Interaction of two solitons in an external field

Colloque Franco-Tunisien d’équations aux dérivées partielles

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October 2, 2009
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A double soliton solution of \textit{mKdV} with speeds \( c_1 = 6 \) and \( c_2 = 9 \).
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\[ q(x, a, c) = \frac{\det M_1}{\det M} \]

where

\[
M = \begin{bmatrix}
\frac{1+\gamma^2_1}{2c_1} & \frac{1+\gamma_1\gamma_2}{c_1+c_2} \\
\frac{1+\gamma_1\gamma_2}{c_1+c_2} & \frac{1+\gamma^2_2}{2c_2}
\end{bmatrix}, \quad M_1 = \begin{bmatrix}
M & \gamma_1 \\
1 & \gamma_2
\end{bmatrix}
\]

and

\[ \gamma_1 = e^{-c_1(x-a_1)}, \quad \gamma_2 = -e^{-c_2(x-a_2)}. \]
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$$M = \begin{bmatrix} \frac{1+\gamma_1^2}{2c_1} & \frac{1+\gamma_1 \gamma_2}{c_1+c_2} \\ \frac{1+\gamma_1 \gamma_2}{c_1+c_2} & \frac{1+\gamma_2^2}{2c_2} \end{bmatrix}, \quad M_1 = \begin{bmatrix} M & \gamma_1 \\ 1 & \gamma_2 \\ 1 & 0 \end{bmatrix}$$

and

$$\gamma_1 = e^{-c_1(x-a_1)}, \quad \gamma_2 = -e^{-c_2(x-a_2)}.$$  

Then remarkably the following solves mKdV:

$$u(x, t) = q(x, a_1 + c_1^2 t, a_2 + c_2^2 t, c_1, c_2)$$
Singular behaviour at \( c_1 = \pm c_2 \).
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In particular, at $c_1 = 0$, $c_2 = c_2^\ast > 0$ we recover the 1-soliton: 

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If $a_1 < a_2$, then

$$q(x, a, c) \approx \eta(x, a_1 + \alpha_1^-, c_1) + \eta(x, a_2 + \alpha_2^-, c_2)$$

where

$$\alpha_1^- = -\frac{1}{c_1} \ln \left( \frac{c_1 + c_2}{c_1 - c_2} \right) < 0, \quad \alpha_2^- = -\frac{1}{c_2} \ln \left( \frac{c_1 - c_2}{c_1 + c_2} \right) > 0$$
shifted left to $a_1 + \alpha_1^-$

shifted right to $a_2 + \alpha_2^-$
If $a_1 > a_2$, then

$$q(x, a, c) \approx \eta(x, a_1 + \alpha_1^+, c_1) + \eta(x, a_2 + \alpha_2^+, c_2)$$

where the shifts are

$$\alpha_1^+ = \frac{1}{c_1} \ln \left( \frac{c_1 + c_2}{c_1 - c_2} \right) > 0, \quad \alpha_2^+ = \frac{1}{c_2} \ln \left( \frac{c_1 - c_2}{c_1 + c_2} \right) < 0$$
shifted left to $a_2 + \alpha_2^+$

$a_2 = -3$
$c_2 = 1$

shifted right to $a_1 + \alpha_1^+$

$a_1 = 3$
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\[ u(x, t) = q(x, a_1 + c_1^2 t, a_2 + c_2^2 t, c_1, c_2) \]
We study the dynamics of 2-soliton initial data for the perturbed mKdV equation

$$\partial_t u + \partial_x (\partial_x^2 u + 2u^3 - bu) = 0$$

with a slowly-varying potential

$$b(x, t) = b_0(hx, ht), \quad 0 < h \ll 1$$
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mKdV (say as opposed to KdV or NLS) seems to provide the simplest setting in which to study 2-solitons.
Motivation:

Formation and propagation of matter-wave soliton trains, K.E. Strecker et al. Nature, May, 2002. This is modeled by NLS + potential but mKdV is a simpler model: the manifold of 2-solitons in four dimensional rather than eight dimensional.
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For $N$-solitons we need to consider solutions in $H^N$: $$
abla_t u + \nabla_x (\nabla_x^2 u + 2u^3 - bu) = 0$$ $u_0 \in H^N$, $k \geq 1$.

Local well-posedness in $H^N$, $N \geq 1$, follows from local smoothing estimate of Kenig-Ponce-Vega (1993) provided $\nabla^\alpha_t \nabla^\beta_x b \in L^\infty_t (L^2_x \cap L^\infty_x)$, $0 \leq \alpha \leq 1$, $0 \leq \beta \leq N + 1$.

Upgraded to global well-posedness by computing $\nabla_t I_j (u)$ and estimating using the Gronwall inequality.
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Take soliton initial data:

1-soliton case : \( u_0(x) = \eta(x, a_0, c_0) \)

2-soliton case : \( u_0(x) = q(x, a_0, c_0) \)
Theorem (HPZ (2009) 1-soliton case)

Suppose that $a(t)$, $c(t)$ satisfy

\begin{align*}
\dot{a} &= c^2 - \frac{1}{2} \partial_c B(a, c, t), \\
\dot{c} &= \frac{1}{2} \partial_a B(a, c, t),
\end{align*}

with initial data $a(0) = a_0$, $c(0) = c_0$.

Suppose that $0 < \delta < c(t) < \delta - \frac{1}{2}$.

Then for $t \leq \delta^{-1} \log \left( \frac{1}{\delta} \right)$,

the solution $u(t)$ to mKdV with initial data $u(\cdot, 0) = \eta(\cdot, a_0, c_0)$ satisfies

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\| u(\cdot) - \eta(\cdot, a(t), c(t)) \|_{H^1} \leq Ch^{2 - \delta}.
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Suppose that $0 < \delta < c(t) < \delta - 1$.

Then for $t \leq \delta - 1 \log(1/h)$, the solution $u(t)$ to mKdV with initial data $u(\cdot, 0) = \eta(\cdot, a_0, c_0)$ satisfies

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B(a, c, t) = \int b(x, t) \eta^2(x, a, c) \, dx.
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This is an improvement of Dejak-Jonsson (2006) who obtained a similar result with $O(h^2)$ errors in the ODE and the conclusion

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Our above result is modeled on our previous work (Holmer-Zworski (2008)) for NLS, which was an improvement of a result of Fröhlich-Gustafson-Jonsson-Sigal (2004).
Other related work:

  Fröhlich–Jonsson-Lenzmann (2007): dynamics of boson stars (as solitons)
  Dejak-Sigal (2006) gKdV.

We are not aware of any result giving effective dynamics for interacting 2-solitons in the presence of a slowly-varying potential for any equation.
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\end{align*}

for $j = 1, 2$.

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If $0 < \delta < |c_1(t) \pm c_2(t)| < \delta - 1$, then for $t \leq \delta h - \frac{1}{2} \log(1/h)$, the solution $u(t)$ to mKdV with initial data

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\|u(\cdot) - q(\cdot, a(t), c(t))\|_{H^2} \leq Ch^{2-\delta}.
$$
Here is an example of soliton motion in an external field:

\[ b = 100 \cos^2(x + 1 - 10^3 t) + 50 \sin(2x + 2 + 10^3 t), \]

\[ c_1 = 6, \quad c_2 = -11, \quad a_1 = 0, \quad a_2 = -2. \]
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Comparison with the effective dynamics:

\[ h_{\text{eff}} \approx 1, \quad t_{\text{eff}} \approx 50 \gg \log(1/h)/h \]
The case to which the theorem does not quite apply:
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\[ b = 40 \cos(2x + 3 - 10^2 \cdot t) + 30 \sin(x + 1 + 10^2 \cdot t), c_1 = 6, c_2 = 9, a_1 = -1, a_2 = -2 \]
Where do the effective equations of motion come from?
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Hamiltonian structure:

\[ J = \partial_x, \quad J^{-1}f(x) = \partial_x^{-1}f(x) = \frac{1}{2} \left( \int_{-\infty}^{x} - \int_{x}^{+\infty} \right) f(y) \, dy \]

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Note that \( \partial_x^{-1} \) is not defined on all of \( H^2 \). Not a problem in our analysis for mKdV, but a problem for KdV.
Suppose we assume that the mKdV flow remains close to the manifold of solitons

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$$H|_M = l_3(q) + \int b q^2 = -\frac{1}{3} c_1^3 - \frac{1}{3} c_2^3 + B(a, c, t)$$

$$\omega|_M = da_1 \wedge dc_1 + da_2 \wedge dc_2$$

Computed using the magic identities for $q$. 
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Computed using the magic identities for \( q \).

The equations in the theorem statement are just the flow equations on \( M \).
To prove the theorem we begin with properties of free mKdV.

\[ \partial_t u = -\partial_x (\partial_x^2 u + 2u^3) \]

with \( u : \mathbb{R}^{1+1} \to \mathbb{R} \).

Infinite number of conservation laws.

\[ I_1(u) = \int u^2 \, dx \]

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From the asymptotics

\[ I_j(q) = 2(-1)^{\frac{j-1}{2}} \frac{c_1^j + c_2^j}{j} \]
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1-soliton:  $\partial_x l'_1(\eta) = \partial_x \eta = -\partial_a \eta$
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$L_c$ is used as a Lyapunov functional in the orbital stability theory of Weinstein, Grillakis-Shatah-Strauss, Bona-Souganidis-Strauss (1985–1990). Notice we get some information about $\mathcal{L}_{c,a}$, namely

$$\mathcal{L}_{c,a}(\partial_a \eta) = 0$$

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$$\mathcal{K}_{c,a}(\partial_{a_1} q) = 0, \quad \mathcal{K}_{c,a}(\partial_{a_2} q) = 0$$

$$\mathcal{K}_{c,a}(\partial_{c_1} q) = c_1^2 l'_3(q) + 2c_1 c_2^2 l'_1(q), \quad \mathcal{K}_{c,a}(\partial_{c_2} q) = c_2^2 l'_3(q) + 2c_1^2 c_2 l'_1(q)$$
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This makes \( q \) the symplectic orthogonal projection of \( u \) onto the manifold of solitons \( M \).
Since $u = q + v$ and $u$ solves mKdV, we have

$$
\partial_t v = \partial_x \mathcal{L}_{c,a} v - 6qv^2 - 2v^3 + \partial_x (bv) - F_0
$$

where $F_0$ results from the perturbation and $\partial_t$ landing on the parameters:

$$
F_0 = \sum_{j=1}^{2} (\dot{a}_j - c_j^2) \partial_{a_j} q + \sum_{j=1}^{2} \dot{c}_j \partial_{c_j} q - \partial_x (bq)
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$F_2$ is the symplectic projection onto $TM^\perp$. 
$F_1$ contains the alleged equations of motion as coefficients:

$$F_1 = \sum_{j=1}^{2} (\dot{a}_j - c_j^2 - \frac{1}{2} \partial c_j B) \partial a_j q + \sum_{j=1}^{2} (\dot{c}_j - \frac{1}{2} \partial a_j B) \partial c_j q$$

$$F_2 = -\partial_x (b q) + \frac{1}{2} \sum_{j=1}^{2} (\partial c_j B) \partial a_j q + (\partial a_j B) \partial c_j q$$
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Using the magic identities, can show that $F_2$ is $O(h^2)$, in fact get a specific form for the $O(h^2)$ term that is amenable to finding the correction term needed later.
The equations of motion are then recovered in approximate form using the symplectic orthogonality properties of \( v \). For example,

\[
0 = \langle v, \partial^{-1}_{x} \partial_{a} q \rangle
\]

\[
\implies 0 = \partial_{t} \langle v, \partial^{-1}_{x} \partial_{a} q \rangle = \langle \partial_{t} v, \partial^{-1}_{x} \partial_{a} q \rangle + \langle v, \partial_{t} \partial^{-1}_{x} \partial_{a} q \rangle
\]

This can be manipulated (again using the identities) to show

\[
|F_1| \leq Ch^2 \| v \|_{H^2} + \| v \|_{H^2}^2
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Assume that initially $v = O_{H^2}(h^2)$. Want to show that on time interval of length $h^{-1}$ that $v$ at most doubles.
Next step is to estimate \( \nu \).

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**Lyapunov functional**

\[
\mathcal{E}(t) = L_{c(t)}(q + \nu) - L_{c(t)}(q)
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where \( L \) was defined before in terms of \( l_5, l_3, l_1 \).
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and $\mathcal{K}_{c,a}$ has a kernel and one negative eigenvalue.

However, the symplectic orthogonality conditions on $\nu$ imply that we project far enough away from these eigenspaces and hence

$$\delta \| \nu \|_{H^2}^2 \leq \mathcal{E}(t)$$
To get the upper bound, we need to compute using that
\[ L'_C(v + q) \approx L''_c(q)v \]

\[
\frac{d}{dt} \mathcal{E}(t) = 2(c_1 \dot{c}_1 + c_2 \dot{c}_2)(l_3(q + v) - l_3(q)) \quad \leftarrow \text{I}
\]
\[ + 2(c_1 \dot{c}_1 c_2^2 + c_1^2 c_2 \dot{c}_2)(l_1(q + v) - l_1(q)) \quad \leftarrow \text{II}
\]
\[ + \langle \mathcal{K}_{c,a}v, \partial_x(bv) \rangle \quad \leftarrow \text{III}
\]
\[ + \langle \mathcal{K}_{c,a}v, F_1 \rangle \quad \leftarrow \text{IV}
\]
\[ + \langle \mathcal{K}_{c,a}v, F_2 \rangle \quad \leftarrow \text{V}
\]
To get the upper bound, we need to compute using that
$L'_C(v + q) \approx L''_C(q)v$

$$\frac{d}{dt} \mathcal{E}(t) = 2(c_1 \dot{c}_1 + c_2 \dot{c}_2)(l_3(q + v) - l_3(q)) \leftarrow I$$
$$+ 2(c_1 \dot{c}_1 c_2^2 + c_1^2 c_2 \dot{c}_2)(l_1(q + v) - l_1(q)) \leftarrow II$$
$$+ \langle \mathcal{K}_c, a v, \partial_x(bv) \rangle \leftarrow III$$
$$+ \langle \mathcal{K}_c, a v, F_1 \rangle \leftarrow IV$$
$$+ \langle \mathcal{K}_c, a v, F_2 \rangle \leftarrow V$$

Terms I, II, III are $\lesssim h \|v\|^2_{H^2}$ and by the good estimate on $F_1$, Term IV is controlled.
To get the upper bound, we need to compute using that $L'_C(v + q) \approx L''_c(q)v$

$$\frac{d}{dt} \mathcal{E}(t) = 2 \left( c_1 \dot{c}_1 + c_2 \dot{c}_2 \right) \left( l_3(q + v) - l_3(q) \right) \leftarrow \text{I}$$
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Terms I, II, III are $\lesssim h \|v\|^2_{H^2}$ and by the good estimate on $F_1$, Term IV is controlled.

However, $|F_2| \lesssim h^2$ only. We improve this to $h^3$ using a correction term to $v$. 
Then obtain on $[0, T]$

$$\|v\|_{H^2}^2 \lesssim \|v(0)\|_{H^2}^2 + T(|F_1\|_{H^2} + h^2 \|v\|_{H^2} + \|v\|_{H^2}^2)$$
Then obtain on $[0, T]$

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Recap the two key estimates:

$$\|v\|_{H^2}^2 \lesssim \|v(0)\|_{H^2}^2 + T(|F_0| \|v\|_{H^2} + h^2 \|v\|_{H^2} + \|v\|_{H^2}^2)$$

$$|F_1| \leq Ch^2 \|v\|_{H^2} + \|v\|_{H^2}^2$$
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Combine to give

$$\|v\|_{H^2} \lesssim h^2, \quad |F_1| \lesssim h^4, \quad \text{on } [0, h^{-1}]$$
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$\delta \log(1/h)$ iterations give the slightly weaker bound on $[0, \delta h^{-1} \log(1/h)]$. 
Then obtain on $[0, T]$

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$\delta \log(1/h)$ iterations give the slightly weaker bound on $[0, \delta h^{-1} \log(1/h)]$.

The $O(h^4)$ errors in the ODEs can be removed without affecting the bound on $v$. 

Remarks:

(1) The idea of adding a correction term to $\nu$ to improve $\|F_2\|$ from $h^2$ to $h^3$ was used by Holmer-Zworski (2007) for NLS 1-solitons. Together with the symplectic projection interpretation, it is key to sharpening the results in earlier works.
Remarks:

(1) The idea of adding a correction term to $v$ to improve $\|F_2\|$ from $h^2$ to $h^3$ was used by Holmer-Zworski (2007) for NLS 1-solitons. Together with the symplectic projection interpretation, it is key to sharpening the results in earlier works.

Implementing the same idea here is a little more subtle. The 2-soliton is treated as if it were the sum of two decoupled 1-solitons, corrections are introduced for each piece, and the result is that $F_2$ is corrected so that

$$\|F_2\|_{H^2} \lesssim h^3 + h^2 e^{-\gamma |a_1 - a_2|}$$

That is, when $|a_1 - a_2| = O(1)$, no improvement. However, can only have $|a_1(t) - a_2(t)| = O(1)$ on an $O(1)$ time scale.
(2) The method is based on Hamiltonian / spectral techniques, which are applicable whether the underlying model is integrable or not. However, the existence and magical properties of $N$-solitons are typically only available for integrable equations.
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Recentl results on interacting solitons for nonintegrable equations:

Martel-Merle (2008) show for gKdV-4, describe the interaction of an $O(1)$ scale soliton with a very broad scale $c \ll 1$ soliton.

Perelman (2009) shows for the NLS with nonlinearity close to cubic, a fast soliton interacting with a stationary high mass soliton ($\delta_0$-like) splits into two solitons described using the scattering matrix of the high soliton.