Interaction of two solitons in an external field

Colloque Franco-Tunisien d'équations aux dérivées partielles

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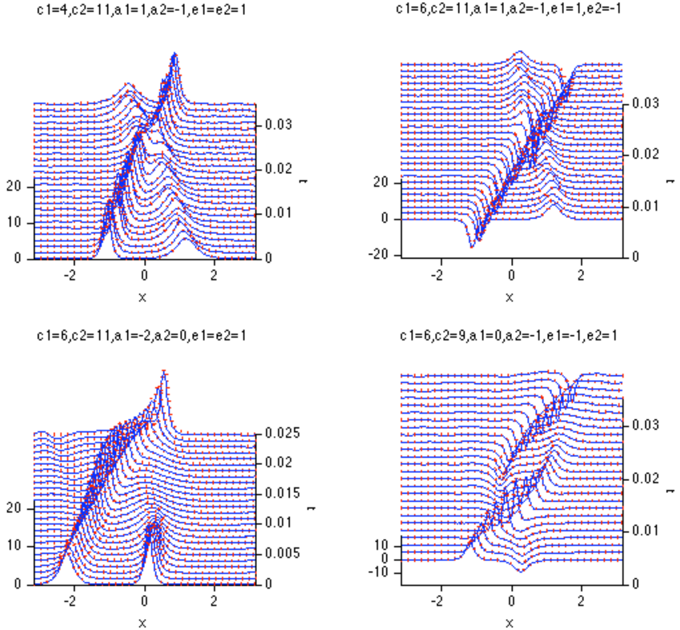
October 2, 2009



c1=4,c2=11,a1=1,a2=-1,e1=e2=1
c1=6,c2=11,a1=1,a2=-1,e1=1,e2=-1
c1=6,c2=11,a1=-2,a2=0,e1=e2=1
c1=6,c2=9,a1=0,a2=-1,e1=-1,e2=1
x
x
x
x

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A double soliton solution of mKdV with speeds $c_1 = 6$ and $c_2 = 9$.

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where

$$M = \begin{bmatrix} \frac{1+\gamma_1^2}{2c_1} & \frac{1+\gamma_1\gamma_2}{c_1+c_2} \\ \\ \frac{1+\gamma_1\gamma_2}{c_1+c_2} & \frac{1+\gamma_2^2}{2c_2} \end{bmatrix}, \qquad M_1 = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{M}} & \gamma_1 \\ & & \gamma_2 \\ \hline 1 & 1 & 0 \end{array}\right]$$

and

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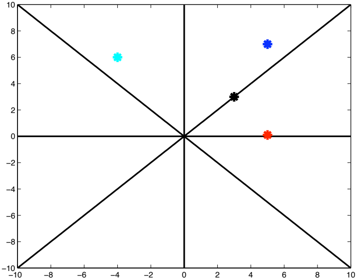
$$\gamma_1 = e^{-c_1(x-a_1)}, \quad \gamma_2 = -e^{-c_2(x-a_2)}.$$

Then remarkably the following solves mKdV:

$$u(x, t) = q(x, a_1 + c_1^2 t, a_2 + c_2^2 t, c_1, c_2)$$

Singular behaviour at $c_1 = \pm c_2$.



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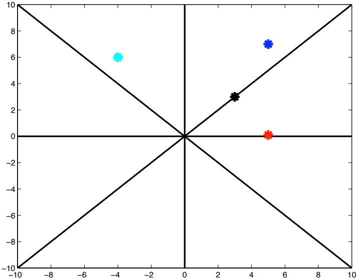

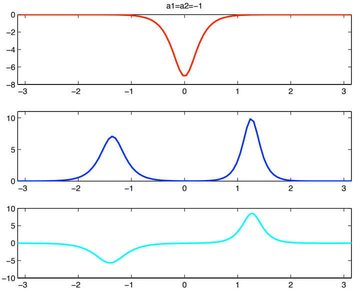

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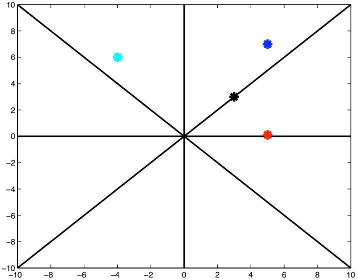

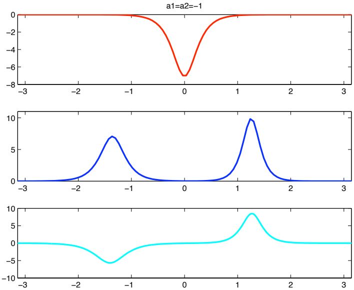

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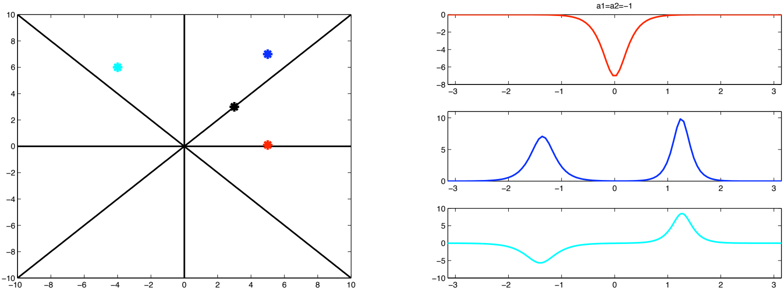

In particular, at $c_1 = 0$, $c_2 = c > 0$ we recover the 1-soliton:

$$\eta(x, a, c) = c\,\mathrm{sech}(c(x - a))\,.$$

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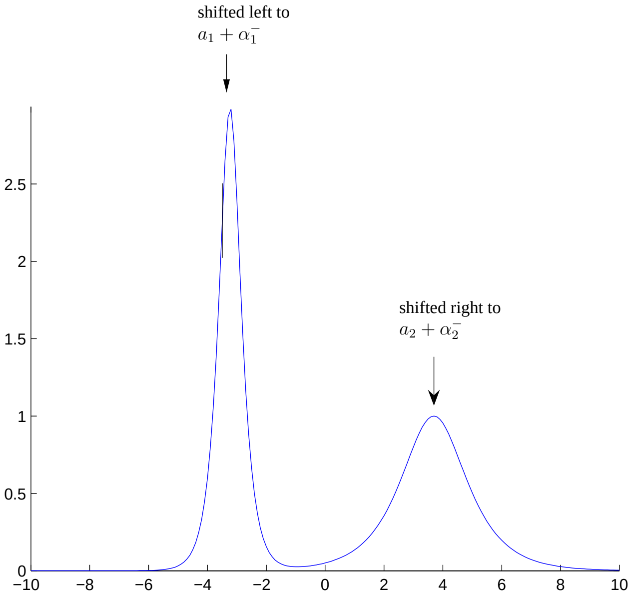
If $a_1 < a_2$, then

$$q(x, a, c) \approx \eta(x, a_1 + \alpha_1^-, c_1) + \eta(x, a_2 + \alpha_2^-, c_2)$$

where

$$\alpha_1^- = -\frac{1}{c_1} \ln\left(\frac{c_1 + c_2}{c_1 - c_2}\right) < 0\,, \qquad \alpha_2^- = -\frac{1}{c_2} \ln\left(\frac{c_1 - c_2}{c_1 + c_2}\right) > 0$$

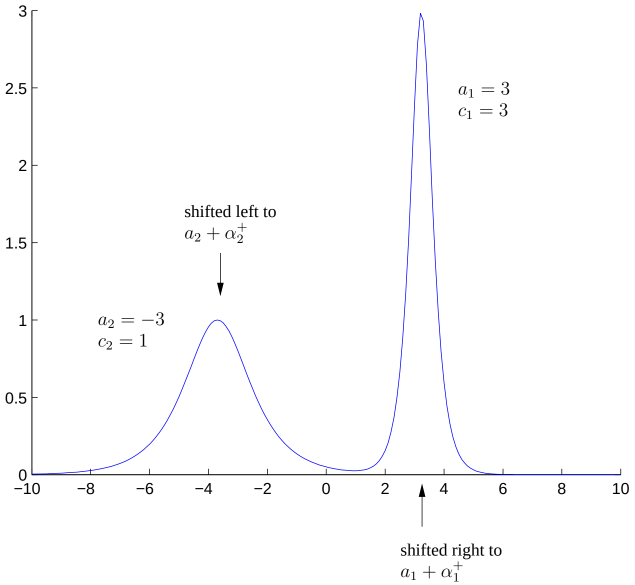

shifted left to
$a_1 + \alpha_1^-$
shifted right to
$a_2 + \alpha_2^-$
2.5
2
1.5
1
0.5
0
−10
−8
−6
−4
−2
0
2
4
6
8
10

If $a_1 > a_2$, then

$$q(x, a, c) \approx \eta(x, a_1 + \alpha_1^+, c_1) + \eta(x, a_2 + \alpha_2^+, c_2)$$

where the shifts are

$$\alpha_1^+ = \frac{1}{c_1} \ln\left(\frac{c_1 + c_2}{c_1 - c_2}\right) > 0\,, \qquad \alpha_2^+ = \frac{1}{c_2} \ln\left(\frac{c_1 - c_2}{c_1 + c_2}\right) < 0$$



3
2.5
2
1.5
1
0.5
0
−10
−8
−6
−4
−2
0
2
4
6
8
10
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We study the dynamics of 2-soliton initial data for the perturbed mKdV equation

$$\partial_t u + \partial_x(\partial_x^2 u + 2u^3 - bu) = 0$$

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mKdV (say as opposed to KdV or NLS) seems to provide the simplest setting in which to study 2-solitons.

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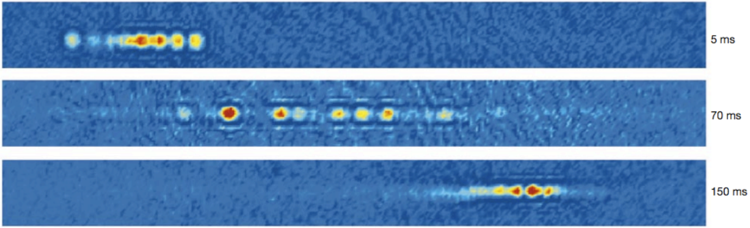

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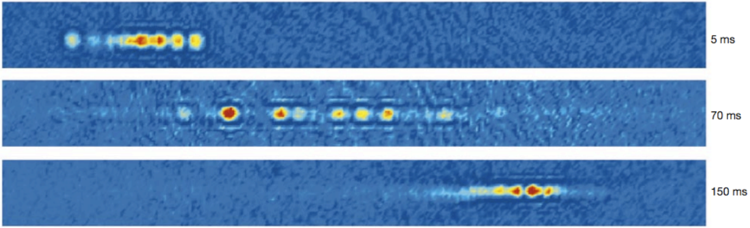

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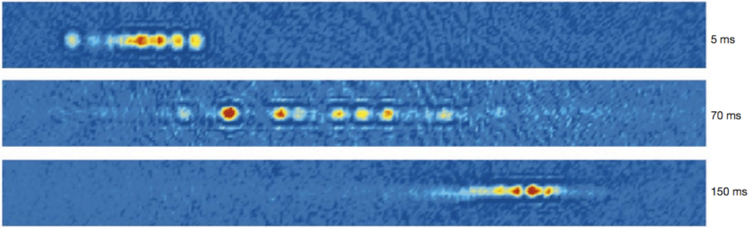

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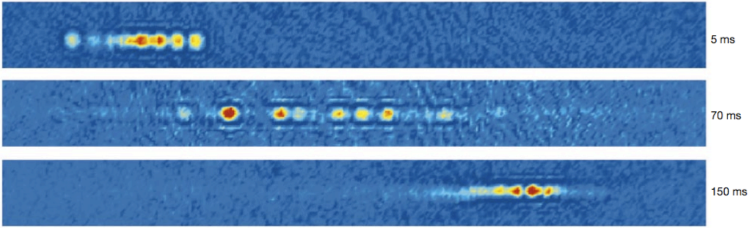

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Local well-posedness in H^N, $N \geq 1$, follows from local smoothing estimate of Kenig-Ponce-Vega (1993) provided

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Upgraded to global well-posedness by computing $\partial_t I_j(u)$ and estimating using the Gronwall inequality.

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Take soliton initial data:

$$\text{1-soliton case}: u_0(x) = \eta(x, a_0, c_0)$$

$$\text{2-soliton case}: u_0(x) = q(x, a_0, c_0)$$

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Our above result is modeled on our previous work (Holmer-Zworski (2008)) for NLS, which was an improvement of a result of Fröhlich-Gustafson-Jonsson-Sigal (2004).

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We are not aware of any result giving effective dynamics for interacting 2-solitons in the presence of a slowly-varying potential for any equation.

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Theorem (HPZ (2009) 2-soliton case)

Suppose that $a(t)$, $c(t)$ *satisfy*

$$\dot{a}_j = c_j^2 - \frac{1}{2}\partial_{c_j} B(a,c,t)\,, \quad \dot{c}_j = \frac{1}{2}\partial_{a_j} B(a,c,t)\,, \qquad j = 1,2.$$

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$$\|u(\cdot) - q(\cdot, a(t), c(t))\|_{H^2} \leq C h^{2-\delta}\,.$$

Here is an example of soliton motion in an external field:

$$b = 100\cos^2(x + 1 - 10^3 t) + 50\sin(2x + 2 + 10^3 t)\,,$$

$$c_1 = 6, \quad c_2 = -11, \quad a_1 = 0, \quad a_2 = -2\,.$$

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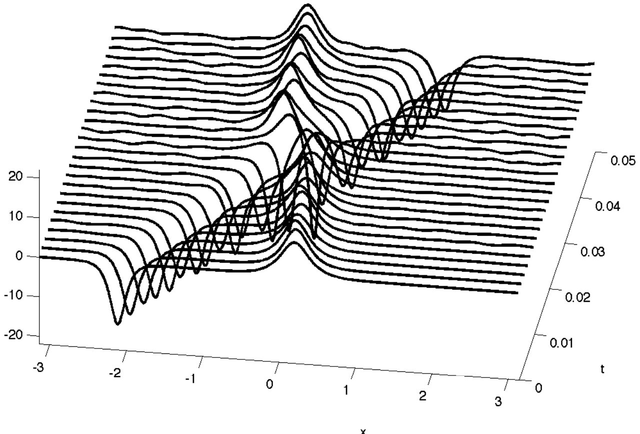

Comparison with the effective dynamics:

$$h_{\text{eff}} \approx 1\,,\;\; t_{\text{eff}} \approx 50 \gg \log(1/h)/h$$

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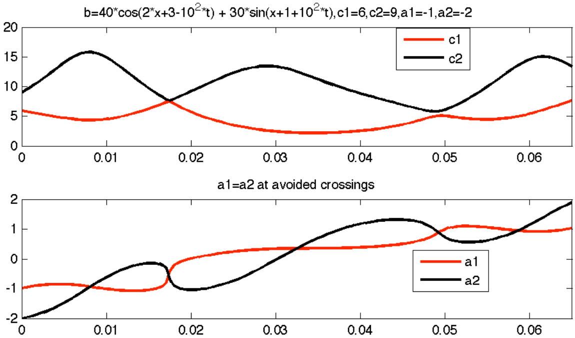

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$$J = \partial_x\,, \qquad J^{-1}f(x) = \partial_x^{-1}f(x) = \frac{1}{2}\left(\int_{-\infty}^{x} - \int_{x}^{+\infty}\right) f(y)\,dy$$

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Suppose we assume that the mKdV flow remains close to the manifold of solitons

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The equations in the theorem statement are just the flow equations on M.

To prove the theorem we begin with properties of free mKdV.

$$\partial_t u = -\partial_x(\partial_x^2 u + 2u^3)$$

with $u : \mathbb{R}^{1+1} \to \mathbb{R}$.
Infinite number of conservation laws.

$$I_1(u) = \int u^2 \, dx$$

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From the asymptotics

$$I_j(q) = 2(-1)^{\frac{j-1}{2}} \frac{c_1^j + c_2^j}{j}$$

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L_c is used as a Lyapunov functional in the orbital stability theory of Weinstein, Grillakis-Shatah-Strauss, Bona-Souganidis-Strauss (1985–1990). Notice we get some information about $\mathcal{L}_{c,a}$, namely

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$$\mathcal{K}_{c,a}(\partial_{a_1} q) = 0 \,, \qquad \mathcal{K}_{c,a}(\partial_{a_2} q) = 0$$

$$\mathcal{K}_{c,a}(\partial_{c_1} q) = c_1^2 I_3'(q) + 2 c_1 c_2^2 I_1'(q) \,, \;\; \mathcal{K}_{c,a}(\partial_{c_2} q) = c_2^2 I_3'(q) + 2 c_1^2 c_2 I_1'(q)$$

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This makes q the symplectic orthogonal projection of u onto the manifold of solitons M.

Since $u = q + v$ and u solves mKdV, we have

$$\partial_t v = \partial_x \mathcal{L}_{c,a} v - 6qv^2 - 2v^3 + \partial_x(bv) - F_0$$

where F_0 results from the perturbation and ∂_t landing on the parameters:

$$F_0 = \sum_{j=1}^{2} (\dot{a}_j - c_j^2)\partial_{a_j} q + \sum_{j=1}^{2} \dot{c}_j \partial_{c_j} q - \partial_x(bq)$$

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F_2 is the symplectic projection onto $TM^{\perp}$.

F_1 contains the alleged equations of motion as coefficients:

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Using the magic identities, can show that F_2 is $O(h^2)$, in fact get a specific form for the $O(h^2)$ term that is amenable to finding the correction term needed later.

The equations of motion are then recovered in approximate form using the symplectic orthogonality properties of v. For example,

$$0 = \langle v, \partial_x^{-1} \partial_{a_j} q \rangle$$

$$\implies 0 = \partial_t \langle v, \partial_x^{-1} \partial_{a_j} q \rangle = \langle \underbrace{\partial_t v}_{\substack{\uparrow \\ \text{substitute equation} \\ \text{for } v}}, \partial_x^{-1} \partial_{a_j} q \rangle + \langle v, \partial_t \partial_x^{-1} \partial_{a_j} q \rangle$$

This can be manipulated (again using the identities) to show

$$|F_1| \leq Ch^2 \|v\|_{H^2} + \|v\|_{H^2}^2$$

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and $\mathcal{K}_{c,a}$ has a kernel and one negative eigenvalue.

However, the symplectic orthogonality conditions on v imply that we project far enough away from these eigenspaces and hence

$$\delta \|v\|_{H^2}^2 \le \mathcal{E}(t)$$

To get the upper bound, we need to compute using that $L'_C(v+q) \approx L''_c(q)v$

$$
\begin{aligned}
\frac{d}{dt}\mathcal{E}(t) = \; & 2(c_1\dot{c}_1 + c_2\dot{c}_2)(I_3(q+v) - I_3(q)) && \leftarrow \mathrm{I} \\
& + 2(c_1\dot{c}_1 c_2^2 + c_1^2 c_2\dot{c}_2)(I_1(q+v) - I_1(q)) && \leftarrow \mathrm{II} \\
& + \langle \mathcal{K}_{c,a} v, \; \partial_x(bv) \rangle && \leftarrow \mathrm{III} \\
& + \langle \mathcal{K}_{c,a} v, \; F_1 \rangle && \leftarrow \mathrm{IV} \\
& + \langle \mathcal{K}_{c,a} v, \; F_2 \rangle && \leftarrow \mathrm{V}
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& + \langle \mathcal{K}_{c,a}v,\ \partial_x(bv)\rangle & \leftarrow \text{III} \\
& + \langle \mathcal{K}_{c,a}v,\ F_1\rangle & \leftarrow \text{IV} \\
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Terms I, II, III are $\lesssim h\|v\|_{H^2}^2$ and by the good estimate on F_1, Term IV is controlled.

To get the upper bound, we need to compute using that $L_C'(v+q) \approx L_c''(q)v$

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Terms I, II, III are $\lesssim h\|v\|_{H^2}^2$ and by the good estimate on F_1, Term IV is controlled.

However, $|F_2| \lesssim h^2$ only. We improve this to h^3 using a correction term to v.

Then obtain on $[0, T]$

$$\|v\|_{H^2}^2 \lesssim \|v(0)\|_{H^2}^2 + T(|F_1|\|v\|_{H^2} + h^2\|v\|_{H^2} + \|v\|_{H^2}^2)$$

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The $O(h^4)$ errors in the ODEs can be removed without affecting the bound on v.

Remarks:

(1) The idea of adding a correction term to v to improve $\|F_2\|$ from h^2 to h^3 was used by Holmer-Zworski (2007) for NLS 1-solitons. Together with the symplectic projection interpretation, it is key to sharpening the results in earlier works.

Remarks:

(1) The idea of adding a correction term to v to improve $\|F_2\|$ from h^2 to h^3 was used by Holmer-Zworski (2007) for NLS 1-solitons. Together with the symplectic projection interpretation, it is key to sharpening the results in earlier works.

Implementing the same idea here is a little more subtle. The 2-soliton is treated as if it were the sum of two decoupled 1-solitons, corrections are introduced for each piece, and the result is that F_2 is corrected so that

$$\|F_2\|_{H^2} \lesssim h^3 + h^2 e^{-\gamma|a_1 - a_2|}$$

That is, when $|a_1 - a_2| = O(1)$, no improvement. However, can only have $|a_1(t) - a_2(t)| = O(1)$ on an $O(1)$ time scale.

(2) The method is based on Hamiltonian / spectral techniques, which are applicable whether the underlying model is integrable or not. However, the existence and magical properties of N-solitons are typically only available for integrable equations.

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Recentl results on interacting solitons for nonintegrable equations:

Martel-Merle (2008) show for gKdV-4, describe the interaction of an $O(1)$ scale soliton with a very broad scale $c \ll 1$ soliton.

Perelman (2009) shows for the NLS with nonlinearity close to cubic, a fast soliton interacting with a stationary high mass soliton (δ_0-like) splits into two solitons described using the scattering matrix of the high soliton.