FRACTAL UNCERTAINTY FOR TRANSFER OPERATORS

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Abstract. We show directly that the fractal uncertainty principle of Bourgain–Dyatlov [BD16] implies that there exists $\sigma > 0$ for which the Selberg zeta function (1.2) for a convex co-compact hyperbolic surface has only finitely many zeros with $\operatorname{Re} s \geq \frac{1}{2} - \sigma$. That eliminates advanced microlocal techniques of Dyatlov–Zahl [DZ16] though we stress that these techniques are still needed for resolvent bounds and for possible generalizations to the case of non-constant curvature.

The purpose of this note is to give a new explanation of the connection between the *fractal uncertainty principle*, which is a statement in harmonic analysis, and the existence of *zero free strips for Selberg zeta functions*, which is a statement in geometric spectral theory/dynamical systems. The connection is proved via the *transfer operator* which is a well known object in thermodynamical formalism of chaotic dynamics.

To explain the fractal uncertainty principle we start with its *flat* version, given by (1.1) below. Let $X \subset [-1, 1]$ be a δ-regular set in the following sense: there exists a Borel measure μ supported on X and a constant C_R such that for each interval I centered on X of size $|I| \leq 1$, we have $C_R^{-1}|I|^\delta \leq \mu(I) \leq C_R|I|^\delta$.

Bourgain–Dyatlov [BD16, Theorem 4] proved that when $\delta < 1$, there exist $\beta > 0$ and C_1 depending only on δ, C_R such that for all $f \in L^2(\mathbb{R})$

$$\operatorname{supp} \mathcal{F}_h f \subset X(h) \implies \|f\|_{L^2(X(h))} \leq C_1 h^\beta \|f\|_{L^2(\mathbb{R})}, \tag{1.1}$$

where $\mathcal{F}_h f := \hat{f}(\xi/h)$ is the semiclassical Fourier transform (2.25) and

$$X(h) := X + [-h, h]$$

denotes the h-neighborhood of X. Roughly speaking, (1.1) quantifies the statement that a function and its Fourier transform cannot both concentrate on a fractal set.

To explain the spectral geometry side, let $M = \Gamma\backslash\mathbb{H}^2$ be a convex co-compact hyperbolic surface, that is, a non-compact hyperbolic surface with a finite number of funnel ends and no cusps – see [Bo16, §2.4]. The Selberg zeta function is defined by

$$Z_M(s) = \prod_{\ell \in \mathscr{L}_M} \prod_{k=0}^{\infty} \left(1 - e^{-(s+k)\ell}\right), \quad \operatorname{Re} s \gg 1, \tag{1.2}$$

and it continues as an entire function to $\mathbb{C}$. Here $\mathscr{L}_M$ denotes the set of lengths of primitive closed geodesics on M with multiplicity – see [Bo16, Chapter 10]. The zeros of Z_M enjoy many interpretations, in particular as *quantum resonances* of the Laplacian on M – see [Zw17] for a general introduction and references. In particular, finding *resonance free* regions has a long tradition and applications in many settings.

The *limit set*, Λ_Γ, is defined as the set of accumulation points of orbits of Γ acting on $\mathbb{H}^2$, see also (2.6). It is a subset of the boundary of $\mathbb{H}^2$ at infinity, so in the Poincaré disk model of $\mathbb{H}^2$ we have $\Lambda_\Gamma \subset \mathbb{S}^1$. The set Λ_Γ is δ-regular where $\delta \in [0, 1)$ is the exponent of convergence of Poincaré series, see [Bo16, Lemma 14.13].

The *hyperbolic* version of fractal uncertainty principle was formulated by Dyatlov–Zahl [DZ16, Definition 1.1]. Define the operator $\mathcal{B}_\chi = \mathcal{B}_\chi(h)$ on $L^2(\mathbb{S}^1)$ by

$$\mathcal{B}_\chi(h)f(y) = (2\pi h)^{-1/2} \int_{\mathbb{S}^1} |y - y'|^{-2i/h} \chi(y, y') f(y')\, dy', \tag{1.3}$$

where $|y - y'|$ is the Euclidean distance between $y, y' \in \mathbb{S}^1 \subset \mathbb{R}^2$.† Here

$$\chi \in C_c^\infty(\mathbb{S}^1_\Delta), \quad \mathbb{S}^1_\Delta := \{(y, y') \in \mathbb{S}^1 \times \mathbb{S}^1 \mid y \neq y'\}. \tag{1.4}$$

We say that Λ_Γ satisfies *(hyperbolic) fractal uncertainty principle with exponent* $\beta \geq 0$ if for each $\varepsilon > 0$ there exists $\rho < 1$ such that for all $C_0 > 0$, $\chi \in C_c^\infty(\mathbb{S}^1_\Delta)$ we have

$$\| \mathbb{1}_{\Lambda_\Gamma(C_0 h^\rho)}\, \mathcal{B}_\chi(h)\, \mathbb{1}_{\Lambda_\Gamma(C_0 h^\rho)} \|_{L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)} = \mathcal{O}(h^{\beta - \varepsilon}) \quad \text{as } h \to 0. \tag{1.5}$$

This hyperbolic fractal uncertainty principle with $\beta = \beta(\Gamma) > 0$ was established for arbitrary convex co-compact groups by Bourgain–Dyatlov [BD16] by proving the flat version (1.1) and showing that it implies (1.5). It followed earlier partial results of Dyatlov–Zahl [DZ16].

With this in place we can now state our main result:

Theorem. *Assume that $M = \Gamma\backslash\mathbb{H}^2$ is a convex co-compact hyperbolic surface and the limit set Λ_Γ satisfies fractal uncertainty principle with exponent β in the sense of* (1.5)*. Then M has an **essential spectral gap** of size $\beta-$, that is for each $\varepsilon > 0$ the Selberg zeta function of M has only finitely many zeroes in $\{\operatorname{Re} s \geq \frac{1}{2} - \beta + \varepsilon\}$.*

Remark. Bourgain–Dyatlov [BD17] showed that when $\delta > 0$, the set Λ_Γ also satisfies the fractal uncertainty principle with exponent $\beta > \frac{1}{2} - \delta$ which only depends on δ. The resulting essential spectral gap is an improvement over the earlier work of Naud [Na05] which gave a gap of size $\beta > \frac{1}{2} - \delta$ which also depends on the surface. See also Dyatlov–Jin [DJ17].

A stronger theorem was proved in [DZ16] using fine microlocal methods which included second microlocalization and Vasy's method for meromorphic continuation (see [Zw17, §3.1] and references given there). In addition to showing a zero free strip, it provided a bound on the scattering resolvent, see [DZ16, Theorem 3]. Having such bounds is essential to applications – see [Zw17, §3.2]. It would be interesting to see if

†Note the sign change from [DZ16, (1.6)]. It is convenient for us and it does not change the norm.

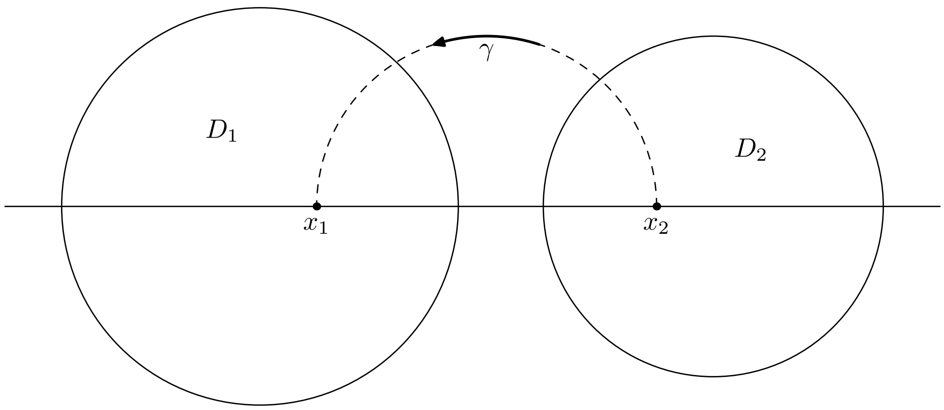

FIGURE 1. The disks D_1, D_2 and the transformation γ in the trivial case (1.7).

one can use the methods of this paper to obtain bounds on the scattering resolvent. However, generalizations to non-constant curvature are likely to be based on the microlocal techniques of [DZ16]. The result of [DZ16] also applies to higher-dimensional hyperbolic quotients. The proof of the present paper can be adapted to higher dimensional cases where the limit set is totally discontinuous (such as Schottky quotients).

The present paper instead uses *transfer operators* – see the outline below and §2.3. The identification of M with a quotient by a Schottky group – see §2.1 – allows to combine simple semiclassical insights with the combinatorial structure of the Schottky data.

Outline of the proof. The proof is based on a well known identification of zeros of $Z_M(s)$ with the values of s at which the *Ruelle transfer operator*, $\mathcal{L}_s$, has eigenvalue 1 – see (2.13). Referring to §2.3 and [Bo16, §15.3] for precise definitions, we have

$$\mathcal{L}_s u(z) = \sum_{w:\ Bw=z} B'(w)^{-s} u(w), \tag{1.6}$$

where B is an expanding *Bowen–Series map* and u is a holomorphic function on a family of disjoint disks symmetric with respect to the real axis.

We outline the idea of the proof in the *trivial* case (the zeros of $Z_M(s)$ can be computed explicitly – see [Bo16, (10.32)]) when there are only two disks D_1, D_2 and $\gamma \in \mathrm{SL}(2,\mathbb{R})$, a linear fractional transformation preserving the upper half plane, maps one disk onto the complement of the other disk, see Figure 1:[‡]

$$D_2 = \dot{\mathbb{C}} \setminus \gamma^{-1}(D_1^\circ), \quad B(z) = \begin{cases} \gamma^{-1}(z), & z \in D_1; \\ \gamma(z), & z \in D_2. \end{cases} \tag{1.7}$$

[‡]This trivial case can be reduced to $\gamma(z) = kz$, $k > 1$, but we do not want to stress that point.

Denote $I_j := D_j \cap \mathbb{R}$. In the case (1.7) the limit set Λ_Γ consists of the fixed points of γ,

$$\Lambda_\Gamma = \{x_1, x_2\} \subset \dot{\mathbb{R}}, \quad x_j \in I_j, \quad \gamma(x_j) = x_j,$$

and the fractal uncertainty principle (1.5) holds with $\beta = \frac{1}{2}$ by a *trivial volume bound*:

$$\begin{aligned} \| \mathbb{1}_{\Lambda_\Gamma(C_0 h^\rho)} \mathcal{B}_\chi(h) \mathbb{1}_{\Lambda_\Gamma(C_0 h^\rho)} \|_{L^2 \to L^2} & \\ \le \| \mathbb{1}_{\Lambda_\Gamma(C_0 h^\rho)} \|_{L^\infty \to L^2} \| \mathcal{B}_\chi(h) \|_{L^1 \to L^\infty} \| \mathbb{1}_{\Lambda_\Gamma(C_0 h^\rho)} \|_{L^2 \to L^1} & \\ \le C h^{\frac{\rho}{2}} \times h^{-\frac{1}{2}} \times h^{\frac{\rho}{2}} = \mathcal{O}(h^{\frac{1}{2} - \varepsilon}), \quad \varepsilon = 1 - \rho. & \end{aligned} \tag{1.8}$$

The operator (1.6) splits as a direct sum over operators on the two disks and hence we only need to consider a simplified transfer operator

$$\mathcal{L}_s u(z) = \gamma'(z)^s u(\gamma(z)), \quad u \in \mathcal{H}(D_1), \tag{1.9}$$

where $\mathcal{H}(D_1)$ is the Bergman space of holomorphic functions in $L^2(D_1)$.

Given (1.8) we want to show, in a complicated way which generalizes, that the equation $\mathcal{L}_s u = u$, $u \in \mathcal{H}(D_1)$, has no non-trivial solutions for

$$s = \tfrac{1}{2} - \nu + i h^{-1} \quad \text{with} \quad \nu < \tfrac{1}{2} - \varepsilon.$$

Thus assume that for such an s, $\mathcal{L}_s u = u$. The first observation is that $u|_{\mathbb{R}}$ is semiclassicaly localized to bounded frequencies: that means that for all $\chi \in C_c^\infty(I_1)$,

$$\mathcal{F}_h(\chi u)(\xi) = \mathcal{O}(h^\infty |\xi|^{-\infty}) \quad \text{for } |\xi| \ge C. \tag{1.10}$$

It is here that holomorphy of u is used: using the maximum principle we show that $\sup_{D_1} |u| \le e^{C/h} \sup_{I_1} |u|$ and derive the Fourier transform bound from this. See Lemmas 3.2 and 3.3.

From now on we work only on the real axis. In the outline we will use concepts from semiclassical analysis but we stress that the actual proofs in the paper are self-contained.

To connect the model used here to $\mathcal{B}_\chi(h)$, which acts on $\mathbb{S}^1$, we identify $\mathbb{S}^1$ with the extended real axis $\dot{\mathbb{R}}$ – see §2.5. Transplanted to $\dot{\mathbb{R}}$, $\mathcal{B}_\chi(h)$ is a semiclassical Fourier integral operator with the phase $\Phi(x, x')$ defined in (2.24) and hence associated to the canonical transformation

$$\varkappa : T^*\dot{\mathbb{R}} \to T^*\dot{\mathbb{R}}, \quad \varkappa : (x', -\partial_{x'}\Phi) \mapsto (x, \partial_x \Phi).$$

We stress that $\varkappa$ is a global diffeomorphism and for $(x, \xi) = \varkappa(x', \xi')$ we have $x \ne x'$. That means that for the action on compactly microlocalized functions, the singularity removed by the cutoff (1.4) is irrelevant and we can consider a simpler operator $\mathcal{B}(s)$ defined by, essentially, removing χ. In the circle model it has the form

$$\mathcal{B}(s) f(y) = (2\pi h)^{-1/2} \int_{\mathbb{S}^1} |y - y'|^{-2s} f(y') dy'.$$

See §2.5 for details. This operator has a nice equivariance property which is particularly simple for the operator (1.9): denoting $\langle x\rangle := \sqrt{1+x^2}$ we have

$$\widetilde{\mathcal{L}}_s\mathcal{B}(s) = \mathcal{B}(s)\widetilde{\mathcal{L}}_{1-s} \quad \text{where} \quad \widetilde{\mathcal{L}}_s := \langle x\rangle^{2s}\mathcal{L}_s\langle x\rangle^{-2s} : L^2(\dot{\mathbb{R}}) \to L^2(\dot{\mathbb{R}}). \tag{1.11}$$

See Lemma 2.5 for the general version. Here we only say that what lies behind this identity is the following formula valid for linear fractional transformations φ:

$$|\varphi(x)-\varphi(y)|^{-2}|\varphi'(x)| = |x-y|^{-2}|\varphi'(y)|^{-1}. \tag{1.12}$$

To use (1.11), we put $\gamma_N := \gamma^N$, the Nth iterate where $N \sim \log(1/h)$ is chosen so that $|\gamma_N(I_1)| \sim h^\rho$. Then

$$u(x) = \mathcal{L}_s^N u(x) = \gamma_N'(x)^s u(\gamma_N(x)), \quad x \in I_1.$$

Choose a cutoff function $\chi_N \in C_c^\infty((x_1 - C_1h^\rho, x_1 + C_1h^\rho))$ which is equal to 1 in a neighbourhood of $\gamma_N(I_1)$. We then put $u_N := \langle x\rangle^{2s}\chi_N u$, so that $\langle x\rangle^{2s}u = \widetilde{\mathcal{L}}_s^N u_N$ on I_1.

Since $\rho < 1$, u_N remains compactly microlocalized in the sense of (1.10) but in addition it is concentrated at $x = x_1$:

$$\mathrm{WF}_h(u_N) \subset \{x = x_1,\ |\xi| \le C\}.$$

The operator $\mathcal{B}(s)$ is elliptic and we denote its *microlocal inverse* by $\mathcal{B}(s)^{-1}$ noting that it is a Fourier integral operator associated to $\varkappa^{-1}$. Hence,

$$u_N = \mathcal{B}(s)v_N + \mathcal{O}(h^\infty), \quad v_N := \mathcal{B}(s)^{-1}u_N,$$
$$\mathrm{WF}_h(v_N) \subset \varkappa^{-1}(\mathrm{WF}_h(u_N)) \subset \{x \ne x_1\}.$$

Therefore, changing v_N by $\mathcal{O}(h^\infty)$, we may assume that $\operatorname{supp} v_N \subset J \subset \dot{\mathbb{R}}$ where J is a fixed 'interval' on $\dot{\mathbb{R}} \simeq \mathbb{S}^1$ and $x_1 \notin J$.

Now we use the equivariance property (1.11): modulo an $\mathcal{O}(h^\infty)$ error, we have

$$\langle x\rangle^{2s}u = \widetilde{\mathcal{L}}_s^N u_N = \widetilde{\mathcal{L}}_s^N\mathcal{B}(s)v_N = \mathcal{B}(s)w_N \quad \text{on } I_1,$$
$$w_N := \widetilde{\mathcal{L}}_{1-s}^N v_N, \quad \operatorname{supp} w_N \subset \widetilde{J} := \gamma^{-N}(J).$$

Since J lies a fixed distance away from x_1, $\widetilde{J} = \gamma^{-N}(J)$ lies in an h^ρ sized interval centered at the repelling point, x_2, of the transformation γ. The change of variables in the integrals shows that

$$\|w_N\|_{L^2} \sim h^{-\rho\nu}\|v_N\|_{L^2} \sim h^{-\rho\nu}\|u_N\|_{L^2}.$$

Now we have

$$u_N = \chi_N\mathcal{B}(s)\,\mathbb{1}_{\widetilde{J}}\,w_N + \mathcal{O}(h^\infty)$$

so the fractal uncertainty principle (1.8) gives

$$\|u_N\|_{L^2} \le \|\chi_N\mathcal{B}(s)\,\mathbb{1}_{\widetilde{J}}\|_{L^2\to L^2}\|w_N\|_{L^2} \le Ch^{1/2-\varepsilon-\rho\nu}\|u_N\|_{L^2} + \mathcal{O}(h^\infty).$$

Since $\nu < \frac{1}{2} - \varepsilon$ and we can take $\rho = 1 - \varepsilon$, we obtain $u_N \equiv 0$ if h is small enough, thus $u \equiv 0$ and the proof is finished.

In the case of non-trivial Schottky groups similar ideas work but with combinatorial complications. We only make a general comment that for iterates γ_N (or more generally iterates $\gamma_{\mathbf{a}}$ – see §2.1), $\partial_x \log |\gamma_N'(x)|$ for x in a small h-independent neighbourhood of x_1 is essentially equal to $-2\partial_x \log |x - \gamma_N^{-1}(x_0)|$, $x_0 \neq x_1$. This follows from (1.12) with $y := \gamma_N^{-1}(x_0)$, $\varphi := \gamma_N$. Any generalization of our method has to replace this explicit formula by writing $\partial_x \log |\gamma_N'(x)|$ approximately as $\partial_x \Phi(x, y)$, $y = \gamma_N^{-1}(x_0)$, $x_0 \neq x_1$, with Φ generating a canonical tranformation.

Notation: We write $f = \mathcal{O}(g)_H$ for $\|f\|_H \leq g$. In particular, $f = \mathcal{O}(h^\infty)_H$ means that for any N there exists C_N such that $\|f\|_H \leq C_N h^N$. The norm $\| \bullet \|$ refers to the L^2 norm but different L^2 norms are used ($L^2(\mathbb{R})$, $L^2(\mathbb{S}^1)$, $L^2(\Omega)$) and they are either specified or clear from the context. With some of abuse of notation, C_Γ denotes large constants which only depend on a fixed Schottky data of Γ (see §2.1) and whose exact value may vary from place to place. We denote $\langle x \rangle := \sqrt{1 + x^2}$.

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2. Ingredients

2.1. Schottky groups.

In this section we briefly review properties of Schottky groups, referring the reader to [Bo16, §15.1] for more details. We use notation similar to [BD17, §2.1].

The group $\mathrm{SL}(2, \mathbb{R})$ acts on the extended complex plane $\dot{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ by Möbius transformations:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}),\ z \in \dot{\mathbb{C}} \quad \Longrightarrow \quad \gamma(z) = \frac{az + b}{cz + d}.$$

Denote by $\mathbb{H}^2 \subset \mathbb{C}$ the upper half-plane model of the hyperbolic plane. Then $\mathrm{SL}(2, \mathbb{R})$ acts on $\mathbb{H}^2$ by isometries.

A *Schottky group* is a convex co-compact subgroup $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$ constructed in the following way (see Figure 2):

- Fix $r \in \mathbb{N}$ and nonintersecting closed disks $D_1, \dots, D_{2r} \subset \mathbb{C}$ centered on the real line.

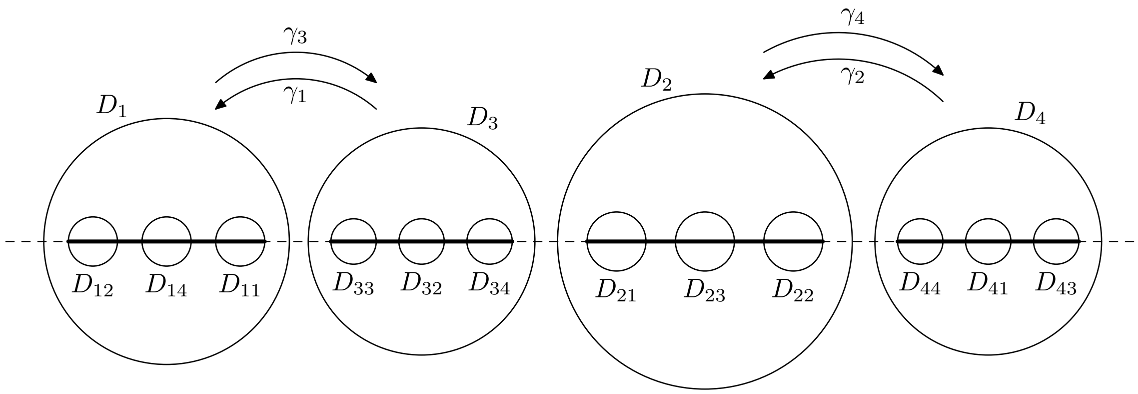

FIGURE 2. A Schottky structure with $r = 2$. The real line is dashed. The solid intervals are I'_1, I'_3, I'_2, I'_4.

- Define the *alphabet* $\mathcal{A} = \{1, \dots, 2r\}$ and for each $a \in \mathcal{A}$, denote

$$\overline{a} := \begin{cases} a + r, & 1 \le a \le r; \\ a - r, & r + 1 \le a \le 2r. \end{cases}$$

- Fix group elements $\gamma_1, \dots, \gamma_{2r} \in \mathrm{SL}(2, \mathbb{R})$ such that for all $a \in \mathcal{A}$

$$\gamma_a(\dot{\mathbb{C}} \setminus D^\circ_{\overline{a}}) = D_a, \quad \gamma_{\overline{a}} = \gamma_a^{-1}. \tag{2.1}$$

  (In the notation of [Bo16, §15.1] we have $\gamma_a = S_a^{-1}$.)
- Let $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$ be the group generated by $\gamma_1, \dots, \gamma_r$; it is a free group on r generators [Bo16, Lemma 15.2].

Every convex co-compact hyperbolic surface M can be viewed as the quotient of $\mathbb{H}^2$ by a Schottky group Γ, see for instance [Bo16, Theorem 15.3]. Note that the complement of $\bigsqcup_a D_a$ in $\mathbb{H}^2$ is a fundamental domain. We fix a Schottky representation for a given hyperbolic surface M and refer to $D_1, \dots, D_{2r}$ and $\gamma_1, \dots, \gamma_{2r}$ as *Schottky data*.

In several places we will use results of [BD17, §2] which can be read independently of the rest of [BD17]. In particular, we use combinatorial notation for indexing the words in the free group Γ and the corresponding disks:

- For $n \in \mathbb{N}_0$, define $\mathcal{W}_n$, the set of words of length n, as follows:

$$\mathcal{W}_n := \{a_1 \dots a_n \mid a_1, \dots, a_n \in \mathcal{A}, \quad a_{j+1} \neq \overline{a_j} \quad \text{for } j = 1, \dots, n-1\}. \tag{2.2}$$

- Denote by $\mathcal{W} := \bigcup_n \mathcal{W}_n$ the set of all words. For $\mathbf{a} \in \mathcal{W}_n$, put $|\mathbf{a}| := n$. Denote the empty word by $\emptyset$ and put $\mathcal{W}^\circ := \mathcal{W} \setminus \{\emptyset\}$.
- For $\mathbf{a} = a_1 \dots a_n \in \mathcal{W}$, put $\overline{\mathbf{a}} := \overline{a_n} \dots \overline{a_1} \in \mathcal{W}$. For a set $Z \subset \mathcal{W}$, put

$$\overline{Z} := \{\overline{\mathbf{a}} \mid \mathbf{a} \in Z\}. \tag{2.3}$$

- For $\mathbf{a} = a_1 \dots a_n \in \mathcal{W}^\circ$, put $\mathbf{a}' := a_1 \dots a_{n-1} \in \mathcal{W}$. Note that $\mathcal{W}$ forms a tree with root $\emptyset$ and each $\mathbf{a} \in \mathcal{W}^\circ$ having parent $\mathbf{a}'$.

- For $\mathbf{a} = a_1 \dots a_n, \mathbf{b} = b_1 \dots b_m \in \mathcal{W}$, we write $\mathbf{a} \to \mathbf{b}$ if either $\mathbf{a} = \emptyset$, or $\mathbf{b} = \emptyset$, or $a_n \neq \overline{b_1}$. Under this condition the concatenation $\mathbf{ab}$ is a word. We write $\mathbf{a} \rightsquigarrow \mathbf{b}$ if $\mathbf{a}, \mathbf{b} \in \mathcal{W}^\circ$ and $a_n = b_1$. In the latter case $\mathbf{a}'\mathbf{b} \in \mathcal{W}$.
- For $\mathbf{a}, \mathbf{b} \in \mathcal{W}$, we write $\mathbf{a} \prec \mathbf{b}$ if $\mathbf{a}$ is a prefix of $\mathbf{b}$, i.e. $\mathbf{b} = \mathbf{ac}$ for some $\mathbf{c} \in \mathcal{W}$.
- Define the following one-to-one correspondence between $\mathcal{W}$ and the group Γ:
$$\mathcal{W} \ni a_1 \dots a_n = \mathbf{a} \quad \longmapsto \quad \gamma_{\mathbf{a}} := \gamma_{a_1} \cdots \gamma_{a_n} \in \Gamma.$$
Note that $\gamma_{\mathbf{ab}} = \gamma_{\mathbf{a}}\gamma_{\mathbf{b}}$ when $\mathbf{a} \to \mathbf{b}$, $\gamma_{\overline{\mathbf{a}}} = \gamma_{\mathbf{a}}^{-1}$, and $\gamma_\emptyset$ is the identity.
- For $\mathbf{a} = a_1 \dots a_n \in \mathcal{W}^\circ$, define the disk centered on the real line (see Figure 2)
$$D_{\mathbf{a}} := \gamma_{\mathbf{a}'}(D_{a_n}) \subset \mathbb{C}.$$
If $\mathbf{a} \prec \mathbf{b}$, then $D_{\mathbf{b}} \subset D_{\mathbf{a}}$. On the other hand, if $\mathbf{a} \not\prec \mathbf{b}$ and $\mathbf{b} \not\prec \mathbf{a}$, then $D_{\mathbf{a}} \cap D_{\mathbf{b}} = \emptyset$. Define the interval
$$I_{\mathbf{a}} := D_{\mathbf{a}} \cap \mathbb{R}$$
and denote by $|I_{\mathbf{a}}|$ its length (which is equal to the diameter of $D_{\mathbf{a}}$).
- For $a \in \mathcal{A}$, define the interval $I'_a \subset I^\circ_a$ as the convex hull of the union $\bigsqcup_{b \in \mathcal{A},\, a \to b} I_{ab}$, see Figure 2. More generally, for $\mathbf{a} = a_1 \dots a_n \in \mathcal{W}^\circ$ define
$$I'_{\mathbf{a}} := \gamma_{\mathbf{a}'}(I'_{a_n}) \subset I^\circ_{\mathbf{a}}. \tag{2.4}$$
Note that $I'_{\mathbf{a}} \supset I_{\mathbf{b}}$ for any $\mathbf{b} \in \mathcal{W}^\circ$ such that $\mathbf{a} \prec \mathbf{b}$, $\mathbf{a} \neq \mathbf{b}$.
- Denote
$$\mathbf{D} := \bigsqcup_{a \in \mathcal{A}} D_a \subset \mathbb{C}, \quad \mathbf{I} := \bigsqcup_{a \in \mathcal{A}} I_a = \mathbf{D} \cap \mathbb{R}, \quad \mathbf{I}' := \bigsqcup_{a \in \mathcal{A}} I'_a \subset \mathbf{I}^\circ. \tag{2.5}$$
- The *limit set* is given by
$$\Lambda_\Gamma := \bigcap_{n \geq 1} \bigsqcup_{\mathbf{a} \in \mathcal{W}_n} D_{\mathbf{a}} \subset \mathbb{R}. \tag{2.6}$$
The fact that $\Lambda_\Gamma \subset \mathbb{R}$ follows from the *contraction property* [BD17, §2.1]
$$|I_{\mathbf{a}}| \leq C_\Gamma (1 - C_\Gamma^{-1})^{|\mathbf{a}|} \quad \text{for all } \mathbf{a} \in \mathcal{W}^\circ. \tag{2.7}$$

We finish this section with a few estimates. We start with the following derivative bound which is the complex version of [BD17, Lemma 2.5]:

Lemma 2.1. *For all* $\mathbf{a} = a_1 \dots a_n \in \mathcal{W}^\circ$ *and* $z \in D_{a_n}$ *we have*
$$C_\Gamma^{-1} |I_{\mathbf{a}}| \leq |\gamma'_{\mathbf{a}'}(z)| \leq C_\Gamma |I_{\mathbf{a}}|. \tag{2.8}$$

Proof. Define the intervals $I := I_{a_n}$, $J := I_{\mathbf{a}} = \gamma_{\mathbf{a}'}(I)$. Let $\gamma_I, \gamma_J \in \mathrm{SL}(2, \mathbb{R})$ be the unique affine transformations such that $\gamma_I([0,1]) = I$, $\gamma_J([0,1]) = J$. Following [BD17, §2.2], we write
$$\gamma_{\mathbf{a}'} = \gamma_J \gamma_\alpha \gamma_I^{-1}, \quad \gamma_\alpha = \begin{pmatrix} e^{\alpha/2} & 0 \\ e^{\alpha/2} - e^{-\alpha/2} & e^{-\alpha/2} \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}),$$

where $\alpha \in \mathbb{R}$ and $|\alpha| \leq C_\Gamma$ by [BD17, Lemma 2.4]. For $z \in D_{a_n}$ we have

$$\gamma_{\mathbf{a}'}(z) = \frac{|J|}{|I|}\gamma'_\alpha(w), \qquad w = \gamma_I^{-1}(z) \in \Big\{\Big|w - \frac{1}{2}\Big| \leq \frac{1}{2}\Big\}.$$

We compute

$$|\gamma'_\alpha(w)| = \frac{e^\alpha}{|(e^\alpha - 1)w + 1|^2} \in [C_\Gamma^{-1}, C_\Gamma].$$

Since $C_\Gamma^{-1}|I_{\mathbf{a}}| \leq |J|/|I| \leq C_\Gamma |I_{\mathbf{a}}|$, (2.8) follows. □

The next lemma bounds the number of intervals $I_{\mathbf{a}}$ of comparable sizes which can contain a given point:

Lemma 2.2. *For all $C_1 \geq 2$ and $\tau > 0$, we have*

$$\sup_{x\in\mathbb{R}} \#\{\mathbf{a} \in \mathcal{W}^\circ \colon \tau \leq |I_{\mathbf{a}}| \leq C_1\tau,\ x \in I_{\mathbf{a}}\} \leq C_\Gamma \log C_1. \tag{2.9}$$

Proof. Fix $x \in \mathbb{R}$. We have

$$\{\mathbf{a} \in \mathcal{W}^\circ \colon \tau \leq |I_{\mathbf{a}}| \leq C_1\tau,\ x \in I_{\mathbf{a}}\} = \{\mathbf{a}_1, \dots, \mathbf{a}_N\}$$

for some $\mathbf{a}_1, \dots, \mathbf{a}_N \in \mathcal{W}^\circ$ such that $\mathbf{a}_1 \prec \mathbf{a}_2 \prec \cdots \prec \mathbf{a}_N$. We have $|I_{\mathbf{a}_{j+1}}| \leq (1 - C_\Gamma^{-1})|I_{\mathbf{a}_j}|$, see [BD17, §2.1], and (2.9) follows. □

2.2. **Functional spaces.**

For any open $\Omega \subset \mathbb{C}$, let $\mathcal{H}(\Omega)$ be the *Bergman space* on Ω, consisting of holomorphic functions $f : \Omega \to \mathbb{C}$ such that $f \in L^2(\Omega)$ (with respect to the Lebesgue measure). Endowing $\mathcal{H}(\Omega)$ with the L^2 norm, we obtain a separable Hilbert space. For general $\Omega \subset \mathbb{C}$, we put $\mathcal{H}(\Omega) := \mathcal{H}(\Omega^\circ)$.

Denote

$$\mathbf{D}_2 := \bigsqcup_{\substack{\mathbf{a}\in\mathcal{W} \\ |\mathbf{a}|=2}} D_{\mathbf{a}} \subset \mathbf{D}^\circ \tag{2.10}$$

and let $\mathbf{D}^\pm := \mathbf{D} \cap \{\pm \operatorname{Im} z \geq 0\}$, $\mathbf{D}_2^\pm := \mathbf{D}_2 \cap \{\pm \operatorname{Im} z \geq 0\}$. See Figure 3.

The following basic estimate is needed for the a priori bounds in §3.1 below:

Lemma 2.3. *There exists $c \in (0, 1]$ such that for all $f \in \mathcal{H}(\mathbf{D}^\pm) \cap C(\mathbf{D}^\pm)$,*

$$\sup_{\mathbf{D}_2^\pm} |f| \leq \big(\sup_{\mathbf{I}} |f|\big)^c \big(\sup_{\mathbf{D}^\pm} |f|\big)^{1-c}. \tag{2.11}$$

Proof. The boundary of $\mathbf{D}^\pm$ consists of $\mathbf{I}$ and a union of half-circles, which we denote $\mathbf{S}^\pm$. Let $F_\pm : (\mathbf{D}^\pm)^\circ \to [0, 1]$ be the harmonic function with boundary values

$$F_\pm|_{\mathbf{I}} \equiv 1, \quad F_\pm|_{\mathbf{S}^\pm} \equiv 0.$$

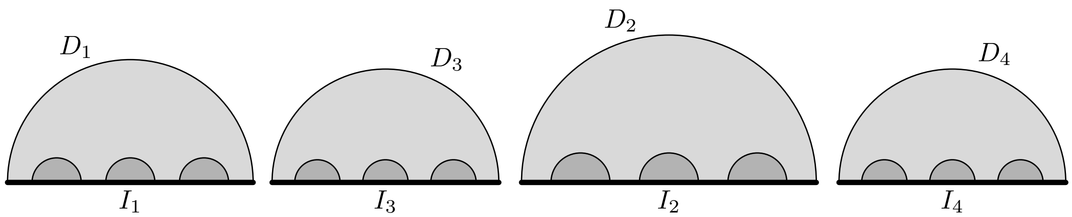

FIGURE 3. An illustration of Lemma 2.3. The lighter shaded region is $\mathbf{D}^+$ and the darker region is $\mathbf{D}_2^+$. The union of the thick lines is $\mathbf{I}$.

Since $F_\pm$ is positive in $(\mathbf{D}^\pm)^\circ$ and $\mathbf{D}_2^\pm$ lies away from $\mathbf{S}^\pm$, the infimum of $F_\pm$ on $\mathbf{D}_2^\pm$ is positive. Denote this infimum by $c := \inf_{\mathbf{D}_2^\pm} F^\pm \in (0,1]$. Since $\log|f|$ is subharmonic in $\mathbf{D}^\pm$ and

$$\log|f| \leq \big(\sup_{\mathbf{I}} \log|f|\big) F_\pm + \big(\sup_{\mathbf{D}^\pm} \log|f|\big)(1-F_\pm) \quad \text{on } \partial\mathbf{D}^\pm,$$

the maximum principle implies that for $z \in \mathbf{D}_2^\pm$,

$$\begin{aligned}\log|f(z)| &\leq \sup_{\mathbf{D}^\pm} \log|f| - F_\pm(z)\big(\sup_{\mathbf{D}^\pm}\log|f| - \sup_{\mathbf{I}}\log|f|\big)\\ &\leq \sup_{\mathbf{D}^\pm}\log|f| - c\big(\sup_{\mathbf{D}^\pm}\log|f| - \sup_{\mathbf{I}}\log|f|\big).\end{aligned}$$

Exponentiating this we obtain (2.11). □

2.3. **Transfer operators and resonances.** The zeros of the Selberg zeta function, that is the resonances of $M = \Gamma\backslash\mathbb{H}^2$, are characterized using a dynamical transfer operator, also called the *Ruelle operator*. Here we follow [Bo16, §15.3],[GLZ04] and consider these operators on Bergman spaces defined in §2.2. We refer to [Ba16] for other approaches to transfer operators and for historical background.

Here, for $s \in \mathbb{C}$ we define the *transfer operator* $\mathcal{L}_s : \mathcal{H}(\mathbf{D}) \to \mathcal{H}(\mathbf{D})$ as follows:

$$\mathcal{L}_s f(z) = \sum_{\substack{a\in\mathcal{A}\\ a\to b}} \gamma_a'(z)^s f(\gamma_a(z)), \quad z \in D_b,\ b \in \mathcal{A}. \tag{2.12}$$

(We note that in the notation of [Bo16, (15.11)], $S_i = \gamma_i^{-1}$.) This is the same as (1.6) if we define $Bz = \gamma_a^{-1}(z)$ for $z \in D_a$.

The derivative satisfies $\gamma_a'(z) > 0$ for $z \in I_b$ and $\gamma_a'(z)^s$ is uniquely defined and holomorphic for $z \in D_b$ and $s \in \mathbb{C}$ such that $\gamma_a'(z)^s > 0$ when $z \in I_b$, $s \in \mathbb{R}$. Since γ_a takes real values on $\mathbb{R}$, the expression (2.12) additionally gives an operator on $L^2(\mathbf{I})$, also denoted $\mathcal{L}_s$. The operators $\mathcal{L}_s$ are of trace class (see [Bo16, Lemma 15.7]) and depend holomorphically on $s \in \mathbb{C}$. Hence the determinant $\det(I - \mathcal{L}_s)$ defines an entire function of $s \in \mathbb{C}$. The connection to the Selberg zeta function defined in (1.2) is a special case of Ruelle theory and and can be found in [Bo16, Theorem 15.10]:

$$Z_M(s) = \det(I - \mathcal{L}_s).$$

It also follows that $I - \mathcal{L}_s$ is a Fredholm operator of index 0 and its invertibility is equivalent to $\ker(I - \mathcal{L}_s) = \{0\}$. Hence,

$$Z_M(s) = \det(I - \mathcal{L}_s) = 0 \iff \exists\, u \in \mathcal{H}(\mathbf{D}),\ u \not\equiv 0,\ u = \mathcal{L}_s u, \tag{2.13}$$

see [Bo16, Theorem A.34].

2.4. **Partitions and refined transfer operators.** Our proof uses refined transfer operators which are generalizations of powers of the standard transfer operator $\mathcal{L}_s$ given by (2.12). To introduce these we use the notion of a partition:

- A finite set $Z \subset \mathcal{W}^\circ$ is called a *partition* if there exists N such that for each $\mathbf{a} \in \mathcal{W}$ with $|\mathbf{a}| \geq N$, there exists unique $\mathbf{b} \in Z$ such that $\mathbf{b} \prec \mathbf{a}$. See Figure 4. In terms of the limit set, this means that
$$\Lambda_\Gamma = \bigsqcup_{\mathbf{b} \in Z} (I_\mathbf{b} \cap \Lambda_\Gamma). \tag{2.14}$$
- The alphabet $\mathcal{A}$ is a partition, as is the set $\mathcal{W}_n$ of words of length $n \geq 1$. Another important example is the set of words discretizing to some resolution $\tau > 0$:
$$Z(\tau) := \big\{\mathbf{a} \in \mathcal{W}^\circ \colon |I_\mathbf{a}| \leq \tau < |I_{\mathbf{a}'}|\big\} \tag{2.15}$$
where we put $|I_\emptyset| := \infty$. The set $Z(\tau)$ is a partition due to (2.7).
- Note however that if Z is a partition, this does not imply that the set $\overline{Z}$ defined in (2.3) is a partition.

If $Z \subset \mathcal{W}^\circ$ is a finite set, we define the refined transfer operator $\mathcal{L}_{Z,s} : \mathcal{H}(\mathbf{D}) \to \mathcal{H}(\mathbf{D})$ as follows:

$$\mathcal{L}_{Z,s} f(z) = \sum_{\substack{\mathbf{a} \in Z \\ \mathbf{a} \rightsquigarrow b}} \gamma'_{\mathbf{a}'}(z)^s f(\gamma_{\mathbf{a}'}(z)), \quad z \in D_b,\ b \in \mathcal{A}. \tag{2.16}$$

As in the case of $\mathcal{L}_s$, we can also consider $\mathcal{L}_{Z,s}$ as an operator on $L^2(\mathbf{I})$.

Here are some basic examples of refined transfer operators:

- If $Z = \mathcal{A}$ then $\mathcal{L}_{Z,s}$ is the identity operator.
- If $Z = \mathcal{W}_2$, the set of words of length 2, then $\mathcal{L}_{Z,s} = \mathcal{L}_s$, the standard transfer operator defined in (2.12).
- More generally if $Z = \mathcal{W}_N$ for some $N \geq 1$, then $\mathcal{L}_{Z,s} = \mathcal{L}_s^{N-1}$.

Lemma 2.4. *Assume that Z is a partition; define $\overline{Z}$ by (2.3). Then for all $u \in \mathcal{H}(\mathbf{D})$ and $s \in \mathbb{C}$*

$$\mathcal{L}_s u = u \quad \Longrightarrow \quad \mathcal{L}_{\overline{Z},s} u = u. \tag{2.17}$$

Proof. We argue by induction on $\sum_{\mathbf{b} \in Z} |\mathbf{b}|$. If $Z = \mathcal{A}$ then $\mathcal{L}_{\overline{Z},s}$ is the identity operator so (2.17) holds. Assume that $Z \neq \mathcal{A}$. Choose a longest word $\mathbf{d}c \in Z$, where $\mathbf{d} \in \mathcal{W}^\circ$ and $c \in \mathcal{A}$. Then Z has the form (see Figure 4)

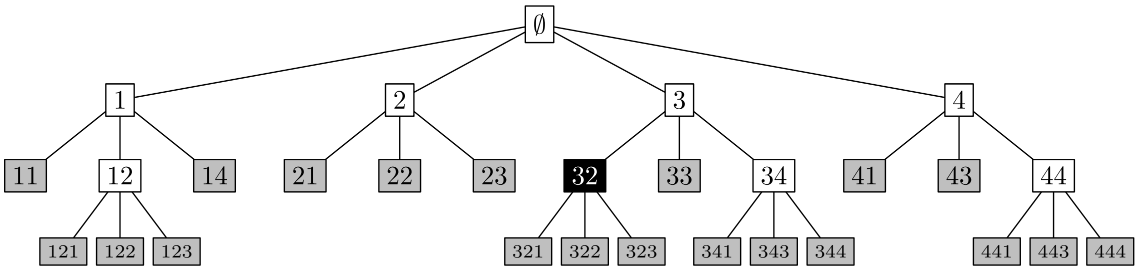

FIGURE 4. An example of a partition Z, with elements of Z shaded grey in the tree of words. The solid black word is one possible choice of $\mathbf{d}$ in the proof of Lemma 2.4.

$$Z = \big(Z' \setminus \{\mathbf{d}\}\big) \sqcup \{\mathbf{d}a \mid a \in \mathcal{A},\ \mathbf{d} \to a\}$$

where Z' is a partition containing $\mathbf{d}$. By the inductive hypothesis we have $\mathcal{L}_{\overline{Z'},s} u = u$, thus is remains to prove that

$$\mathcal{L}_s u = u \quad \Longrightarrow \quad \mathcal{L}_{\overline{Z'},s} u = \mathcal{L}_{\overline{Z},s} u. \tag{2.18}$$

We write for each $z \in D_b$, $b \in \mathcal{A}$

$$\mathcal{L}_{\overline{Z},s} u(z) = \sum_{\substack{\mathbf{a} \in Z \\ \overline{\mathbf{a}} \rightsquigarrow b}} \gamma'_{\overline{\mathbf{a}}'}(z)^s u(\gamma_{\overline{\mathbf{a}}'}(z))$$

and similarly for Z'. The condition $\overline{\mathbf{a}} \rightsquigarrow b$ simply means $b = \overline{a_1}$ where a_1 is the first letter of $\mathbf{a}$.

For $b \neq \overline{d_1}$ the expressions for $\mathcal{L}_{\overline{Z},s} u(z)$ and $\mathcal{L}_{\overline{Z'},s} u(z)$ are identical. Assume now that $b = \overline{d_1}$. Removing identical terms from the conclusion of (2.18) and using that $\overline{\mathbf{d}a}' = \overline{a}\overline{\mathbf{d}}'$ we reduce (2.18) to

$$\gamma'_{\overline{\mathbf{d}}'}(z)^s u(\gamma_{\overline{\mathbf{d}}'}(z)) = \sum_{\substack{a \in \mathcal{A} \\ \mathbf{d} \to a}} \gamma'_{\overline{a}\overline{\mathbf{d}}'}(z)^s u(\gamma_{\overline{a}\overline{\mathbf{d}}'}(z)) \quad \text{for all } z \in D_{\overline{d_1}}.$$

Using the chain rule and dividing by $\gamma'_{\overline{\mathbf{d}}'}(z)^s$ this reduces to

$$u(\gamma_{\overline{\mathbf{d}}'}(z)) = \sum_{\substack{a \in \mathcal{A} \\ \overline{a} \to \overline{\mathbf{d}}}} \gamma'_{\overline{a}}(\gamma_{\overline{\mathbf{d}}'}(z))^s u(\gamma_{\overline{a}}(\gamma_{\overline{\mathbf{d}}'}(z))) \quad \text{for all } z \in D_{\overline{d_1}}.$$

The latter follows from the equality $\mathcal{L}_s u(w) = u(w)$ where $w := \gamma_{\overline{\mathbf{d}}'}(z) \in D_{\overline{\mathbf{d}}}$. □

2.5. **An integral operator.** We now introduce an integral operator $\mathcal{B}(s)$ similar to $\mathcal{B}_\chi(h)$ from (1.3). That operator has a simpler definition than $\mathcal{B}_\chi(h)$ but one pays by introducing singularities.

In our approach we use the upper half plane model while $\mathcal{B}_\chi(h)$ acts on functions on a circle rather than a line. Hence, we will use the extended real line $\dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$

which is identified with the circle $\mathbb{S}^1 \subset \mathbb{C}$ by the map

$$x \in \dot{\mathbb{R}} \ \mapsto \ y = \frac{i-x}{i+x} \in \mathbb{S}^1. \tag{2.19}$$

The standard volume form on $\mathbb{S}^1$, pulled back by (2.19), is

$$dP(x) = 2\langle x\rangle^{-2}\,dx.$$

Denote $L^2(\dot{\mathbb{R}}) := L^2(\dot{\mathbb{R}}, dP) \simeq L^2(\mathbb{S}^1)$. For $x, x' \in \dot{\mathbb{R}}$, let $|x-x'|_{\mathbb{S}}$ be the Euclidean distance between $y(x)$ and $y(x')$, namely

$$|x-x'|_{\mathbb{S}} = \frac{2|x-x'|}{\langle x\rangle\langle x'\rangle}.$$

With this notation in place we define the operator $\mathcal{B}(s)$, depending on $s \in \mathbb{C}$: for $\operatorname{Re} s < \frac{1}{2}$, it is a bounded operator on $L^2(\dot{\mathbb{R}})$ given by the formula

$$\mathcal{B}(s)f(x) = \left|\frac{\operatorname{Im} s}{2\pi}\right|^{1/2} \int_{\dot{\mathbb{R}}} |x-x'|_{\mathbb{S}}^{-2s} f(x')\,dP(x'). \tag{2.20}$$

For general s the integral in (2.20) may diverge however

$$\chi_1\mathcal{B}(s)\chi_2 : L^2(\dot{\mathbb{R}}) \to L^2(\dot{\mathbb{R}}), \quad \chi_1, \chi_2 \in C^\infty(\dot{\mathbb{R}}), \ \ \operatorname{supp}\chi_1 \cap \operatorname{supp}\chi_2 = \emptyset, \tag{2.21}$$

is well defined. In other words, $\mathcal{B}(s)f$ can be defined outside of $\operatorname{supp} f$.

The following equivariance property of $\mathcal{B}(s)$ will be used in the proof of the main theorem in §3.3:

Lemma 2.5. *Let $\gamma \in \mathrm{SL}(2,\mathbb{R})$, $s \in \mathbb{C}$, and consider the operator*

$$T_{\gamma,s} : L^2(\dot{\mathbb{R}}) \to L^2(\dot{\mathbb{R}}), \quad T_{\gamma,s}f(x) := |\gamma'(x)|_{\mathbb{S}}^{s} f(\gamma(x)). \tag{2.22}$$

Here $|\gamma'(x)|_{\mathbb{S}} = \langle x\rangle^2\langle\gamma(x)\rangle^{-2}\gamma'(x)$ is the derivative of the action of γ on the circle defined using (2.19). Then

$$T_{\gamma,s}\mathcal{B}(s) = \mathcal{B}(s)T_{\gamma,1-s}. \tag{2.23}$$

Proof. Take $f \in L^2(\dot{\mathbb{R}})$. We need to show that

$$T_{\gamma,s}\mathcal{B}(s)f(x) = \mathcal{B}(s)T_{\gamma,1-s}f(x)$$

for all x when $\operatorname{Re} s < \frac{1}{2}$ and for all $x \notin \gamma^{-1}(\operatorname{supp} f)$ otherwise. This is equivalent to

$$\int_{\dot{\mathbb{R}}} |\gamma'(x)|_{\mathbb{S}}^{s} \cdot |\gamma(x)-x'|_{\mathbb{S}}^{-2s} f(x')\,dP(x') = \int_{\dot{\mathbb{R}}} |x-x''|_{\mathbb{S}}^{-2s} \cdot |\gamma'(x'')|_{\mathbb{S}}^{1-s} f(\gamma(x''))\,dP(x'').$$

The latter follows by the change of variables $x' = \gamma(x'')$ using the identity

$$|\gamma(x)-\gamma(x'')|_{\mathbb{S}}^2 = |x-x''|_{\mathbb{S}}^2 \cdot |\gamma'(x)|_{\mathbb{S}} \cdot |\gamma'(x'')|_{\mathbb{S}}. \qquad \square$$

We now discuss the properties of $\mathcal{B}(s)$ in the semiclassical limit which means that we put $s := \frac{1}{2} - \nu + ih^{-1}$, where ν is bounded and $0 < h \ll 1$. In the notation of (2.21) we obtain an oscillatory integral representation:

$$\chi_1 \mathcal{B}(s) \chi_2 f(x) = (2\pi h)^{-1/2} \int_{\dot{\mathbb{R}}} e^{\frac{i}{h}\Phi(x,x')} \chi_1(x)\chi_2(x') |x - x'|_{\mathbb{S}}^{2\nu-1} f(x')\, dP(x')$$

where the phase function Φ is defined by

$$\Phi(x, x') := -2 \log |x - x'|_{\mathbb{S}} = -2 \log |x - x'| + 2 \log\langle x\rangle + 2 \log\langle x'\rangle - \log 4. \tag{2.24}$$

Thus $\chi_1 \mathcal{B}(s) \chi_2$ is a semiclassical Fourier integral operator, see for instance [DZ16, §2.2].

The next two lemmas can be derived from the theory of these operators but we present self-contained proofs in the Appendix, applying the method of stationary phase directly. That first gives

Lemma 2.6 (Boundedness of $\mathcal{B}(s)$)**.** *Let $\chi_1, \chi_2 \in C^\infty(\dot{\mathbb{R}})$ satisfy $\operatorname{supp}\chi_1 \cap \operatorname{supp}\chi_2 = \emptyset$. Then there exists C depending only on ν, χ_1, χ_2 such that*

$$\|\chi_1 \mathcal{B}(s) \chi_2\|_{L^2(\dot{\mathbb{R}}) \to L^2(\dot{\mathbb{R}})} \leq C.$$

The next lemma gives partial invertibility of $\mathcal{B}(s)$. To state it we recall the definition of *semiclassical Fourier transform*,

$$\mathcal{F}_h f(\xi) := \int_{\mathbb{R}} f(x) e^{-\frac{i}{h}x\xi} dx, \tag{2.25}$$

see [Zw12, §3.3] for basic properties. We say that an h-dependent family of functions $f = f(h)$ is *semiclassically localized to frequencies* $|\xi| \leq M$ if for every N,

$$|\mathcal{F}_h f(\xi)| \leq C_N h^N |\xi|^{-N} \quad \text{when} \quad |\xi| \geq M. \tag{2.26}$$

We also recall *semiclassical quantization* $a \mapsto \operatorname{Op}_h(a) = a(x, hD_x)$, $D_x := \frac{1}{i}\partial_x$

$$\operatorname{Op}_h(a) u(x) := \frac{1}{2\pi h} \int_{\mathbb{R}} e^{\frac{i}{h}(x-y)\xi} a(x, \xi) u(y)\, dy d\xi, \tag{2.27}$$

stressing that only elementary properties from [Zw12, §§4.2,4.3] will be used.

The partial invertibility of $\mathcal{B}(s)$ means that for $f \in L^2(\mathbb{R})$ which is supported in an interval I and semiclassically localized to frequencies $|\xi| \leq (5|I|)^{-1}$, we have $f = \mathcal{B}(s)g + \mathcal{O}(h^\infty)$ on I for some g which is supported away from I:

Lemma 2.7 (Partial invertibility of $\mathcal{B}(s)$)**.** *Let $I \subset \mathbb{R}$ be an interval and $K \geq 10$ satisfy $10K|I| \leq 1$. Assume that*

$$A = \operatorname{Op}_h(a), \quad a(x, \xi) \in C_{\mathrm{c}}^\infty(\mathbb{R}^2), \quad \operatorname{supp} a \subset \{x \in I^\circ,\ |\xi| < 2K\}.$$

Then for all $\psi_I \in C_{\mathrm{c}}^\infty(I^\circ)$ and $\omega_I \in C_{\mathrm{c}}^\infty(\dot{\mathbb{R}} \setminus I)$ satisfying

$$\operatorname{supp} a \subset \{\psi_I(x) = 1\}, \quad \operatorname{supp}(1 - \omega_I) \subset I + \left[-\tfrac{1}{10K}, \tfrac{1}{10K}\right], \tag{2.28}$$

there exists an operator $Q_I(s) : L^2(\mathbb{R}) \to L^2(\dot{\mathbb{R}})$ *uniformly bounded in* h, *such that*

$$A = \psi_I \mathcal{B}(s) \omega_I Q_I(s) + \mathcal{O}(h^\infty)_{L^2(\mathbb{R}) \to L^2(\mathbb{R})}.$$

We recall that the proofs of both lemmas are given in the Appendix.

3. Proof of the main theorem

We recall that $M = \Gamma \backslash \mathbb{H}^2$ is a convex co-compact hyperbolic quotient and that we fix a Schottky representation for Γ as in §2.1. Finally we assume that the limit set Λ_Γ satisfies fractal uncertainty principle with exponent $\beta \geq 0$ in the sense of (1.5).

As seen in (2.13), the main theorem follows from showing that for

$$s := \frac{1}{2} - \nu + \frac{i}{h}, \quad 0 \leq \nu \leq \beta - 2\varepsilon \tag{3.1}$$

we have

$$u \in \mathcal{H}(\mathbf{D}), \quad \mathcal{L}_s u = u \implies u \equiv 0. \tag{3.2}$$

That follows from progressively obtaining more and more bounds on solutions to the equation $\mathcal{L}_s u = u$, until we can use the fractal uncertainty principle (1.5) to show that $u \equiv 0$.

As shown in [DZ16, §5.1], we always have $\beta \leq \frac{1}{2}$. This implies the inequality $\operatorname{Re} s \geq 0$ which we occasionally use below.

3.1. A priori bounds.

We first use the equation in (3.2) to establish some a priori bounds on u. Note that $u = \mathcal{L}_s u$ implies that u extends holomorphically to a neighbourhood of $\mathbf{D}$ and in particular is smooth up to the boundary of $\mathbf{D}$.

The prefactors $\gamma'_a(z)^s$ in (2.12) are exponentially large in h when z is not on the real line. To balance this growth we introduce the weight

$$w_K : \mathbb{C} \to (0, \infty), \quad w_K(z) = e^{-K|\operatorname{Im} z|/h}.$$

The constant K is chosen so that a sufficiently high power of the transfer operator $\mathcal{L}_s$ is bounded uniformly in h on the weighted space induced by w_K:

Lemma 3.1. *There exist* $K \geq 10$ *and* $n_0 \in \mathbb{N}$ *depending only on the Schottky data so that, with* $\mathbf{D}_2 \subset \mathbf{D}^\circ$ *defined in* (2.10),

$$\sup_{\mathbf{D}} |w_K \mathcal{L}_s^{n_0} f| \leq C_\Gamma \sup_{\mathbf{D}_2} |w_K f| \quad \textit{for all } f \in \mathcal{H}(\mathbf{D}). \tag{3.3}$$

Proof. Using Lemma 2.1 and the contraction property (2.7), we choose n_0 such that

$$\sup_{D_b} |\gamma'_{\mathbf{a}}| \leq \tfrac{1}{2} \quad \text{for all} \quad \mathbf{a} \in \mathcal{W}_{n_0}, \quad \mathbf{a} \to b \in \mathcal{A}.$$

(See (2.2) for definitions.) In particular, since $\gamma_{\mathbf{a}}$ maps the real line to itself, we have

$$|\operatorname{Im}\gamma_{\mathbf{a}}(z)|\leq\tfrac{1}{2}|\operatorname{Im}z|\quad\text{for all}\quad\mathbf{a}\in\mathcal{W}_{n_0},\quad\mathbf{a}\to b\in\mathcal{A},\quad z\in D_b.$$

Take $z\in D_b$ for some $b\in\mathcal{A}$. Then

$$\begin{aligned}|w_K\mathcal{L}_s^{n_0}f(z)|&\leq\sum_{\substack{\mathbf{a}\in\mathcal{W}_{n_0}\\ \mathbf{a}\to b}}w_K(z)\big|\gamma'_{\mathbf{a}}(z)^s f(\gamma_{\mathbf{a}}(z))\big|\leq\sum_{\substack{\mathbf{a}\in\mathcal{W}_{n_0}\\ \mathbf{a}\to b}}\frac{w_K(z)|\gamma'_{\mathbf{a}}(z)^s|}{w_K(\gamma_{\mathbf{a}}(z))}\sup_{\mathbf{D}_2}|w_Kf|\\&\leq C_\Gamma\sum_{\substack{\mathbf{a}\in\mathcal{W}_{n_0}\\ \mathbf{a}\to b}}e^{-(K|\operatorname{Im}z|+2\arg\gamma'_{\mathbf{a}}(z))/2h}\sup_{\mathbf{D}_2}|w_Kf|\end{aligned}$$

where in the last inequality we use the bound $|\gamma'_{\mathbf{a}}(z)^s|\leq C_\Gamma e^{-\arg\gamma'_{\mathbf{a}}(z)/h}$. It remains to choose K large enough so that

$$-\arg\gamma'_{\mathbf{a}}(z)\leq\tfrac{1}{2}K|\operatorname{Im}z|\quad\text{for all}\quad\mathbf{a}\in\mathcal{W}_{n_0},\quad\mathbf{a}\to b\in\mathcal{A},\quad z\in D_b$$

which is possible since $\arg\gamma'_{\mathbf{a}}(z)=0$ when $z\in\mathbb{R}$. □

We next show that for u satisfying (3.2), the norm of u on $\mathbf{D}$ is controlled by its norm on $\mathbf{I}=\mathbf{D}\cap\mathbb{R}$:

Lemma 3.2. *We have*

$$\sup_{\mathbf{D}}|w_Ku|\leq C_\Gamma\sup_{\mathbf{I}}|u|.\tag{3.4}$$

Proof. Let n_0 come from Lemma 3.1. Since $u=\mathcal{L}_s^{n_0}u$, (3.3) shows

$$\sup_{\mathbf{D}}|w_Ku|\leq C_\Gamma\sup_{\mathbf{D}_2}|w_Ku|.\tag{3.5}$$

Applying Lemma 2.3 to the functions $\exp(\pm iKz/h)u(z)$ in $\mathbf{D}^\pm$, we get for some $c\in(0,1]$ depending only on the Schottky data

$$\sup_{\mathbf{D}_2}|w_Ku|\leq\big(\sup_{\mathbf{I}}|u|\big)^c\big(\sup_{\mathbf{D}}|w_Ku|\big)^{1-c}.\tag{3.6}$$

Together (3.5) and (3.6) imply (3.4). □

3.2. **Cutoffs and microlocalization.** Let $Z(h^\rho)\subset\mathcal{W}^\circ$ be the partition defined in (2.15), where $\rho\in(0,1)$ is fixed as in (1.5). By Lemma 2.4

$$\mathcal{L}_s(h^\rho)u=u\quad\text{where}\quad\mathcal{L}_s(h^\rho):=\mathcal{L}_{\overline{Z(h^\rho)},s}.\tag{3.7}$$

That is,

$$u(x)=\mathcal{L}_s(h^\rho)u(x)=\sum_{\substack{\mathbf{a}\in Z(h^\rho)\\ \overline{\mathbf{a}}\rightsquigarrow b}}\gamma'_{\overline{\mathbf{a}}'}(x)^s u(\gamma_{\overline{\mathbf{a}}'}(x)),\quad x\in I_b,\ b\in\mathcal{A}.\tag{3.8}$$

From [BD17, Lemma 2.10] and Lemma 2.1, we have for all $\mathbf{a} = a_1 \dots a_n \in Z(h^\rho)$

$$C_\Gamma^{-1} h^\rho \le |I_{\mathbf{a}}| \le h^\rho, \tag{3.9}$$

$$C_\Gamma^{-1} h^\rho \le |I_{\overline{\mathbf{a}}}| \le C_\Gamma h^\rho, \tag{3.10}$$

$$C_\Gamma^{-1} h^\rho \le \gamma'_{\overline{\mathbf{a}}'} \le C_\Gamma h^\rho \quad \text{on } I_{\overline{a_1}}. \tag{3.11}$$

Let $\mathbf{I}'$ be defined by (2.5); recall also the definition (2.4) of $I'_{\mathbf{a}}$. Since $\operatorname{Re} s \ge 0$, (3.8) gives

$$\sup_{\mathbf{I}} |u| \le C_\Gamma \sup_{\mathbf{I}'} |u| \le C_\Gamma \sum_{\mathbf{a} \in Z(h^\rho)} \sup_{I'_{\overline{\mathbf{a}}}} |u| \tag{3.12}$$

where for the first inequality we used $u = \mathcal{L}_s u$. That is, the behavior of u on $\mathbf{I}$ (and thus by Lemma 3.2, on $\mathbf{D}$) is controlled by its behavior on the intervals $I'_{\overline{\mathbf{a}}}$ for $\mathbf{a} \in Z(h^\rho)$.

We now define pieces of u localized to each interval $I_{\overline{\mathbf{a}}}$. To that aim, for each $\mathbf{a} \in Z(h^\rho)$ we choose a cutoff function

$$\chi_{\mathbf{a}} \in C_c^\infty(I_{\overline{\mathbf{a}}}^\circ; [0,1]), \quad \operatorname{supp}(1 - \chi_{\mathbf{a}}) \cap I'_{\overline{\mathbf{a}}} = \emptyset, \quad \sup |\partial_x^j \chi_{\mathbf{a}}| \le C_{\Gamma,j} h^{-\rho j}. \tag{3.13}$$

This is possible since $d_{\mathbb{R}}(I'_{\overline{\mathbf{a}}}, \mathbb{R} \setminus I_{\overline{\mathbf{a}}}) \ge C_\Gamma^{-1} h^\rho$ by (3.11). We then define

$$u_{\mathbf{a}} := \langle x \rangle^{2s} \chi_{\mathbf{a}} u \in C_c^\infty(I_{\overline{\mathbf{a}}}^\circ), \quad \mathbf{a} \in Z(h^\rho). \tag{3.14}$$

Equation (3.8) implies that, with $T_{\gamma,s}$ given by (2.22),

$$\langle x \rangle^{2s} u(x) = \sum_{\substack{\mathbf{a} \in Z(h^\rho) \\ \overline{\mathbf{a}} \rightsquigarrow b}} T_{\gamma_{\overline{\mathbf{a}}'}, s} u_{\mathbf{a}}(x), \quad x \in I'_b, \ b \in \mathcal{A}. \tag{3.15}$$

The next lemma shows that $u_{\mathbf{a}}$'s are semiclassically localized to a bounded set in ξ. This is where we use that $\rho < 1$: if we instead put $\rho = 1$ then multiplication by $\chi_{\mathbf{a}}$ would inevitably blur the support of the semiclassical Fourier transform.

Lemma 3.3. *Let $K \ge 10$ be chosen in Lemma 3.1. Then for each $\mathbf{a} \in Z(h^\rho)$, $u_{\mathbf{a}}$ is semiclassically localized to frequencies $|\xi| \le \frac{3K}{2}$. More precisely, for all N,*

$$|\mathcal{F}_h u_{\mathbf{a}}(\xi)| \le C_{\Gamma,N} h^N |\xi|^{-N} \sup_{\mathbf{I}} |u| \quad \text{when} \quad |\xi| \ge \tfrac{3K}{2}. \tag{3.16}$$

Remark. A finer compact microlocalization statement can be given using the FBI transform – see [Ji17, Proposition 2.2].

Proof. For each $\mathbf{a} \in Z(h^\rho)$, let $\widetilde{\chi}_{\mathbf{a}} \in C_c^\infty(D_{\overline{\mathbf{a}}}^\circ)$ be an almost analytic extension of $\chi_{\mathbf{a}}$, more precisely for each N

$$\widetilde{\chi}_{\mathbf{a}}|_{\mathbb{R}} = \chi_{\mathbf{a}}, \qquad h^\rho |\bar{\partial}_z \widetilde{\chi}_{\mathbf{a}}(z)| \le C_{\Gamma,N} \big(h^{-\rho} |\operatorname{Im} z|\big)^N.$$

See for instance [Zw12, Theorem 3.6] for a construction of such extension; here we map $D_{\overline{\mathbf{a}}}$ to the unit disk and use the derivative bounds (3.13). Let $\langle z \rangle^{2s} := (1 + z^2)^s$ be the holomorphic extension of $\langle x \rangle^{2s}$ to $D_{\overline{\mathbf{a}}}$. We note that

$$|\langle z \rangle^{2s}| \le C_\Gamma \exp(2 |\operatorname{Im} z| / h), \quad z \in D_{\overline{\mathbf{a}}}.$$

Since u is holomorphic in $D_{\overline{\mathbf{a}}}$, we have

$$\begin{aligned}\mathcal{F}_h u_{\mathbf{a}}(\xi) &= \int_{\mathbb{R}} u(x)\langle x\rangle^{2s} e^{-\frac{i}{h}\xi x}\widetilde{\chi}_{\mathbf{a}}(x)\,dx\\ &= -(\operatorname{sgn}\xi)\int_{D_{\overline{\mathbf{a}}}\cap\{(\operatorname{sgn}\xi)\operatorname{Im}z\leq 0\}} u(z)\langle z\rangle^{2s}e^{-\frac{i}{h}\xi z}\bar\partial_z\widetilde{\chi}_{\mathbf{a}}(z)\,d\bar z\wedge dz.\end{aligned}$$

Using Lemma 3.2 and the fact that $|\xi|\geq 3K/2$ we estimate for all N,

$$\begin{aligned}|\mathcal{F}_h u_{\mathbf{a}}(\xi)| &\leq C_{\Gamma,N}\sup_{\mathbf{I}}|u|\int_0^\infty (h^{-\rho}y)^N e^{(K-|\xi|+2)y/h}\,dy\\ &\leq C_{\Gamma,N}\sup_{\mathbf{I}}|u|\int_0^\infty (h^{-\rho}y)^N e^{-|\xi|y/5h}\,dy \leq C_{\Gamma,N}h^{1+N(1-\rho)}|\xi|^{-1-N}\sup_{\mathbf{I}}|u|.\end{aligned}$$

Since $\rho<1$ and N can be chosen arbitrary, (3.16) follows. □

Lemma 3.3 implies that the sup-norm of $u_{\mathbf{a}}$ can be estimated by its L^2 norm:

Lemma 3.4. *We have for all N and $\mathbf{a}\in Z(h^\rho)$,*

$$\sup|u_{\mathbf{a}}|\leq C_\Gamma h^{-1/2}\|u_{\mathbf{a}}\| + C_{\Gamma,N}h^N\sup_{\mathbf{I}}|u|. \tag{3.17}$$

Proof. We use the Fourier inversion formula: $u_{\mathbf{a}}(x) = (2\pi h)^{-1}\int_{\mathbb{R}} e^{ix\xi/h}\mathcal{F}_h u_{\mathbf{a}}(\xi)\,d\xi$. We split this integral into two parts. The first one is the integral over $\{|\xi|\leq\frac{3K}{2}\}$, which is bounded by $C_\Gamma h^{-1}\|\mathcal{F}_h u_{\mathbf{a}}\| = C_\Gamma h^{-1/2}\|u_{\mathbf{a}}\|$. The second one, the integral over $\{|\xi|\geq\frac{3K}{2}\}$, is bounded by $C_{\Gamma,N}h^N\sup_{\mathbf{I}}|u|$ by Lemma 3.3. □

Since the intervals $I_{\mathbf{a}}$, $\mathbf{a}\in Z(h^\rho)$, do not intersect and are contained in $\mathbf{I}$, we have $\#(Z(h^\rho))\leq C_\Gamma h^{-1}$. Combining this with (3.12) and (3.17), we get the following bound:

$$\sup_{\mathbf{I}}|u|\leq C_\Gamma\sum_{\mathbf{a}\in Z(h^\rho)}\sup|u_{\mathbf{a}}|\leq C_\Gamma h^{-1}\Big(\sum_{\mathbf{a}\in Z(h^\rho)}\|u_{\mathbf{a}}\|^2\Big)^{1/2}. \tag{3.18}$$

3.3. **End of the proof.** We use (3.15) together with equivariance of the operator $\mathcal{B}(s)$ (Lemma 2.5) to obtain a formula for u in terms of $\mathcal{B}(s)$, see (3.26) below. Together with the fractal uncertainty bound (1.5) this will give $u\equiv 0$.

In order to take advantage of the equivariance of $\mathcal{B}(s)$, we approximate $u_{\mathbf{a}}$'s appearing in (3.15) by functions in the range of $\mathcal{B}(s)$. For that, let K be chosen from Lemma 3.1. Define the partition $Z(\frac{1}{10K})$ by (2.15). Since h is small and by (3.10),

$$\forall\,\mathbf{a}\in Z(h^\rho)\quad \exists!\ \widetilde{\mathbf{a}}\in Z\big(\tfrac{1}{10K}\big):\quad \widetilde{\mathbf{a}}\prec\overline{\mathbf{a}},\quad \widetilde{\mathbf{a}}\neq\overline{\mathbf{a}}. \tag{3.19}$$

We stress that $10K|I_{\widetilde{\mathbf{a}}}|\leq 1$ and that $\widetilde{\mathbf{a}}$ lies in $Z(\frac{1}{10K})$ which is a finite h-independent set. Choose h-independent cutoff functions (see (2.4))

$$\begin{aligned}\chi_{\widetilde{\mathbf{a}}}&\in C_c^\infty(I_{\widetilde{\mathbf{a}}}^\circ),\quad \operatorname{supp}(1-\chi_{\widetilde{\mathbf{a}}})\cap I'_{\widetilde{\mathbf{a}}}=\emptyset;\\ \chi_K&\in C_c^\infty((-2K,2K)),\quad \operatorname{supp}(1-\chi_K)\cap\big[-\tfrac{3K}{2},\tfrac{3K}{2}\big]=\emptyset\end{aligned}$$

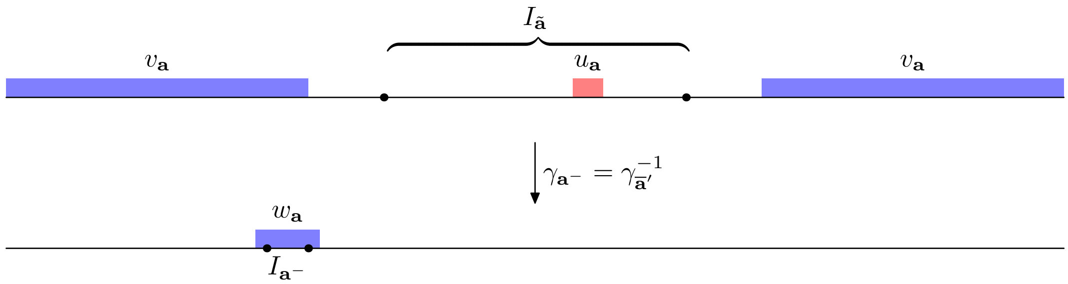

FIGURE 5. The supports of $u_{\mathbf{a}}, v_{\mathbf{a}}, w_{\mathbf{a}}$ (shaded) and the intervals $I_{\tilde{\mathbf{a}}}, I_{\mathbf{a}^-}$ (marked by endpoints). The top picture is mapped to the bottom one by $\gamma_{\mathbf{a}^-}$. The supports of $u_{\mathbf{a}}, w_{\mathbf{a}}$ have size $\sim h^\rho$.

and consider semiclassical pseudodifferential operators

$$A_{\tilde{\mathbf{a}}} := \chi_{\tilde{\mathbf{a}}}(x)\chi_K(hD_x) = \mathrm{Op}_h\big(\chi_{\tilde{\mathbf{a}}}(x)\chi_K(\xi)\big).$$

By Lemma 3.3 and since $\operatorname{supp} u_{\mathbf{a}} \subset I_{\overline{\mathbf{a}}} \subset I'_{\tilde{\mathbf{a}}}$, we have for all $\mathbf{a} \in Z(h^\rho)$ and all N

$$\|u_{\mathbf{a}} - A_{\tilde{\mathbf{a}}} u_{\mathbf{a}}\| \le C_{\Gamma,N} h^N \sup_{\mathbf{I}} |u|. \tag{3.20}$$

We now apply Lemma 2.7 with $I := I_{\tilde{\mathbf{a}}}$ and $A := A_{\tilde{\mathbf{a}}}$ to write

$$A_{\tilde{\mathbf{a}}} = \psi_{\tilde{\mathbf{a}}} \mathcal{B}(s) \omega_{\tilde{\mathbf{a}}} Q_{\tilde{\mathbf{a}}} + \mathcal{O}(h^\infty)_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \tag{3.21}$$

where $Q_{\tilde{\mathbf{a}}} : L^2(\mathbb{R}) \to L^2(\dot{\mathbb{R}})$ is bounded uniformly in h and

$$\psi_{\tilde{\mathbf{a}}} \in C_c^\infty(I_{\tilde{\mathbf{a}}}^\circ), \quad \operatorname{supp} \chi_{\tilde{\mathbf{a}}} \subset \{\psi_{\tilde{\mathbf{a}}} = 1\}, \quad \omega_{\tilde{\mathbf{a}}} \in C^\infty(\dot{\mathbb{R}} \setminus I_{\tilde{\mathbf{a}}}). \tag{3.22}$$

Define (see Figure 5)

$$v_{\mathbf{a}} := \omega_{\tilde{\mathbf{a}}} Q_{\tilde{\mathbf{a}}} u_{\mathbf{a}} \in L^2(\dot{\mathbb{R}}), \quad \|v_{\mathbf{a}}\|_{L^2(\dot{\mathbb{R}})} \le C_\Gamma \|u_{\mathbf{a}}\|_{L^2(\mathbb{R})}. \tag{3.23}$$

Then (3.20) and (3.21) imply that for all N,

$$\|u_{\mathbf{a}} - \psi_{\tilde{\mathbf{a}}} \mathcal{B}(s) v_{\mathbf{a}}\| \le C_{\Gamma,N} h^N \sup_{\mathbf{I}} |u|. \tag{3.24}$$

To approximate u by an element of the image of $\mathcal{B}(s)$ we use the equation (3.15) and the equivariance property (2.23). Let $\mathbf{a} \in Z(h^\rho)$, $b \in \mathcal{A}$, $\overline{\mathbf{a}} \rightsquigarrow b$. By (3.22), we have $\psi_{\tilde{\mathbf{a}}} = 1$ on $\gamma_{\overline{\mathbf{a}}'}(I_b) = I_{\overline{\mathbf{a}}} \subset I'_{\tilde{\mathbf{a}}}$. Thus by (2.23)

$$T_{\gamma_{\overline{\mathbf{a}}'},s} \psi_{\tilde{\mathbf{a}}} \mathcal{B}(s) v_{\mathbf{a}} = \mathcal{B}(s) T_{\gamma_{\overline{\mathbf{a}}'},1-s} v_{\mathbf{a}} \quad \text{on } I_b. \tag{3.25}$$

Substituting (3.24) into (3.15) and using (3.25), we obtain the following approximate formula for u, true for each N and $b \in \mathcal{A}$:

$$\Big\| \langle x \rangle^{2s} u - \mathcal{B}(s) \sum_{\substack{\mathbf{a} \in Z(h^\rho) \\ \overline{\mathbf{a}} \rightsquigarrow b}} w_{\mathbf{a}} \Big\|_{L^2(I'_b)} \le C_{\Gamma,N} h^N \sup_{\mathbf{I}} |u| \tag{3.26}$$

where

$$w_{\mathbf{a}} := T_{\gamma_{\overline{\mathbf{a}}'},1-s} v_{\mathbf{a}} \in L^2(\dot{\mathbb{R}}). \tag{3.27}$$

We now establish a few properties of $w_{\mathbf{a}}$, starting with its support:

Lemma 3.5. *Let $\mathbf{a} = a_1 \dots a_n \in Z(h^\rho)$ and put $\mathbf{a}^- := a_2 \dots a_n \in \mathcal{W}^\circ$. Then*

$$\operatorname{supp} w_{\mathbf{a}} \subset I_{\mathbf{a}^-}(C_\Gamma h^\rho) \tag{3.28}$$

where $I(\tau) = I + [-\tau, \tau]$ denotes the τ-neighbourhood of I.

Proof. From the definition of $w_{\mathbf{a}}$ and (3.22) we have (see Figure 5)

$$\operatorname{supp} w_{\mathbf{a}} \subset \gamma_{\overline{\mathbf{a}}'}^{-1}(\operatorname{supp} v_{\mathbf{a}}) \subset \gamma_{\mathbf{a}^-}(\dot{\mathbb{R}} \setminus I_{\widetilde{\mathbf{a}}}).$$

Recall that $\widetilde{\mathbf{a}} \prec \overline{\mathbf{a}}$, thus $\widetilde{\mathbf{a}} = \overline{a_n}\,\overline{a_{n-1}} \dots \overline{a_{n-m}}$ for some $m \le C_\Gamma$, in particular $m \ll n$. Then by (2.1) we have $\gamma_{a_{n-m}}(\dot{\mathbb{R}} \setminus I_{\overline{a_{n-m}}}) = I^\circ_{a_{n-m}}$ and thus

$$\gamma_{\mathbf{a}^-}(\dot{\mathbb{R}} \setminus I_{\widetilde{\mathbf{a}}}) = \gamma_{a_2} \cdots \gamma_{a_{n-m}}(\dot{\mathbb{R}} \setminus I_{\overline{a_{n-m}}}) = I^\circ_{a_2 \dots a_{n-m}}.$$

We have $I_{a_2 \dots a_{n-m}} \supset I_{\mathbf{a}^-}$ and by [BD17, Lemma 2.7] we have $|I_{a_2 \dots a_{n-m}}| \le C_\Gamma |I_{\mathbf{a}^-}| \le C_\Gamma h^\rho$. Therefore $I_{a_2 \dots a_{n-m}} \subset I_{\mathbf{a}^-}(C_\Gamma h^\rho)$, finishing the proof. □

We next estimate the norm of $w_{\mathbf{a}}$. This is where the value of the parameter ν from (3.1) enters the argument.

Lemma 3.6. *We have*

$$\|w_{\mathbf{a}}\| \le C_\Gamma h^{-\rho\nu} \|u_{\mathbf{a}}\|. \tag{3.29}$$

Proof. The definition (3.27) of $w_{\mathbf{a}}$ gives (recalling (2.22)) $w_{\mathbf{a}}(x) = |\gamma'_{\overline{\mathbf{a}}'}(x)|_{\mathbb{S}}^{1-s} v_{\mathbf{a}}(\gamma_{\overline{\mathbf{a}}'}(x))$ for $x \in \dot{\mathbb{R}}$. Using the notation of the proof of Lemma 3.5, we can restrict to $x \in I_{a_2 \dots a_{n-m}}$. Since $m \le C_\Gamma$ we have

$$C_\Gamma^{-1} |\gamma'_{\overline{a_{n-m-1}} \dots \overline{a_2}}(x)|_{\mathbb{S}} \le |\gamma'_{\overline{\mathbf{a}}'}(x)|_{\mathbb{S}} \le C_\Gamma |\gamma'_{\overline{a_{n-m-1}} \dots \overline{a_2}}(x)|_{\mathbb{S}}.$$

We have for $x \in I_{a_2 \dots a_{n-m}}$

$$|\gamma'_{\overline{a_{n-m-1}} \dots \overline{a_2}}(x)|_{\mathbb{S}} = |\gamma'_{a_2 \dots a_{n-m-1}}(x')|_{\mathbb{S}}^{-1}, \quad x' := \gamma_{\overline{a_{n-m-1}} \dots \overline{a_2}}(x) \in I_{a_{n-m}}.$$

By Lemma 2.1 and [BD17, Lemma 2.7] we have

$$C_\Gamma^{-1} h^\rho \le |\gamma'_{a_2 \dots a_{n-m-1}}(x')|_{\mathbb{S}} \le C_\Gamma h^\rho,$$

therefore

$$C_\Gamma^{-1} h^{-\rho} \le |\gamma'_{\overline{\mathbf{a}}'}(x)|_{\mathbb{S}} \le C_\Gamma h^{-\rho}, \quad x \in I_{a_2 \dots a_{n-m}},$$

which using the change of variables formula and (3.1) implies

$$\|w_{\mathbf{a}}\|_{L^2(\mathbb{R})} \le C_\Gamma h^{-\rho\nu} \|v_{\mathbf{a}}\|_{L^2(\dot{\mathbb{R}})}.$$

Together with (3.23) this gives (3.29). □

We next rewrite the formula (3.26) in terms of the operator $\mathcal{B}_\chi(h)$ defined in (1.3). Let $\chi_0 \in C^\infty(\dot{\mathbb{R}} \times \dot{\mathbb{R}})$ be such that

$$\operatorname{supp} \chi_0 \cap \{(x, x) \mid x \in \dot{\mathbb{R}}\} = \emptyset$$

and

$$a, b \in \mathcal{A},\ a \neq b \implies \operatorname{supp}(1 - \chi_0) \cap (I_a \times I_b) = \emptyset.$$

Put

$$\chi(x, x') := |x - x'|_{\mathbb{S}}^{2\nu-1} \chi_0(x, x'). \tag{3.30}$$

For each $b \in \mathcal{A}$ define the following compactly supported function

$$w^{(b)} := \sum_{\substack{\mathbf{a} \in Z(h^\rho) \\ \overline{\mathbf{a}} \rightsquigarrow b}} w_{\mathbf{a}} \in L^2(\mathbb{R}).$$

Then by (3.28) we have (identifying $\dot{\mathbb{R}}$ with $\mathbb{S}^1$ using (2.19))

$$\mathcal{B}_\chi(h) w^{(b)} = \mathcal{B}(s) \sum_{\substack{\mathbf{a} \in Z(h^\rho) \\ \overline{\mathbf{a}} \rightsquigarrow b}} w_{\mathbf{a}} \quad \text{on } I_b.$$

Thus (3.26) implies

$$\big\| \langle x \rangle^{2s} u - \mathcal{B}_\chi(h) w^{(b)} \big\|_{L^2(I_b')} \leq C_{\Gamma,N} h^N \sup_{\mathbf{I}} |u|. \tag{3.31}$$

The supports of $w_{\mathbf{a}}$, as well as the supports of $u_{\mathbf{a}}$, lie in a $C_\Gamma h^\rho$ neighbourhood of the limit set and have bounded overlaps. Analysing this closely will give us

Lemma 3.7. *Denote by $\Lambda_\Gamma(\tau) = \Lambda_\Gamma + [-\tau, \tau]$ the τ-neighbourhood of the limit set $\Lambda_\Gamma \subset \mathbb{R}$. Then*

$$\bigcup_{b \in \mathcal{A}} \operatorname{supp} w^{(b)} \subset \Lambda_\Gamma(C_\Gamma h^\rho), \tag{3.32}$$

$$\max_{b \in \mathcal{A}} \|w^{(b)}\|^2 \leq C_\Gamma \sum_{\mathbf{a} \in Z(h^\rho)} \|w_{\mathbf{a}}\|^2, \tag{3.33}$$

$$\sum_{\mathbf{a} \in Z(h^\rho)} \|u_{\mathbf{a}}\|^2 \leq C_\Gamma \| \mathbb{1}_{\Lambda_\Gamma(C_\Gamma h^\rho)} \langle x \rangle^{2s} u \|^2. \tag{3.34}$$

Proof. By Lemma 3.5, for each $\mathbf{a} \in Z(h^\rho)$ we have $\operatorname{supp} w_{\mathbf{a}} \subset I_{\mathbf{a}^-}(C_\Gamma h^\rho)$. By [BD17, Lemma 2.7] the interval $I_{\mathbf{a}^-}$ has length $\leq C_\Gamma h^\rho$ and it intersects the limit set, therefore $\operatorname{supp} w_{\mathbf{a}} \subset \Lambda_\Gamma(C_\Gamma h^\rho)$. This proves (3.32).

Next, we have the following multiplicity estimates:

$$\sup_{x \in \mathbb{R}} \#\{\mathbf{a} \in Z(h^\rho) \mid x \in \operatorname{supp} w_{\mathbf{a}}\} \leq C_\Gamma, \tag{3.35}$$

$$\sup_{x \in \mathbb{R}} \#\{\mathbf{a} \in Z(h^\rho) \mid x \in \operatorname{supp} \chi_{\mathbf{a}}\} \leq C_\Gamma. \tag{3.36}$$

Both of these follow from Lemma 2.2. Indeed, from the proof of Lemma 3.5 we see that for $\mathbf{a} = a_1 \dots a_n \in Z(h^\rho)$, we have $\operatorname{supp} w_{\mathbf{a}} \subset I_{a_2\dots a_{n-m}}$ where m is fixed large enough depending only on the Schottky data. By [BD17, Lemma 2.7] we have $C_\Gamma^{-1}h^\rho \le |I_{a_2\dots a_{n-m}}| \le C_\Gamma h^\rho$. Therefore, the number of intervals $I_{a_2\dots a_{n-m}}$ containing a given point x is bounded, which gives (3.35). To prove (3.36), we use that $\operatorname{supp}\chi_{\mathbf{a}} \subset I_{\overline{\mathbf{a}}}$ by (3.13) and $C_\Gamma^{-1}h^\rho \le |I_{\overline{\mathbf{a}}}| \le C_\Gamma h^\rho$ by (3.10).

Now, (3.35) immediately gives (3.33):

$$\max_{b\in\mathcal{A}} \|w^{(b)}\|^2_{L^2(\mathbb{R})} \le \int_{\mathbb{R}} \Big(\sum_{\mathbf{a}\in Z(h^\rho)} |w_{\mathbf{a}}(x)|\Big)^2 dx \le C_\Gamma \int_{\mathbb{R}} \sum_{\mathbf{a}\in Z(h^\rho)} |w_{\mathbf{a}}(x)|^2\,dx.$$

To show (3.34), we first note that for all $\mathbf{a}\in Z(h^\rho)$ we have $\operatorname{supp}\chi_{\mathbf{a}} \subset I_{\overline{\mathbf{a}}} \subset \Lambda_\Gamma(C_\Gamma h^\rho)$, since $I_{\overline{\mathbf{a}}}$ is an interval of size $\le C_\Gamma h^\rho$ intersecting the limit set. We now recall (3.14):

$$\begin{aligned}\sum_{\mathbf{a}\in Z(h^\rho)} \|u_{\mathbf{a}}\|^2_{L^2(\mathbb{R})} &= \int_{\Lambda_\Gamma(C_\Gamma h^\rho)} \sum_{\mathbf{a}\in Z(h^\rho)} |\chi_{\mathbf{a}}(x)|^2 \cdot |\langle x\rangle^{2s} u(x)|^2\,dx \\ &\le C_\Gamma \int_{\Lambda_\Gamma(C_\Gamma h^\rho)} |\langle x\rangle^{2s}u(x)|^2\,dx\end{aligned}$$

where in the last inequality we used (3.36). □

We are now ready to prove (3.2) and hence finish the proof of the main theorem. From (3.31) and the fact that $\Lambda_\Gamma(C_\Gamma h^\rho) \subset \bigcup_{b\in\mathcal{A}} I'_b$ we obtain

$$\|\mathbb{1}_{\Lambda_\Gamma(C_\Gamma h^\rho)}\langle x\rangle^{2s}u\|^2 \le C_\Gamma \max_{b\in\mathcal{A}} \|\mathbb{1}_{\Lambda_\Gamma(C_\Gamma h^\rho)}\mathcal{B}_\chi(h)w^{(b)}\|^2 + \mathcal{O}(h^\infty)\sup_{\mathbf{I}} |u|^2. \tag{3.37}$$

The fractal uncertainty bound (1.5) and the support property (3.32) show that

$$\max_{b\in\mathcal{A}} \|\mathbb{1}_{\Lambda_\Gamma(C_\Gamma h^\rho)}\mathcal{B}_\chi(h)w^{(b)}\|^2 \le C_\Gamma h^{2(\beta-\varepsilon)} \max_{b\in\mathcal{A}} \|w^{(b)}\|^2. \tag{3.38}$$

The estimate (3.33) and Lemma 3.6 give

$$h^{2(\beta-\varepsilon)} \max_{b\in\mathcal{A}} \|w^{(b)}\|^2 \le C_\Gamma h^{2(\beta-\varepsilon-\rho\nu)} \sum_{\mathbf{a}\in Z(h^\rho)} \|u_{\mathbf{a}}\|^2. \tag{3.39}$$

Due to (3.34), the left hand side of (3.37) bounds the sum on the right hand side of (3.39). Putting (3.37), (3.38), and (3.39) together and using (3.18) to remove $\mathcal{O}(h^\infty)\sup_{\mathbf{I}}|u|^2$, we obtain for h small enough

$$\sum_{\mathbf{a}\in Z(h^\rho)} \|u_{\mathbf{a}}\|^2 \le C_\Gamma h^{2(\beta-\varepsilon-\rho\nu)} \sum_{\mathbf{a}\in Z(h^\rho)} \|u_{\mathbf{a}}\|^2. \tag{3.40}$$

From (3.1) we have $2(\beta-\varepsilon-\rho\nu) > 0$, thus (3.40) implies that $\sum_{\mathbf{a}\in Z(h^\rho)} \|u_{\mathbf{a}}\|^2 = 0$ if h is small enough. Recalling (3.14), analyticity of u shows that $u \equiv 0$.

Appendix

We give proofs of the two lemmas about the operator $\mathcal{B}(s)$ introduced in §2.5.

Proof of Lemma 2.6. For simplicity we assume that $\operatorname{supp}\chi_1, \operatorname{supp}\chi_2 \subset \mathbb{R}$; same proof applies to the general case, identifying $\dot{\mathbb{R}}$ with the circle by (2.19). The operator $(\chi_1\mathcal{B}(s)\chi_2)^*\chi_1\mathcal{B}(s)\chi_2$ has integral kernel

$$\begin{aligned}\mathcal{K}(x,x'') = (2\pi h)^{-1}\int_{\dot{\mathbb{R}}} e^{\frac{i}{h}(\Phi(x',x'')-\Phi(x',x))}|x-x'|_{\mathbb{S}}^{2\nu-1}|x'-x''|_{\mathbb{S}}^{2\nu-1}\\ |\chi_1(x')|^2\overline{\chi_2(x)}\chi_2(x'')\,dP(x').\end{aligned} \tag{A.1}$$

We have $\partial_{x'}\Phi(x',x) = 2(x-x')^{-1} + 2x'\langle x'\rangle^{-2}$, thus

$$\big|\partial_{x'}\big(\Phi(x',x'')-\Phi(x',x)\big)\big| \geq |x-x''|/C \quad \text{for } x,x''\in\operatorname{supp}\chi_2,\ x'\in\operatorname{supp}\chi_1.$$

Integrating by parts N times in (A.1) we see that $|\mathcal{K}(x,x'')| \leq C_N h^{N-1}|x-x''|^{-N}$. Therefore $\sup_x \int_{\dot{\mathbb{R}}} |\mathcal{K}(x,x'')|\,dP(x'') \leq C$. (Here we use the case $N=0$ for $|x-x''|\leq h$ and the case $N=2$ for $|x-x''|\geq h$.) By Schur's Lemma (see for instance [Zw12, Proof of Theorem 4.21]) we see that the operator $(\chi_1\mathcal{B}(s)\chi_2)^*\chi_1\mathcal{B}(s)\chi_2$, and thus $\chi_1\mathcal{B}(s)\chi_2$, is bounded on $L^2(\dot{\mathbb{R}})$ uniformly in h. □

The following technical lemma is useful to establish invertibility of $\mathcal{B}(s)$ in Lemma 2.7. To state it we use the standard (left) quantization procedure (2.27).

Lemma A.1. *Assume that $\chi_1,\chi_3 \in C_c^\infty(\mathbb{R})$ and $\chi_2\in C^\infty(\dot{\mathbb{R}})$ satisfy*

$$\operatorname{supp}\chi_1\cap\operatorname{supp}\chi_2 = \operatorname{supp}\chi_2\cap\operatorname{supp}\chi_3 = \emptyset.$$

Let $q(x,\xi)\in C_c^\infty(\mathbb{R}^2)$. Define the operator

$$B = \chi_1\mathcal{B}(s)\chi_2\mathcal{B}(1-s)\chi_3\operatorname{Op}_h(q) : L^2(\mathbb{R})\to L^2(\mathbb{R}).$$

Then $B = \operatorname{Op}_h(b)$ where $b(x,\xi;h)\in\mathscr{S}(\mathbb{R}^2)$ (uniformly in h) admits an asymptotic expansion:

$$\begin{aligned} b(x,\xi;h) \sim \chi_1(x)\sum_{j=0}^\infty h^j L_j\big(\chi_2(x')\chi_3(x)q(x,\xi)\big)\big|_{x'=x'(x,\xi)},\\ x'(x,\xi) := x + \frac{2\langle x\rangle^2}{\langle x\rangle^2\xi - 2x}\in\dot{\mathbb{R}},\end{aligned} \tag{A.2}$$

where L_j is an order $2j$ differential operator on $\mathbb{R}_x\times\dot{\mathbb{R}}_{x'}$ with coefficients depending on x,ξ,ν, and $L_0 = 1/2$.

Proof. For simplicity we assume that $\operatorname{supp}\chi_2\subset\mathbb{R}$. The general case is handled similarly, identifying $\dot{\mathbb{R}}$ with the circle.

For $\xi \in \mathbb{R}$, define the function $e_\xi(x) = \exp(ix\xi/h)$. By oscillatory testing [Zw12, Theorem 4.19], we have $B = \operatorname{Op}_h(b)$ where b is defined by the formula

$$(Be_\xi)(x) = b(x,\xi;h)e_\xi(x).$$

It remains to show that $b \in \mathscr{S}$ and had the expansion (A.2). We compute

$$b(x,\xi;h) = (2\pi h)^{-1} \int_{\mathbb{R}^2} e^{\frac{i}{h}\Psi(x',x'';x,\xi)} p(x',x'';x,\xi)\, dx'dx''$$

where

$$\Psi = \Phi(x,x') - \Phi(x',x'') + (x''-x)\xi,$$
$$p = 4\chi_1(x)\chi_2(x')\chi_3(x'')q(x'',\xi)|x-x'|_{\mathbb{S}}^{2\nu-1}|x'-x''|_{\mathbb{S}}^{-2\nu-1}\langle x'\rangle^{-2}\langle x''\rangle^{-2}.$$

We have

$$\partial_{x'}\Psi = \frac{2(x''-x)}{(x-x')(x''-x')}, \quad \partial_{x''}\Psi = \xi - \frac{2x''}{\langle x''\rangle^2} + \frac{2}{x''-x'}.$$

It follows that Ψ is a Morse function on $\{x' \neq x,\ x' \neq x''\}$ with the unique critical point given by $x'' = x$, $x' = x'(x,\xi)$ where $x'(x,\xi)$ is defined in (A.2).

We have $\operatorname{supp} b \subset \{x \in \operatorname{supp}\chi_1\}$. Next, for $(x',x'') \in \operatorname{supp} p$ and large $|\xi|$, we have $|\partial_{x''}\Psi| \geq \frac{1}{2}|\xi|$. Therefore, repeated integration by parts in x'' shows that $b = \mathcal{O}(h^\infty)_{\mathscr{S}(\mathbb{R}^2)}$ when $|x| + |\xi|$ is large. The expansion (A.2) follows from the method of stationary phase. To compute L_0 we use that the Hessian of Ψ at the critical point has signature 0 and determinant $4(x - x'(x,\xi))^{-4}$. □

Proof of Lemma 2.7. Choose ψ_I, ω_I satisfying the conditions in the statement of the lemma. It follows from the definition of $x'(x,\xi)$ in (A.2) that $|x - x'(x,\xi)| \geq 2/(1+|\xi|)$. Since $10K|I| \leq 1$ we have

$$x'(x,\xi) \notin \operatorname{supp}(1-\omega_I) \quad \text{for all } x \in I, \quad |\xi| \leq 2K. \tag{A.3}$$

We now put

$$Q_I(s) := \omega_I \mathcal{B}(1-s)\psi_I \operatorname{Op}_h(q) : L^2(\mathbb{R}) \to L^2(\dot{\mathbb{R}})$$

where $q(x,\xi;h) \in C_c^\infty(\mathbb{R}^2)$, to be chosen later, satisfies the support condition

$$\operatorname{supp} q \subset \operatorname{supp} a \subset \{\psi_I(x) = 1,\ |\xi| \leq 2K\}$$

and has an asymptotic expansion

$$q(x,\xi;h) \sim \sum_{j=0}^\infty h^j q_j(x,\xi) \quad \text{as } h \to 0.$$

The L^2 boundedness of $Q_I(s)$ follows from Lemma 2.6, whose proof applies to $\mathcal{B}(1-s)$.

By Lemma A.1, we have $\psi_I \mathcal{B}(s)\omega_I Q_I(s) = \operatorname{Op}_h(b)$ where the symbol b satisfies

$$b(x,\xi;h) \sim \sum_{j=0}^\infty h^j b_j(x,\xi), \quad b_j(x,\xi) = \frac{1}{2}q_j(x,\xi) + \dots$$

and '$\ldots$' denotes terms depending on $q_0, \ldots, q_{j-1}$ according to (A.2), and in particular supported in $\operatorname{supp} a$. Here we use that on $\operatorname{supp} q_j \subset \operatorname{supp} a$ we have $\psi_I(x) = 1$ and $\omega_I(x'(x,\xi)) = 1$ by (A.3). We can now iteratively construct $q_0, q_1, \ldots$ such that $b = a + \mathcal{O}(h^\infty)_{\mathscr{S}(\mathbb{R}^2)}$, finishing the proof. □

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