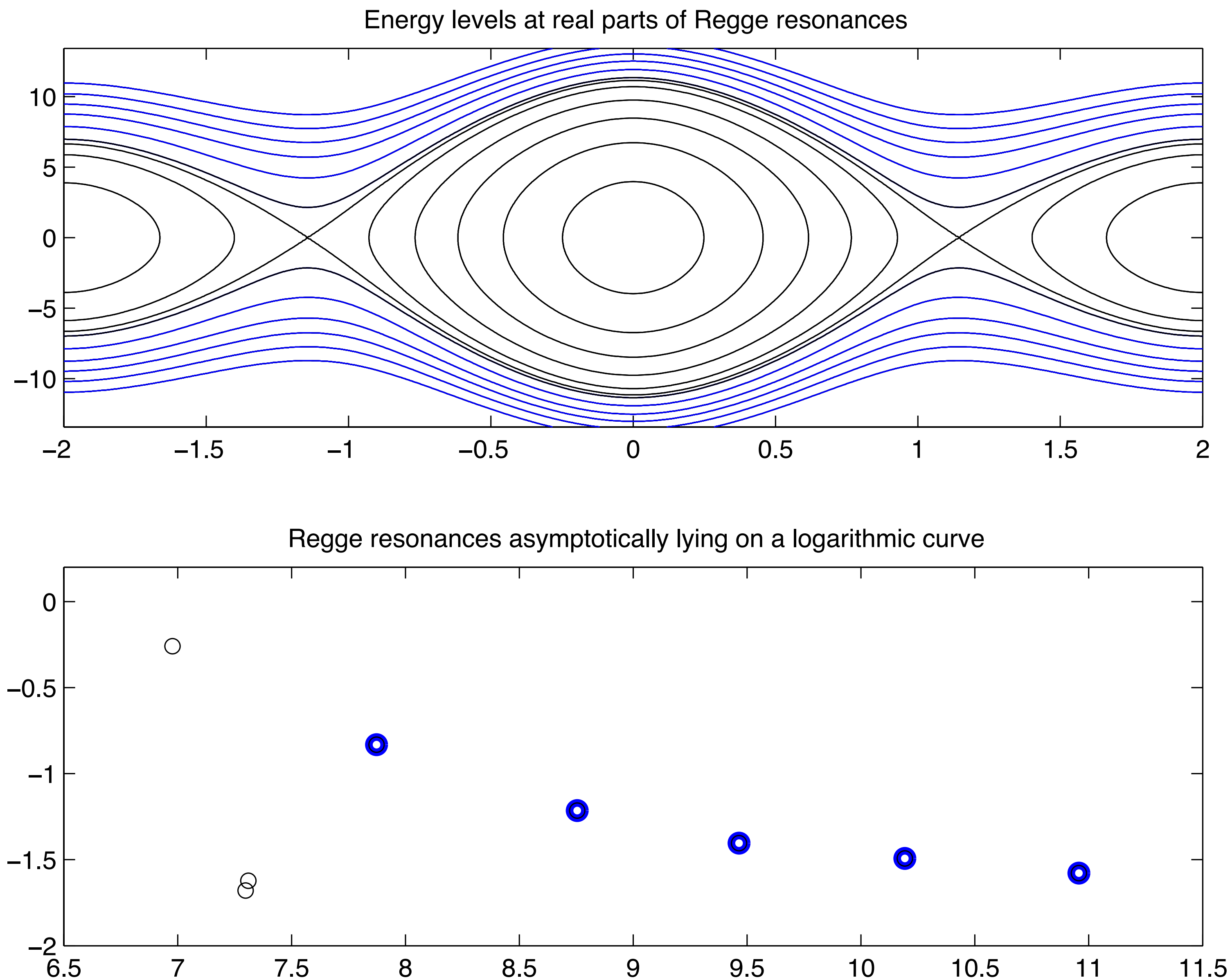
$$S_0(E) = \int_{\mathbf{R}} (V(x) - E)_+^{\frac{1}{2}} dx\,, \quad \log(-\operatorname{Im}\lambda_j^2) \sim -\frac{S_0(\operatorname{Re}\lambda_j^2)}{h}\,.$$

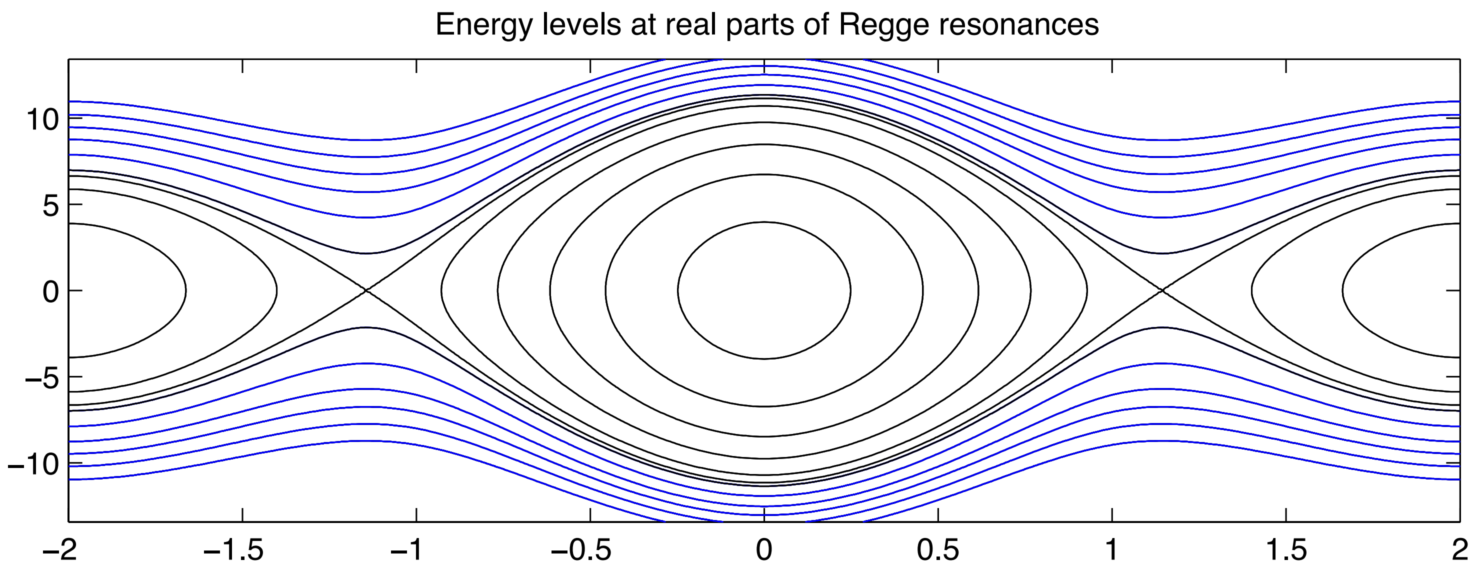


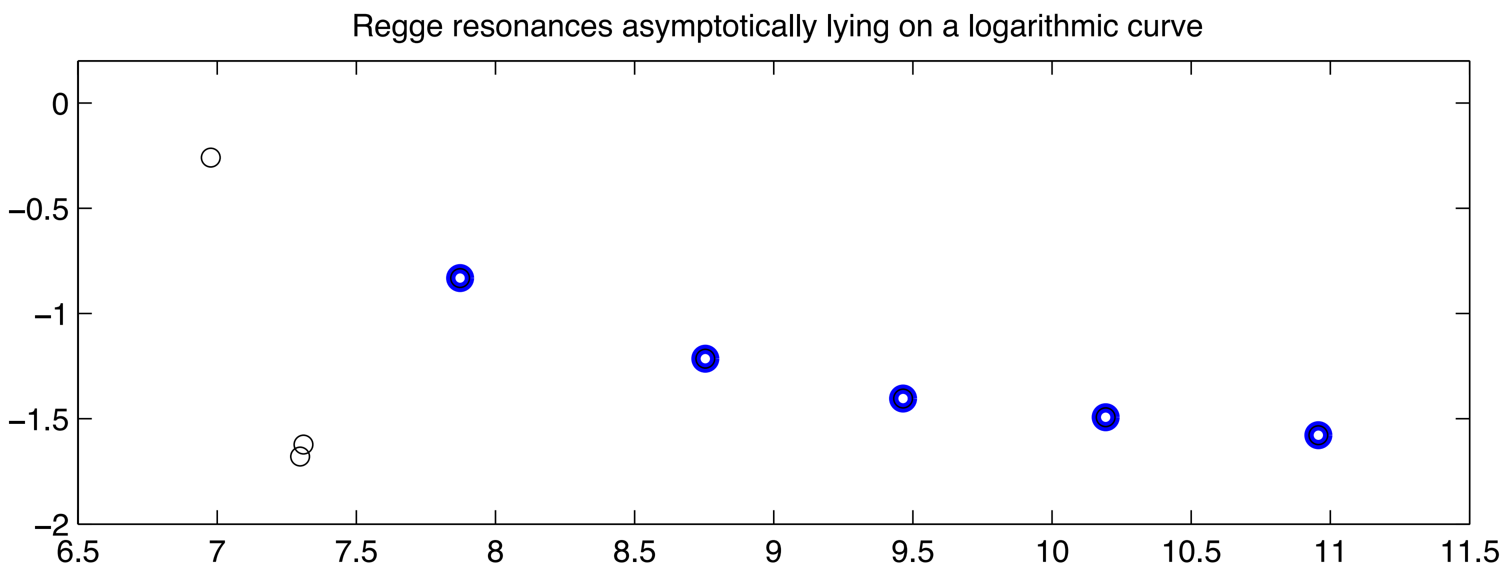
Unstable equilibrim points, the separatrix, and the energy levels of real parts of barrier top resonances
10
5
0
−5
−10
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
Barrier top resonances
0
−0.5
−1
−1.5
−2
3
4
5
6
7
8
9
10



Barrier top resonances corresponding to two unstable equilibria. The actual real parts are corrected due to quantum effects.


Energy levels at real parts of Regge resonances
10
5
0
−5
−10
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
Regge resonances asymptotically lying on a logarithmic curve
0
−0.5
−1
−1.5
−2
6.5
7
7.5
8
8.5
9
9.5
10
10.5
11
11.5





Regge resonances generated by reflections by the singularities at the end points of the support of the potential and lying on logarithmic curves.

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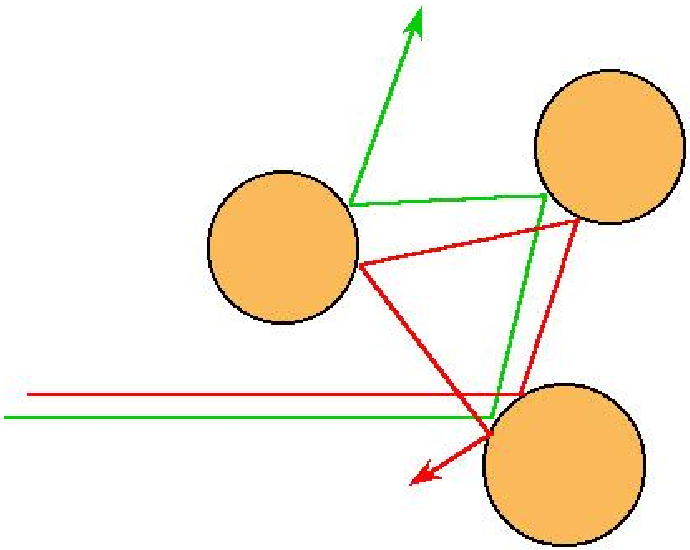
or in fact any dimension...



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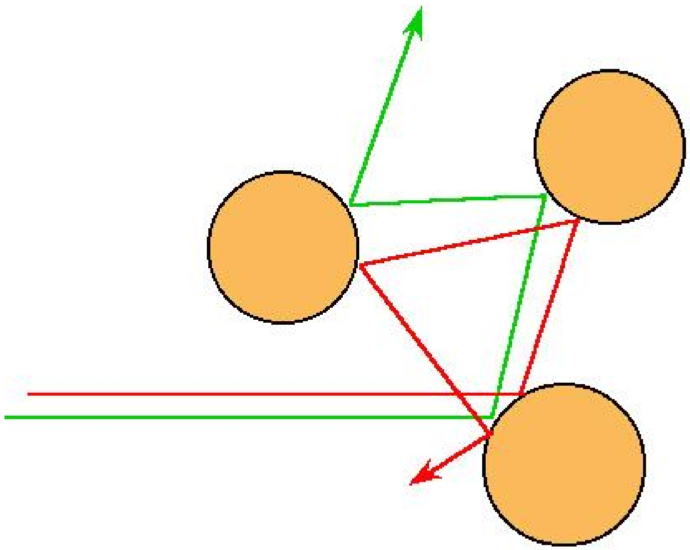

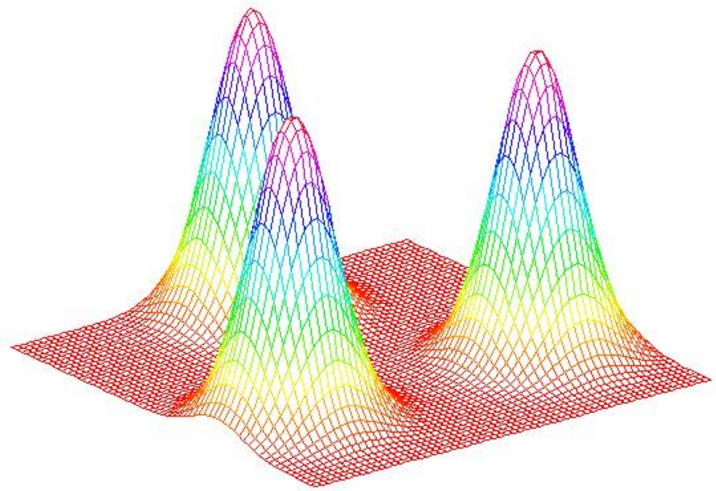

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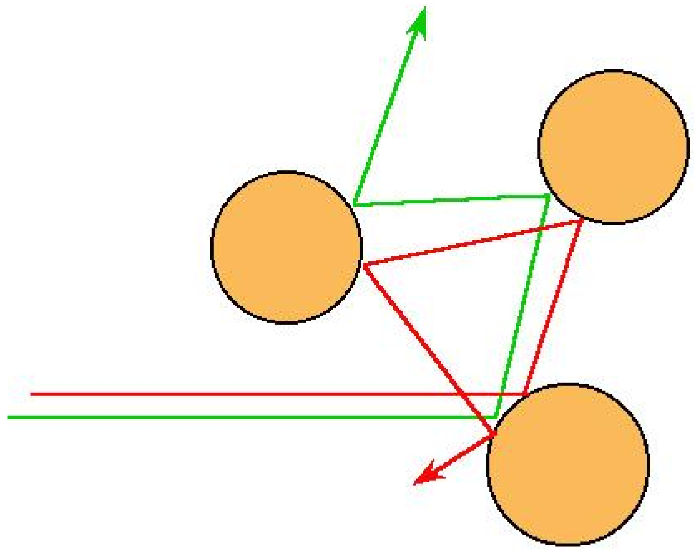

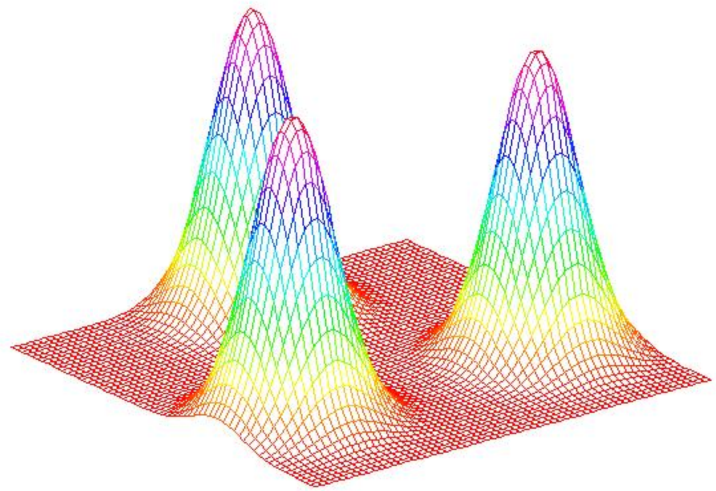

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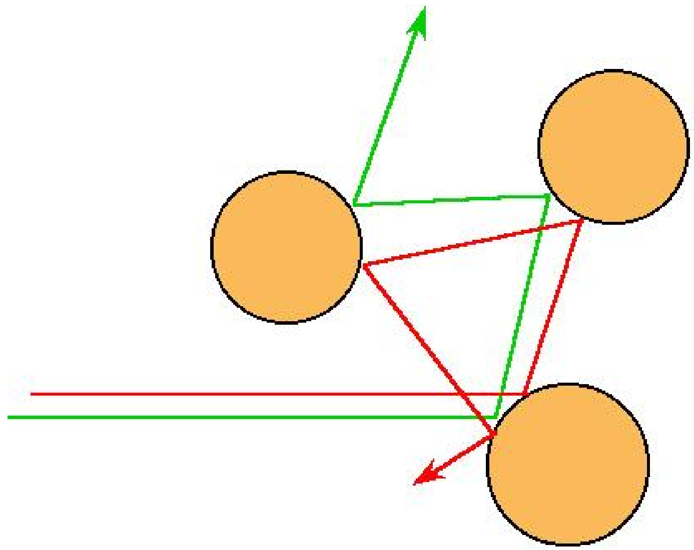

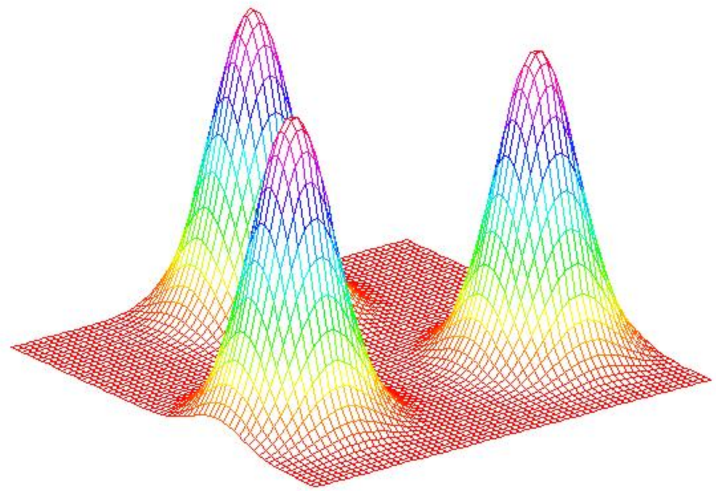

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$$H = \xi^2 + V(x) \, ,$$

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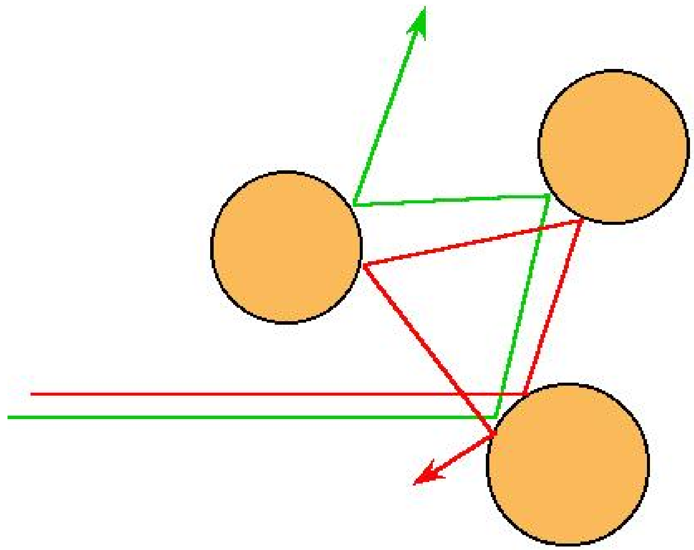

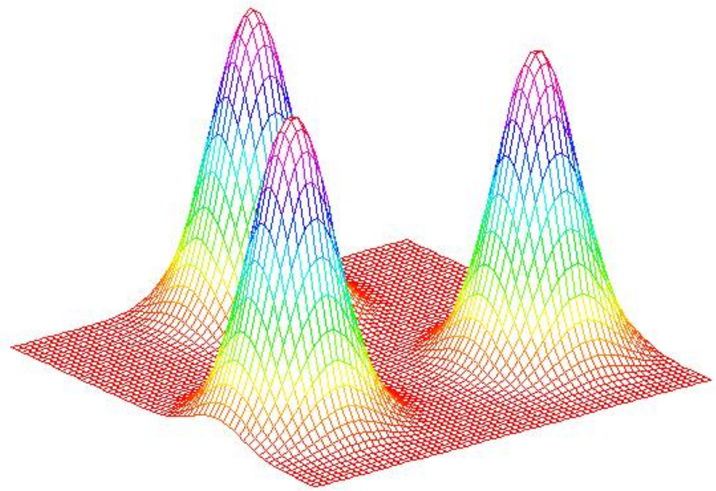

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Now to dimension two...

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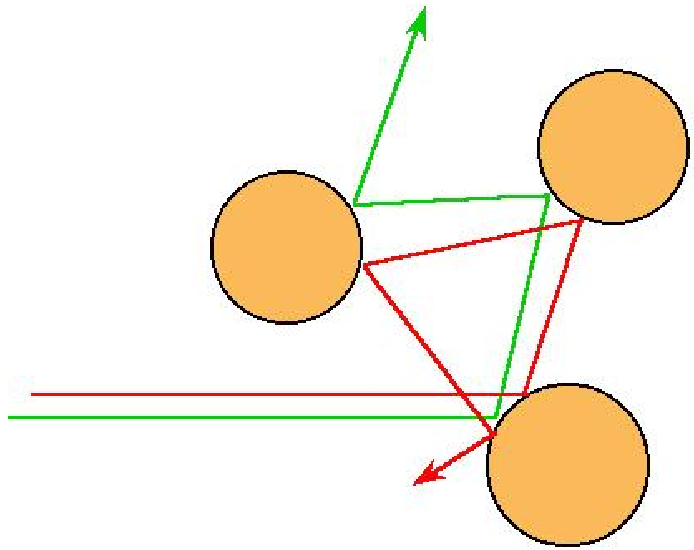

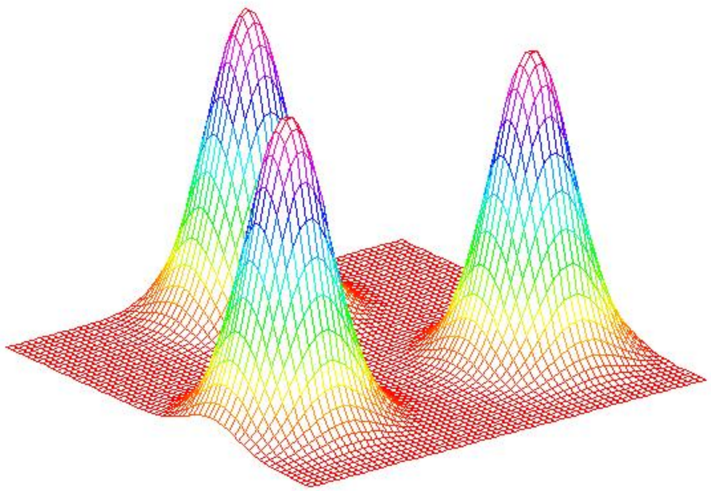

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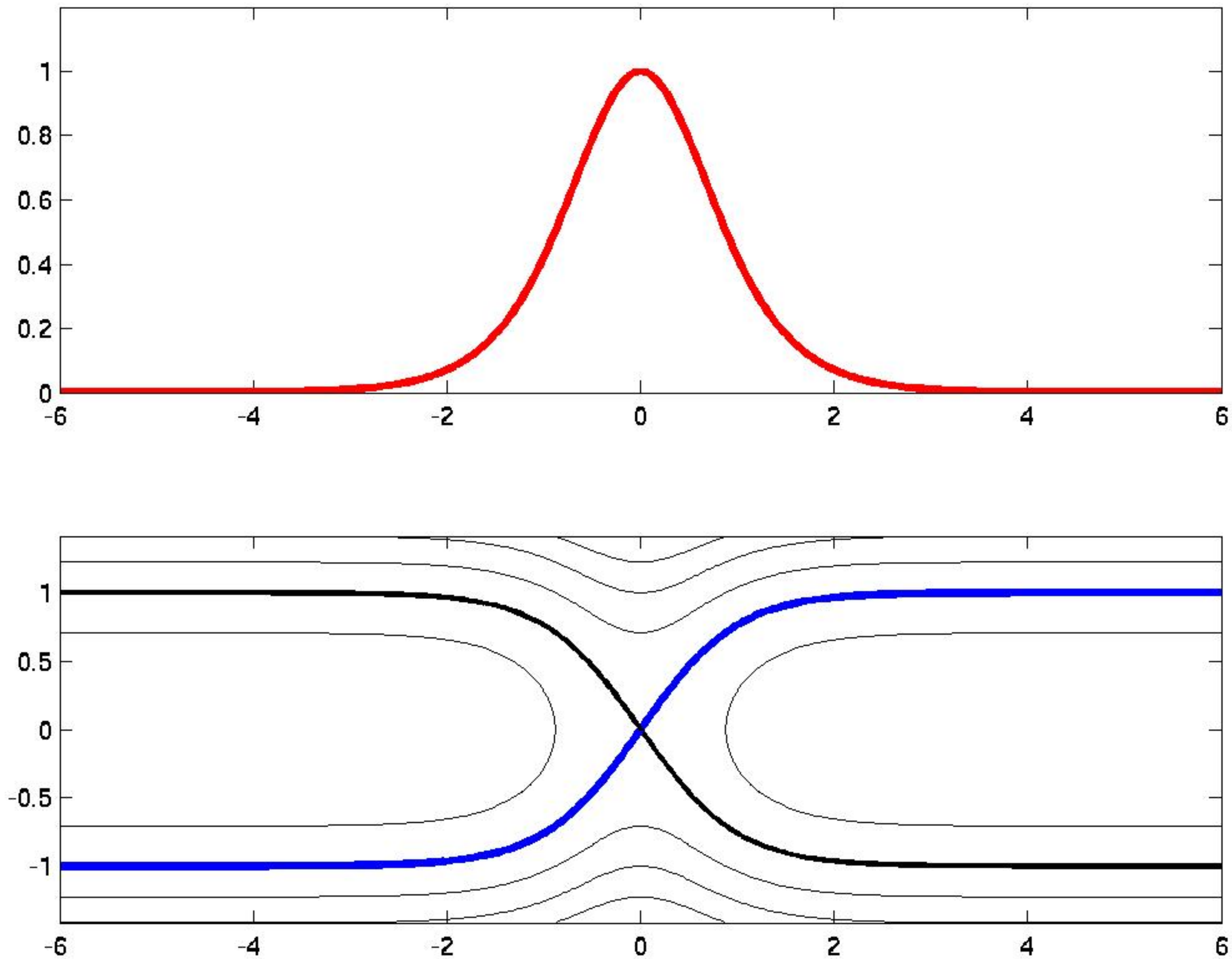
where $\mathcal{L}$ is the Lie derivative along the flow.



1
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0.6
0.4
0.2
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-6
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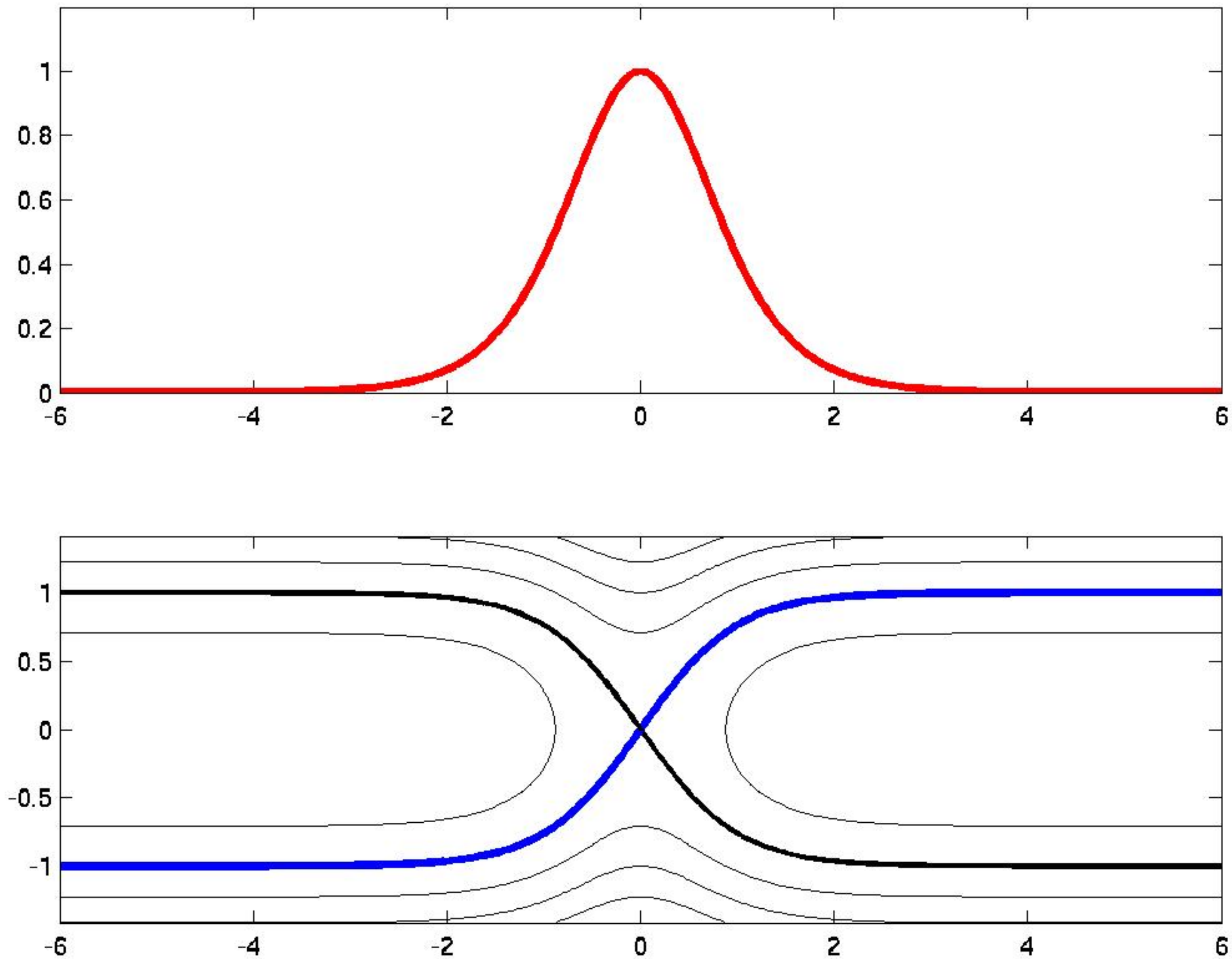
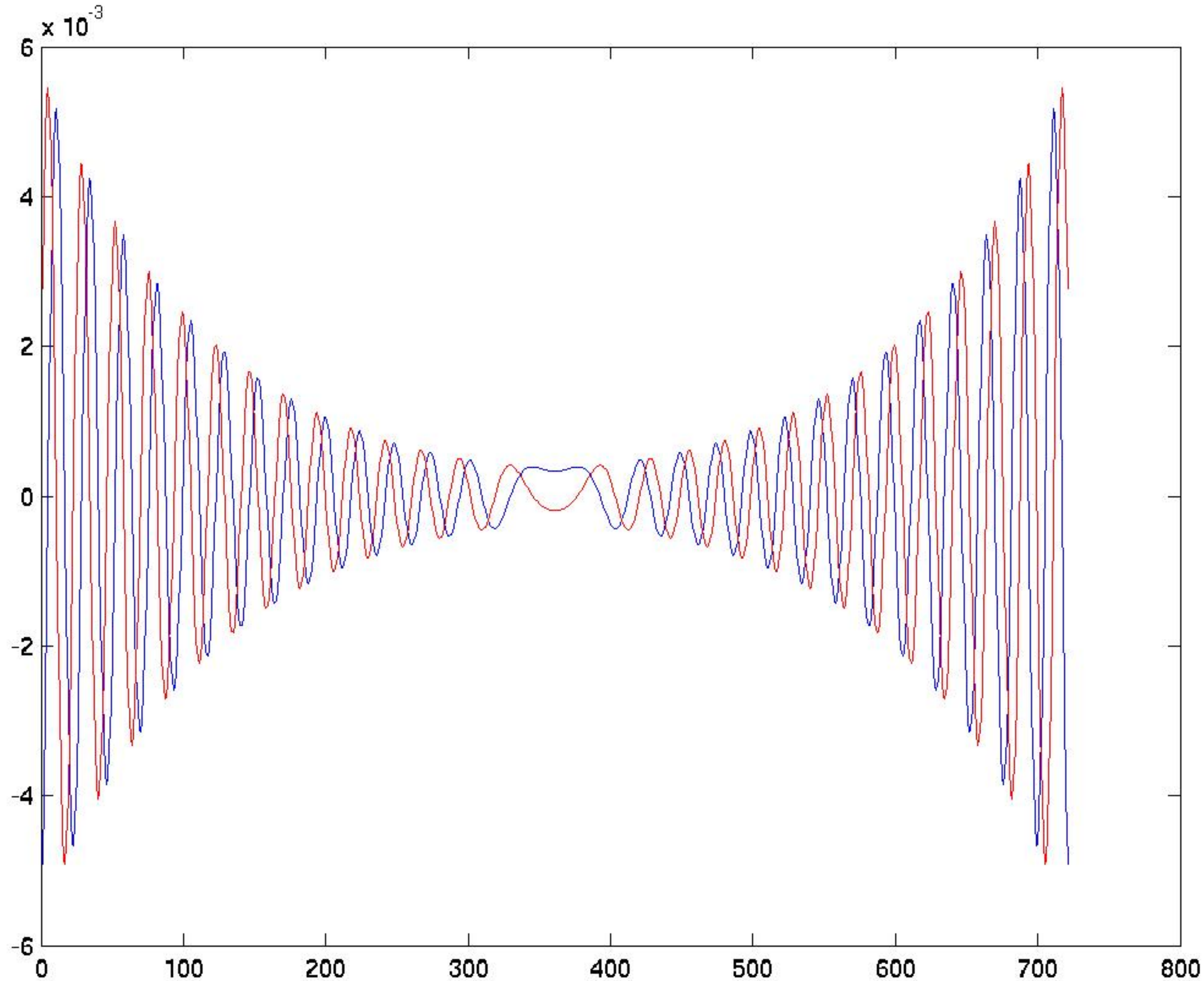


A potential with a simple trapped set.


x 10^-3
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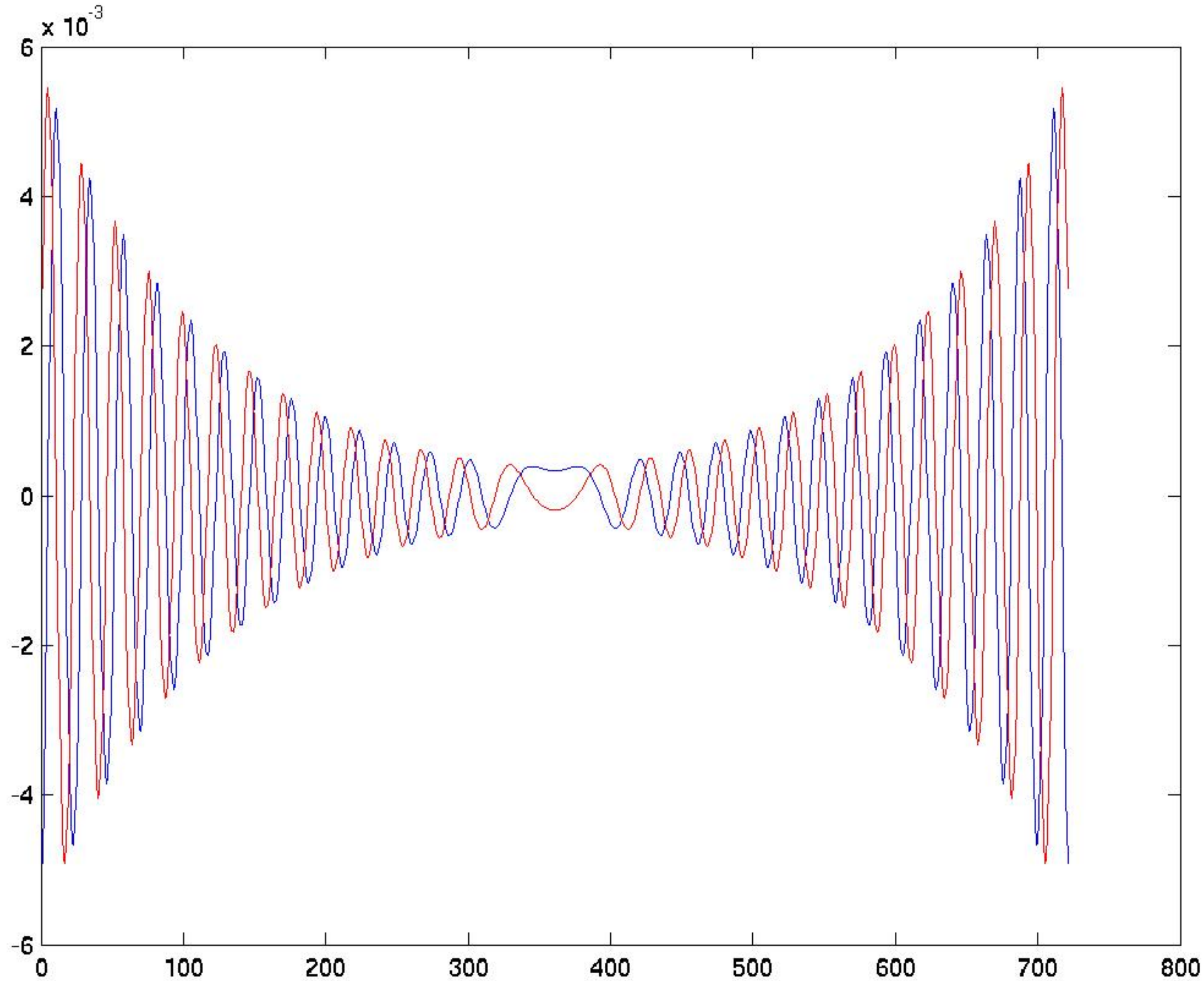
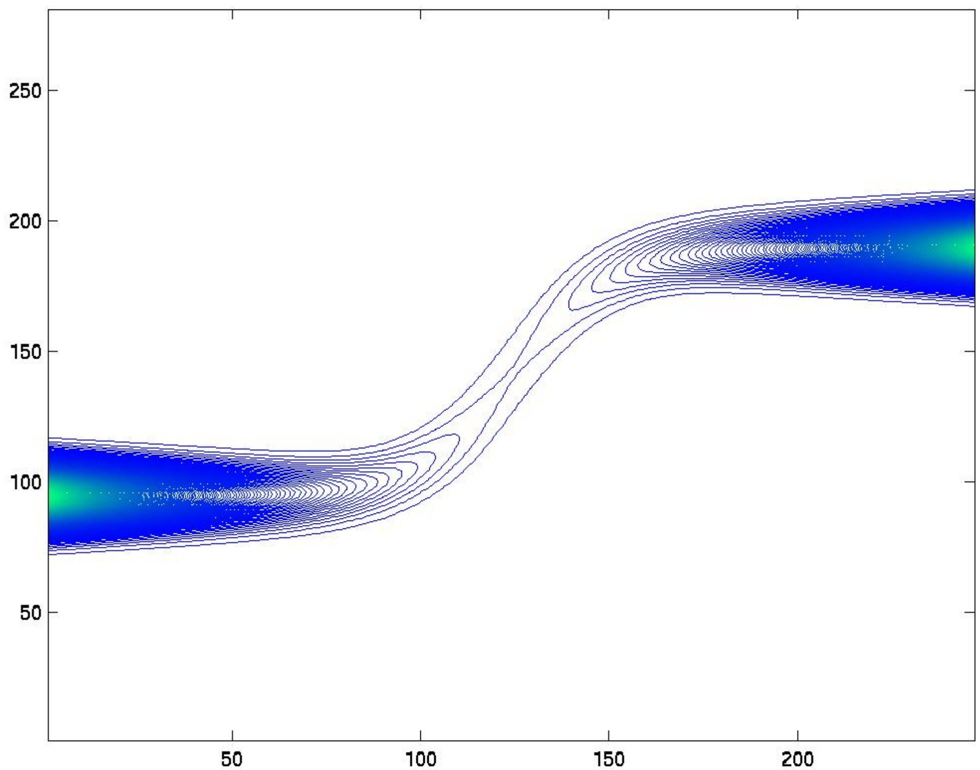


The “first” resonant function for $h = 1/16$.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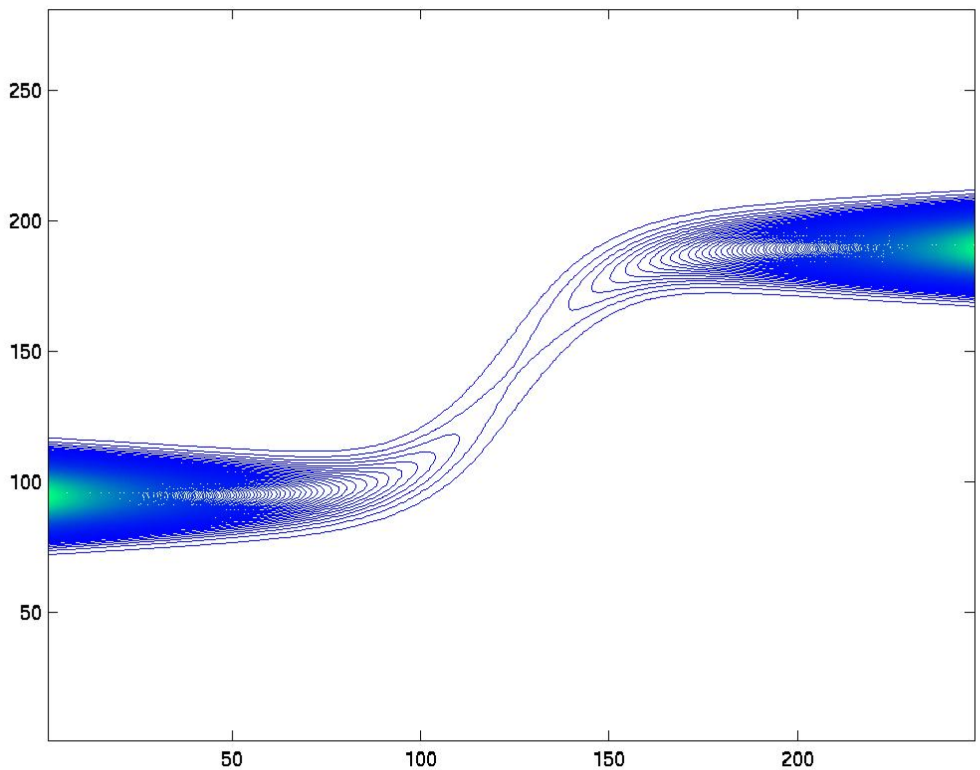
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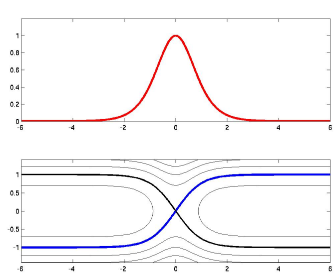

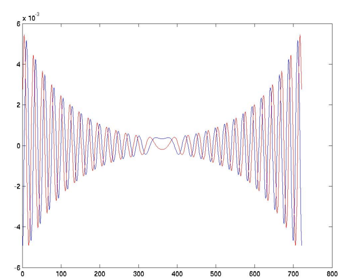

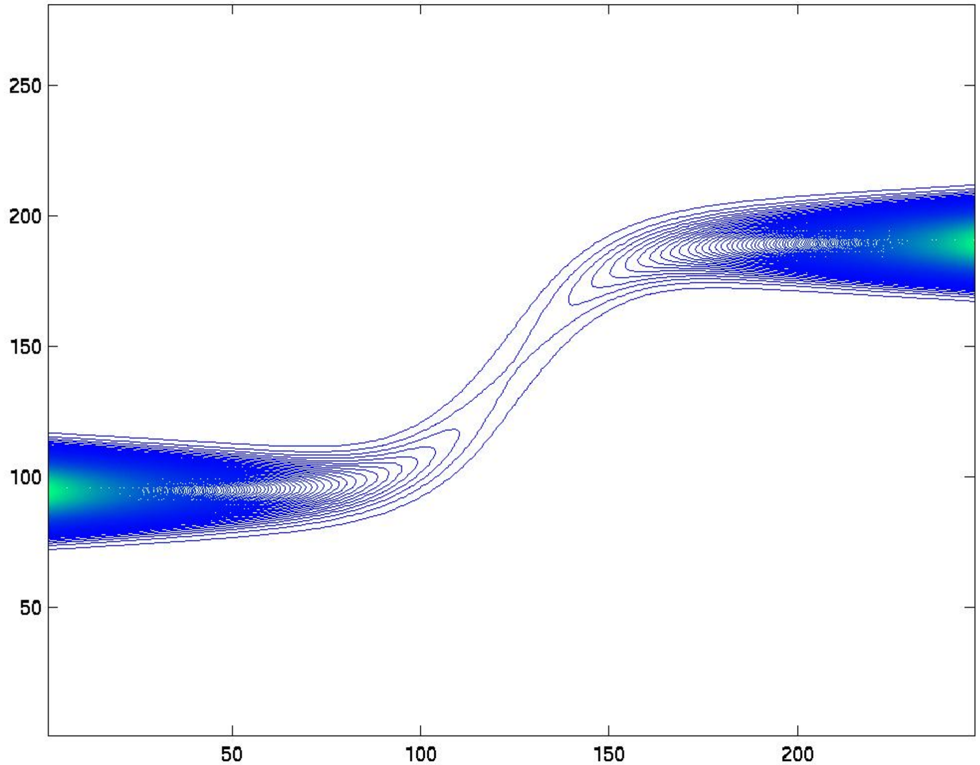

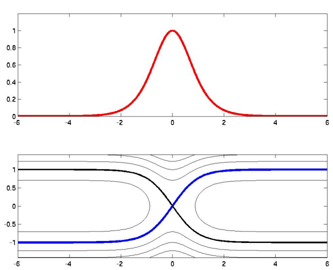

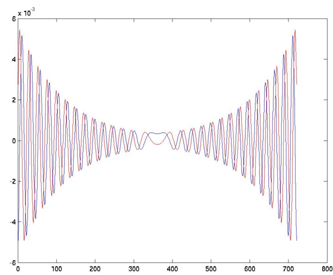

FBI transform thanks to Laurent Demanet

`www.math.stanford.edu/~laurent`

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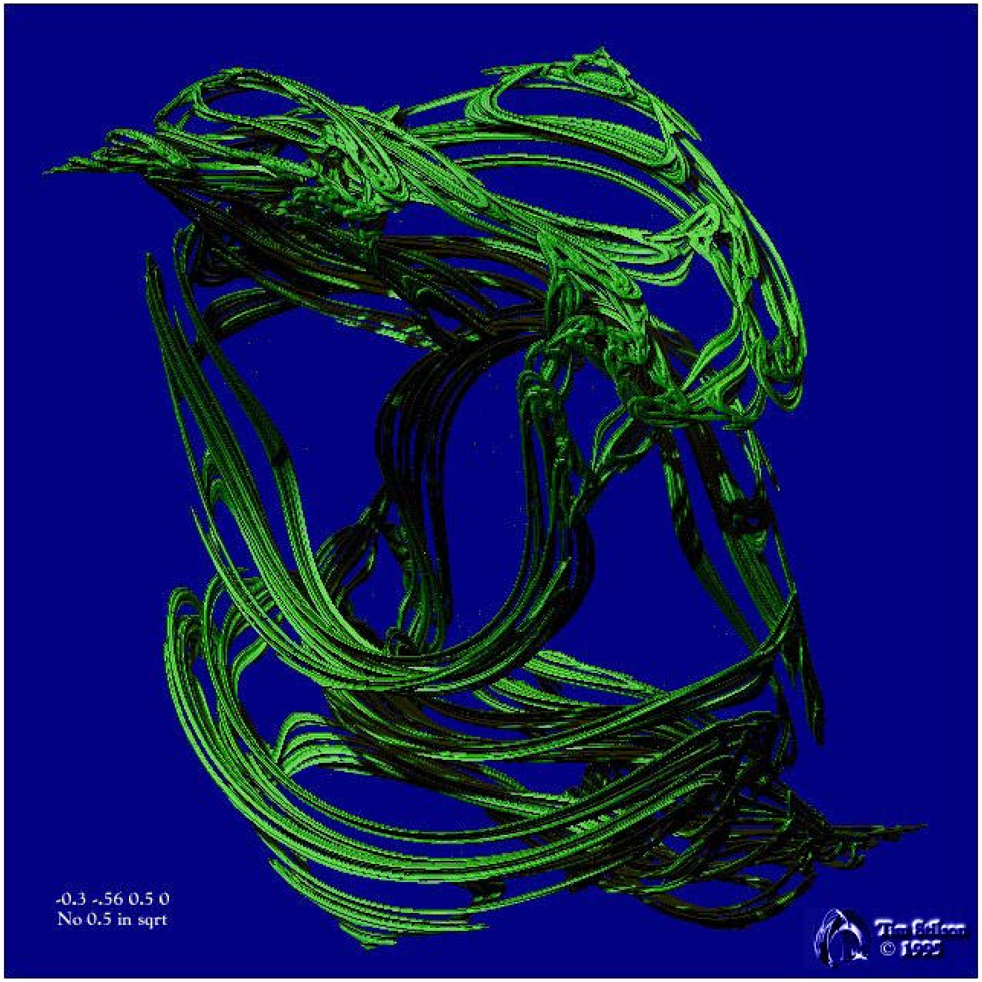

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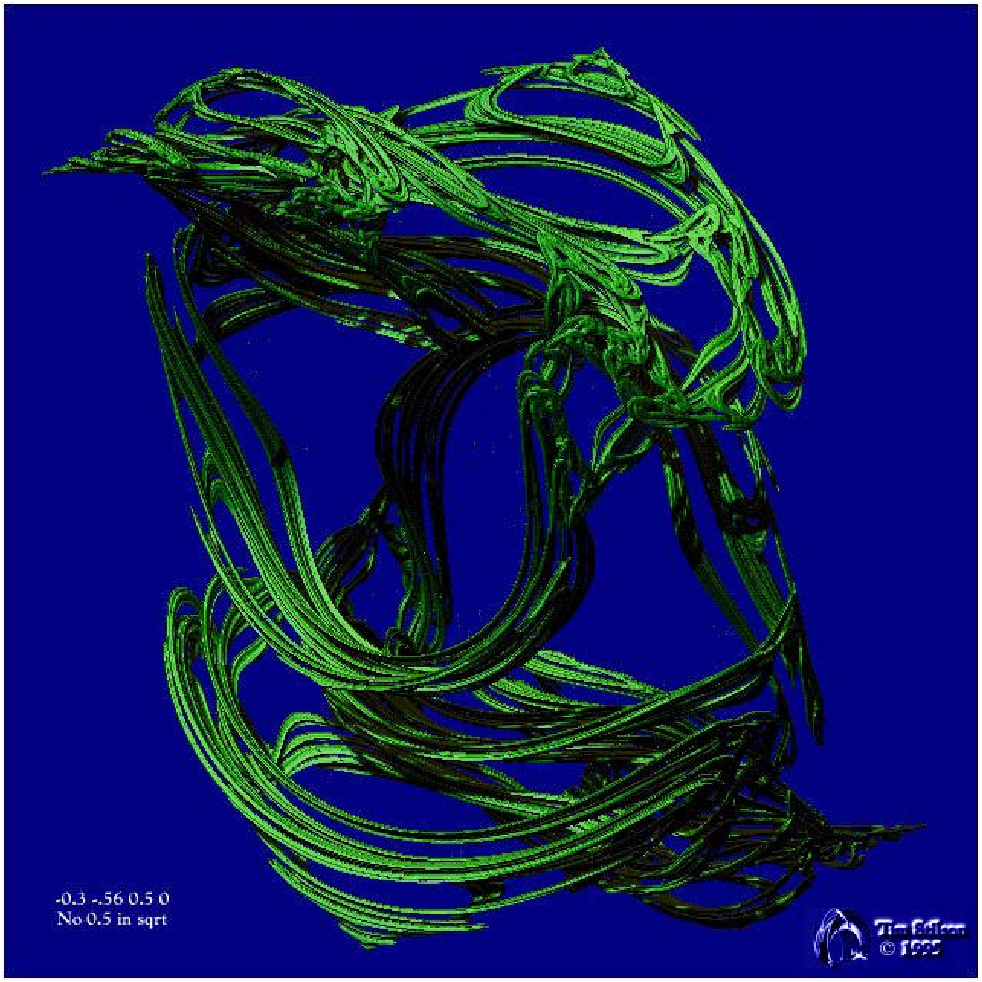

This is actually a picture of a Julia set but the similarity is more than formal and similar ideas apply to zeros of Ruelle zeta functions.

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Verification of this is not easy but in our setting it is available thanks to the work of Sinai, Ikawa, Sjöstrand, and Morita.

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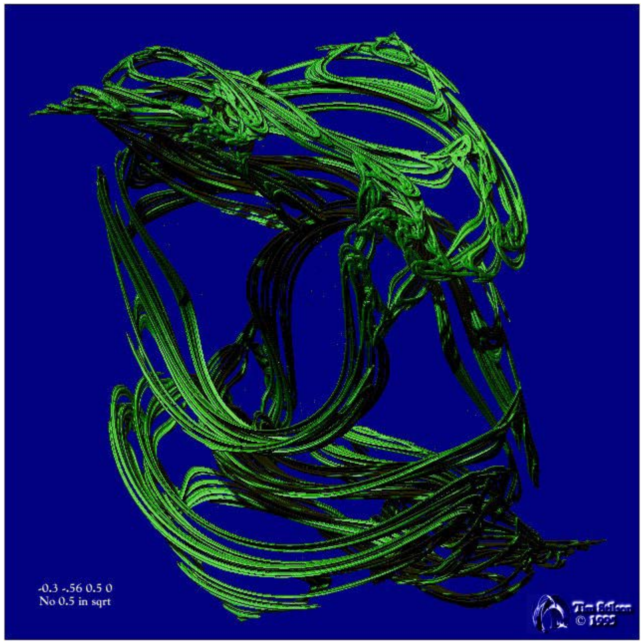

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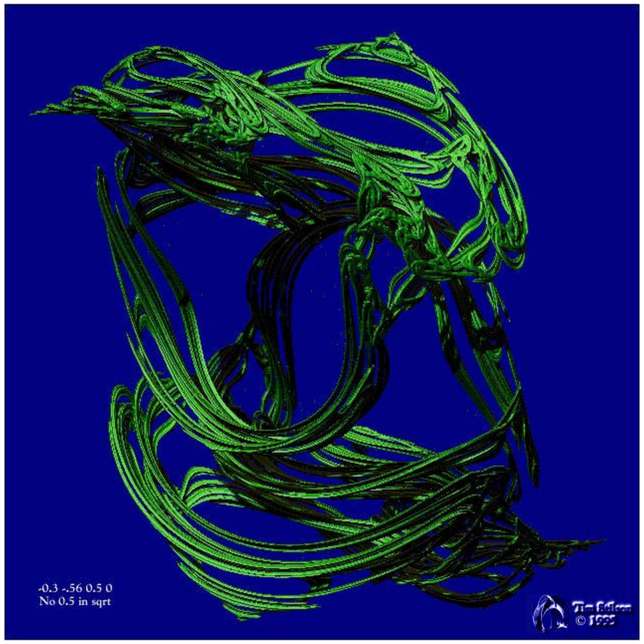

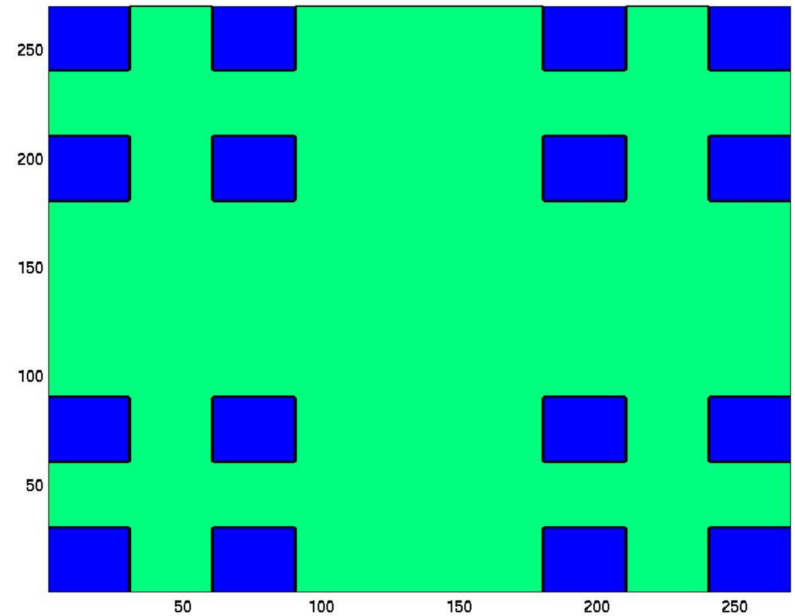

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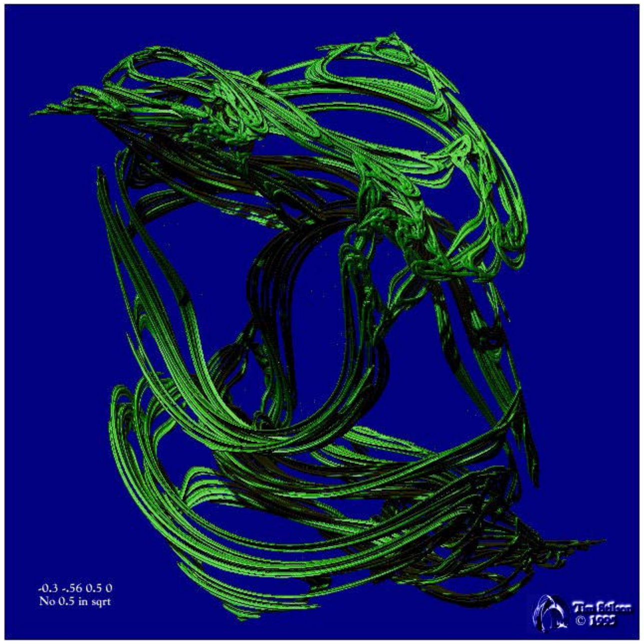

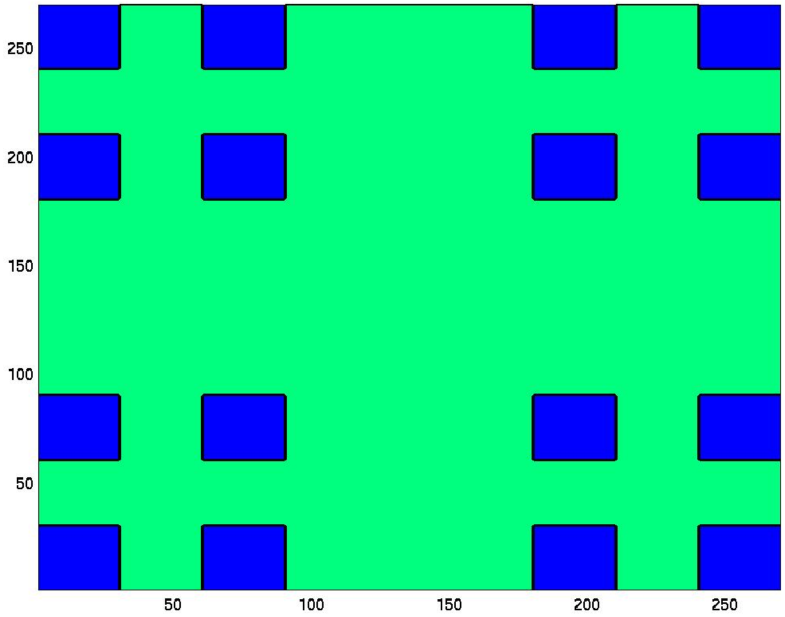

We write the Hausdorff dimension of K_E as

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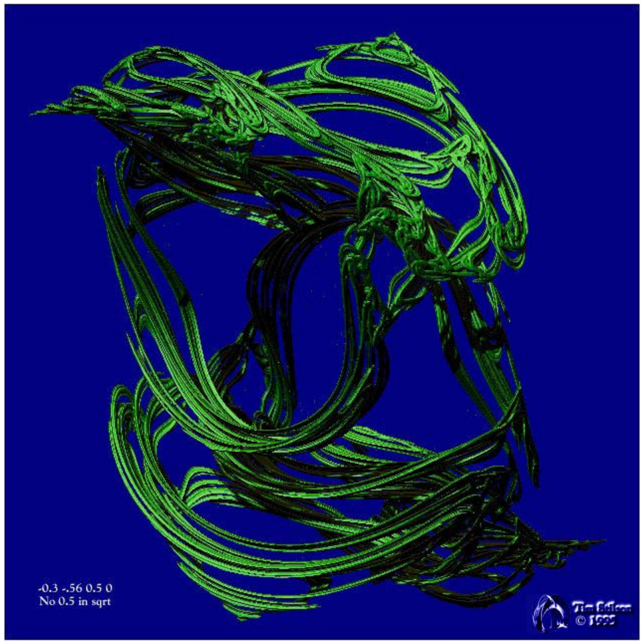

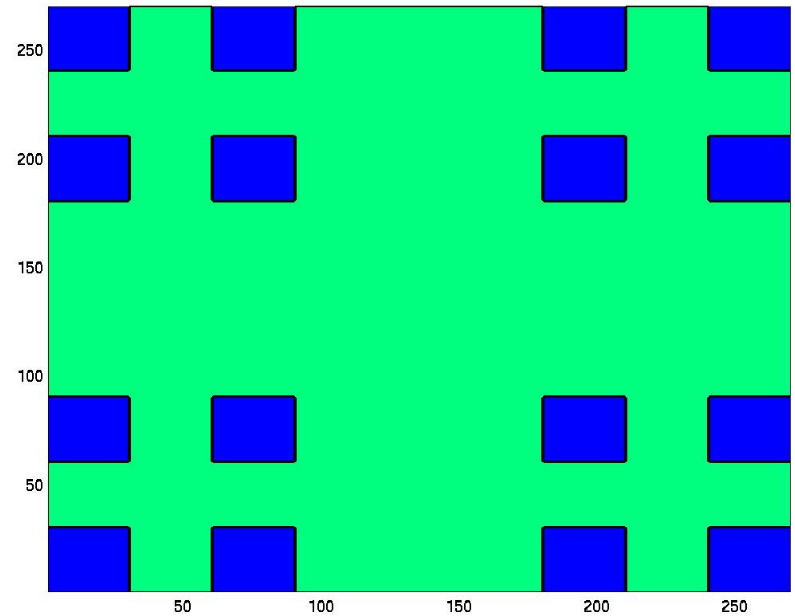

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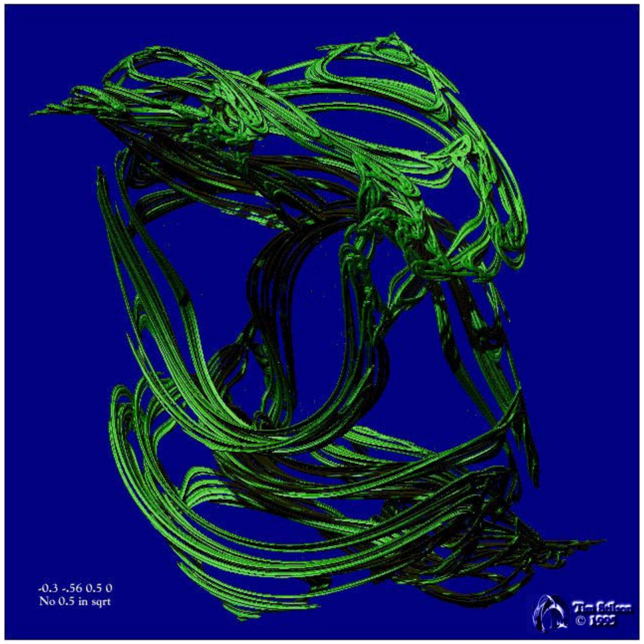

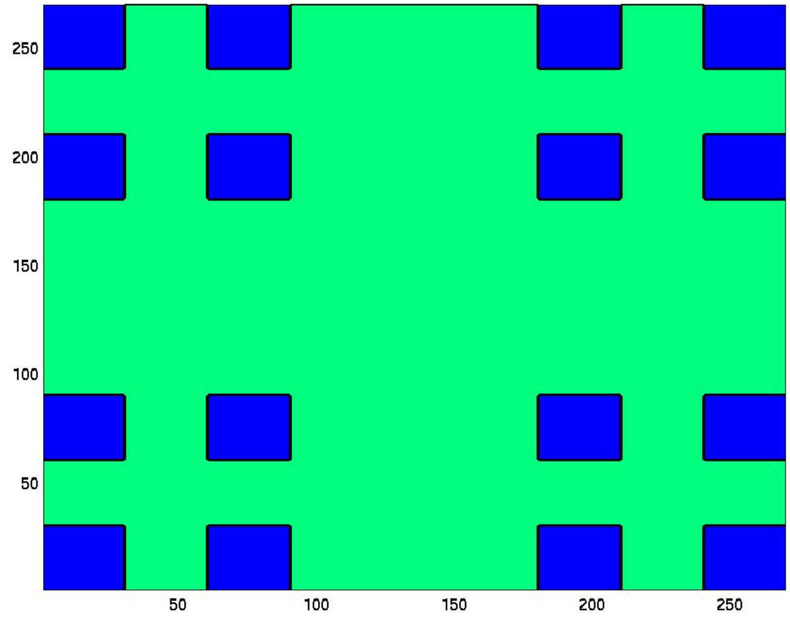

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Pesin-Sadovskaya 2001

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This is the analogue of the counting law for eigenvalues of a closed system. Classically everything is trapped in a closed system, so $\dim K_E = 3$, $d_E = 1$ and the number of eigenvalues is asymptotic to

$$C_E h^{-1}\,.$$

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Interpretation of the imaginary part as decay rate brings us to the next theorem.

Question: What properties of the flow Φ_t, or of K_E alone, imply the existence of a gap $\gamma > 0$ such that, for $h > 0$ sufficiently small,

$$z \in \mathcal{R}(h)\,, \qquad \operatorname{Re} z \sim E \implies \operatorname{Im} z < -\gamma h\,?$$

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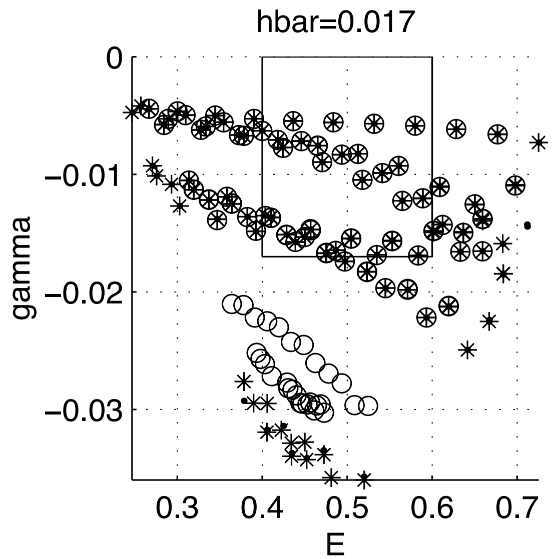

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What is γ?

It can be described using the topological pressure of the flow on K_E.

We can take any γ satisfying

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The existence of a resonance gap depends on the *sign* of the pressure at $s = 1/2$, $P_E(1/2)$.

The connection between the pressure and the quantum decay rate first appeared in the physics/chemistry literature in the work of Gaspard-Rice 1989.

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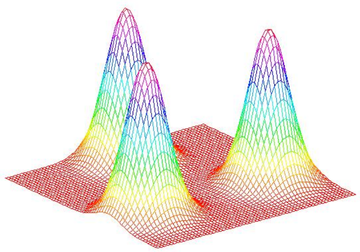

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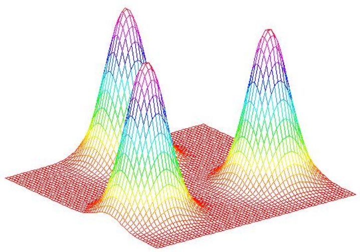

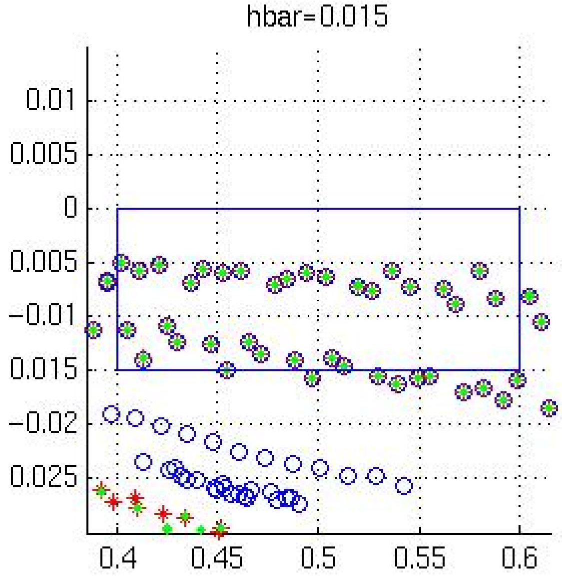

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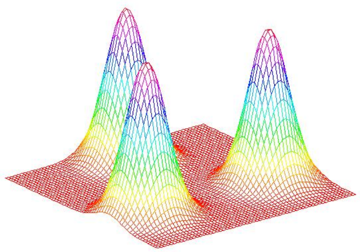

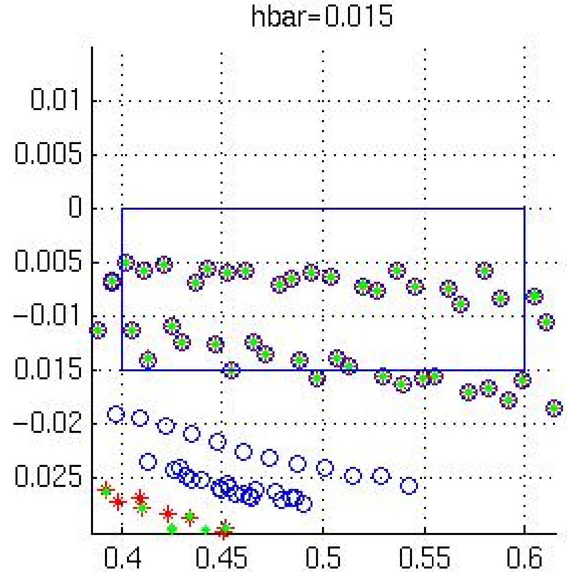

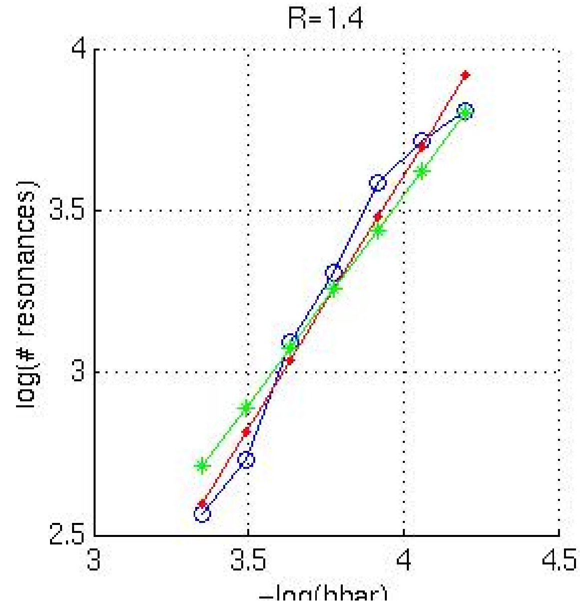

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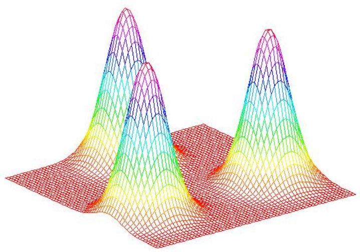

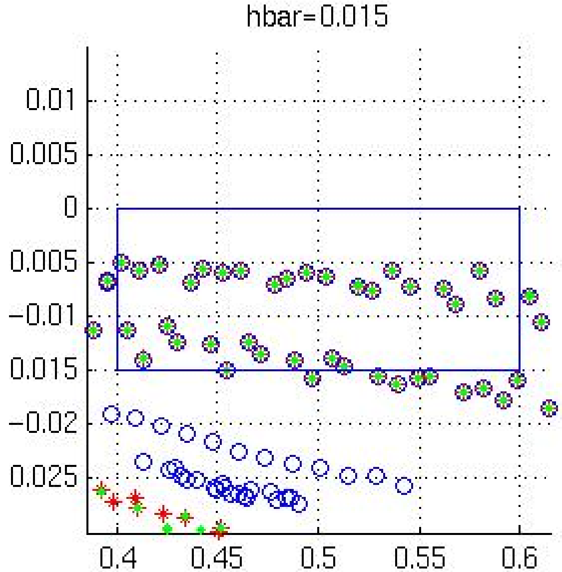

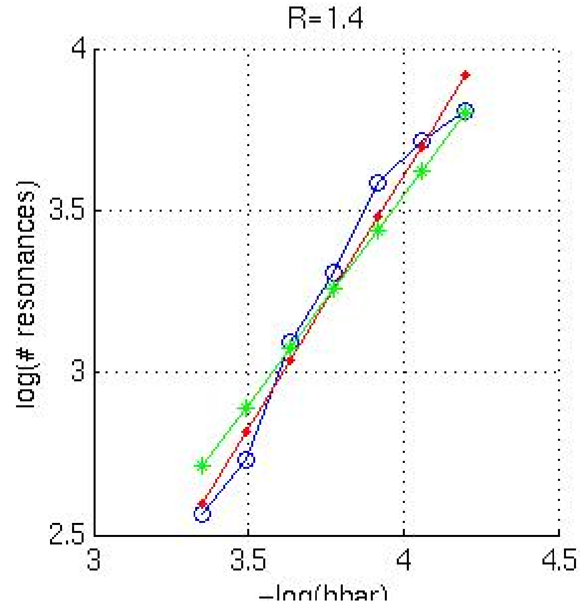

This and also some quantum map rigorous models of Nonnenmacher-Zworski 2005 suggest that Theorem 2 is optimal.

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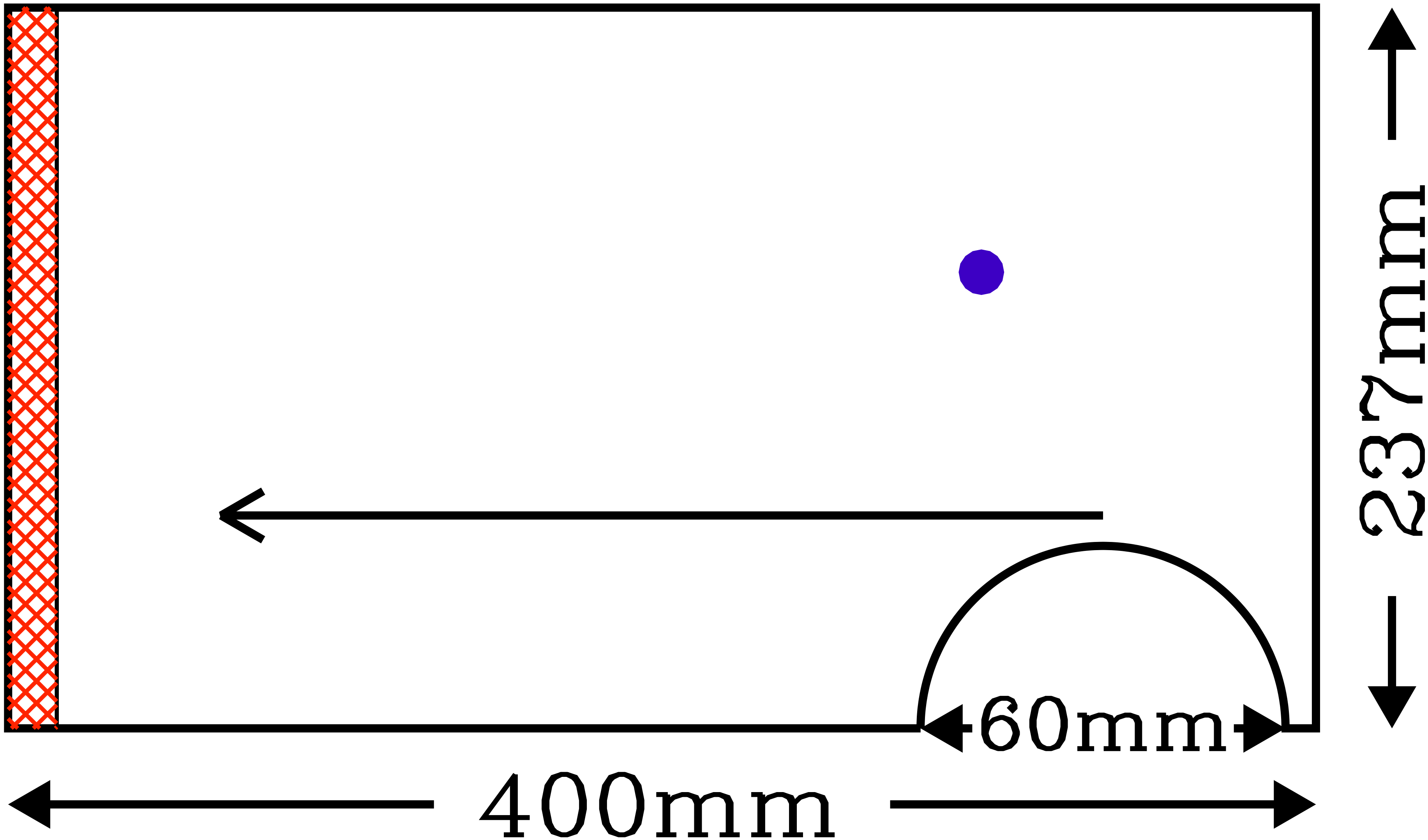

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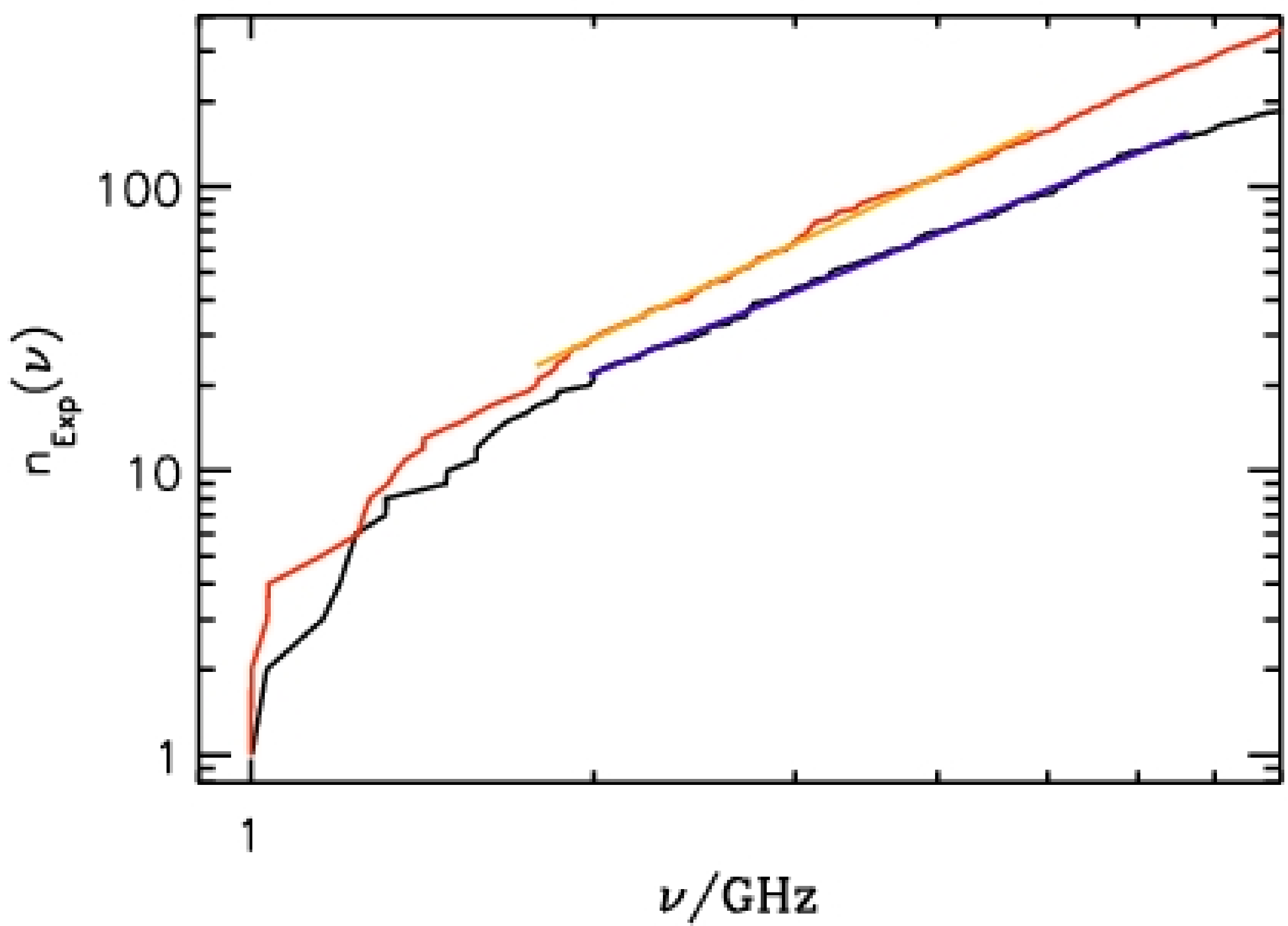

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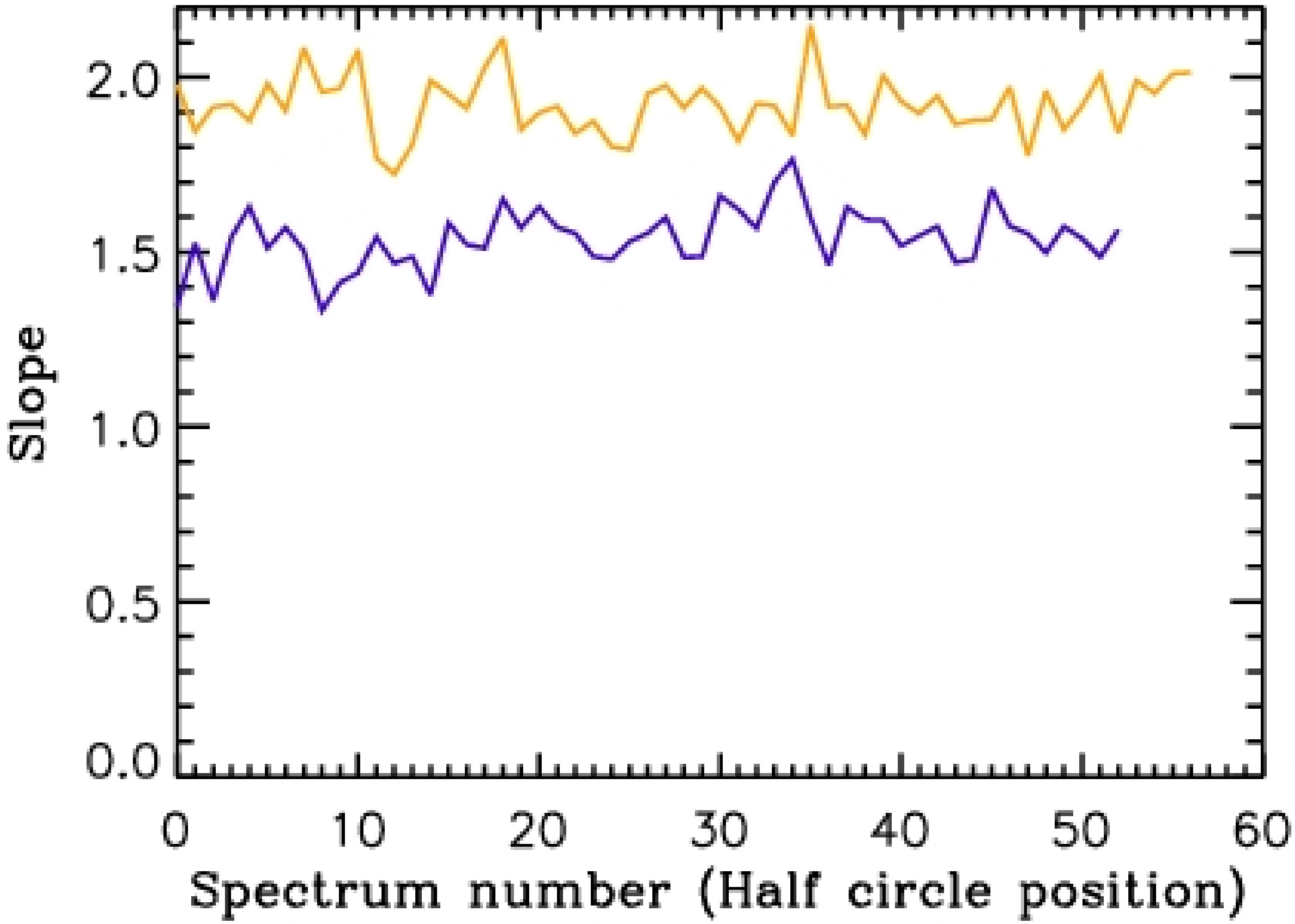

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That relies on delicate zeta function analysis following the work of Dolgopyat: at zero energy there exists a Patterson-Sullivan resonance with the imaginary part (width) given by the pressure but all other resonances have more negative imaginary parts.