QUANTUM MONODROMY AND SEMI-CLASSICAL TRACE FORMULÆ

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1. Introduction

Trace formulæ provide one of the most elegant descriptions of the classical-quantum correspondence. One side of a formula is given by a trace of a quantum object, typically derived from a quantum Hamiltonian, and the other side is described in terms of closed orbits of the corresponding classical Hamiltonian. In algebraic situations, such as the original Selberg trace formula, the identities are exact, while in general they hold only in semi-classical or high-energy limits. We refer to a recent survey [14] for an introduction and references.

In this paper we present an intermediate trace formula in which the original trace is expressed in terms of traces of quantum monodromy operators directly related to the classical dynamics. The usual trace formulæ follow and in addition this approach allows handling effective Hamiltonians.

Let $P = (1/i)h\partial_x$ be the semi-classical differentiation operator on the circle, $x \in \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$, $0 < h < 1$. The classical Poisson formula can be written as follows: if $\hat{f} \in \mathcal{C}^\infty_{\rm c}(\mathbb{R})$ then

$$\operatorname{tr} f(P/h) = \frac{1}{2\pi i} \sum_{|k|\leq N} \int_{\mathbb{R}} f(z/h) \left(e^{2\pi i z/h}\right)^k \frac{d}{dz}\left(e^{2\pi i z/h}\right) dz \,, \tag{1.1}$$

where N depends on the support of $\hat{f}$, and we think of $M(z,h) = e^{2\pi i z/h} : \mathbb{C} \to \mathbb{C}$ as the monodromy operator for the solutions of $P - z$. It acts on functions in one dimension lower (zero dimension here), identified geometrically with the functions on the transversal to the closed curve ($\mathbb{S}^1$ here), and analytically with ker $(P-z)$ ($\mathbb{C}$ here).

Now let P be a semi-classical, self-andjoint, principal type operator, with symbol p (for instance $P = -h^2\Delta + V(x)$, $p = \xi^2 + V(x)$), and let $\gamma \subset p^{-1}(0)$ be a closed primitive orbit of the Hamilton flow of p. We can define the *monodromy operator*, $M(z,h)$ for $P - z$ along γ, acting on functions in one dimension lower, that is, on functions on the transversal to γ in the base. We then have

Theorem 1. *Suppose that there exists a neighbourhood of γ, Ω, satisfying the condition*

$$m \in \Omega \text{ and } \exp tH_p(m) = m \,,\ p(m) = 0 \,,\quad 0 < |t| \leq TN \implies m \in \gamma \,, \tag{1.2}$$

where T is the primitive period of γ. If $\hat{f} \in \mathcal{C}^\infty_{\rm c}(\mathbb{R})$, supp $\hat{f} \subset (-NT + C, NT - C) \setminus \{0\}$, $C = C(p) \geq 0$, $\chi \in \mathcal{C}^\infty_{\rm c}(\mathbb{R})$, and $A \in \Psi^{0,0}_h(X)$ is a microlocal cut-off to a sufficiently small neighbourhood of γ, then

$$\operatorname{tr} f(P/h)\chi(P)A = \frac{1}{2\pi i} \sum_{-N-1}^{N-1} \operatorname{tr} \int_{\mathbb{R}} f(z/h) M(z,h)^k \frac{d}{dz} M(z,h)\chi(z) dz + \mathcal{O}(h^\infty) \,, \tag{1.3}$$

where $M(z,h)$ is the semi-classical monodromy operator associated to γ.

The dynamical assumption on the operator means that in a neighbourhood of γ there are no other closed orbits of period less than TN, on the energy surface $p = 0$. We avoid a neighbourhood of 0 in the support of $\hat{f}$ to avoid the dependence on the microlocal cut-off A.

The monodromy operator quantizes the Poincaré map for γ and its geometric analysis gives the now standard trace formulæ of Selberg, Gutzwiller and Duistermaat-Guillemin (see [1] for a recent proof and a historical discussion, and Sect.7 for a derivation based on Theorem 1). The term $k = -1$ corresponds to the contributions from "not moving at all" and the other terms to contributions from going $|k+1|$ times around γ, in the positive direction when $k \geq 0$, and in the negative direction, when $k < -1$. For non-degenerate orbits we analyse the traces on monodromy operators in Sect.7 and recover the usual semi-classical trace formulæ in our general setting – see Theorem 3.

Theorem 1 is a special case of the more general Theorem 2 presented in Sect.6. Motivated by *effective Hamiltonians* in which the spectral parameter appears non-linearly, we give there a trace formula for a family $P(z)$ with the special case corresponding to $P - z$. For an example of a use of effective Hamiltonians in an interesting physical situation we refer to [7]. The effective Hamiltonian described there comes from the "Peierls substitution", and the celebrated "Onsager rule" is a consequence of a calculation of traces.

The point of view taken here is purely semi-classical but when translated to the special case of $\mathcal{C}^\infty$-singularities/high energy regime, it is close to that of Marvizi-Melrose [10] and Popov [12]. In those works the trace of the wave group was reduced to the study of a trace of an operator quantizing the Poincaré map. In [12] it was used to determine contributions of degenerate orbits and our formula could be used for that as well.

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2. Outline of the proof

To present the idea of our proof we use it to derive the classical Poisson summation formula (1.1). The left hand side there can be written using the usual functional calculus based on Cauchy's formula:

$$\operatorname{tr} f\left(\frac{P}{h}\right) = \frac{1}{2\pi i} \operatorname{tr} \int_\Gamma f\left(\frac{z}{h}\right)(P-z)^{-1} dz\,, \quad \Gamma = \Gamma_+ - \Gamma_-\,, \quad \Gamma_\pm = \mathbb{R} \pm iR\,, \tag{2.1}$$

where we take the positive orientation of $\mathbb{R}$ and $R > 0$ is an arbitrary constant. We make an assumption on the support of the Fourier transform on f:

$$\operatorname{supp} \hat{f} \subset (-2\pi N, 2\pi N)\,. \tag{2.2}$$

We would like to replace $(P-z)^{-1}$ by an *effective Hamiltonian* which measures the obstruction to the solvability of $(P-z)u = f$. For that we introduce a *Grushin problem* (see for instance [6] for applications of this method in spectral problems, and for references):

$$\mathcal{P}(z) \stackrel{\text{def}}{=} \begin{pmatrix} P - z & R_-(z) \\ R_+(z) & 0 \end{pmatrix} \;:\; H^1(\mathbb{S}^1) \times \mathbb{C} \;\longrightarrow\; L^2(\mathbb{S}^1) \times \mathbb{C}\,, \tag{2.3}$$

where $R_\pm(z)$ should be chosen so that $\mathcal{P}(z)$ is invertible. If we put

$$R_+ u \stackrel{\text{def}}{=} u(0)\,,$$

then we can locally solve

$$\begin{cases} (P-z)u = 0 \\ R_+ u = v \end{cases},$$

by putting

$$u = I_+(z)v = \exp(izx/h)v\,, \quad -\epsilon < x < 2\pi - 2\epsilon\,.$$

This is the forward solution, and we can also define the backward one by

$$u = I_-(z)v = \exp(izx/h)v\,, \quad -2\pi + 2\epsilon < x\epsilon\,.$$

The monodromy operator $M(z,h) : \mathbb{C} \to \mathbb{C}$, can be defined by

$$I_+(z)v(\pi) = I_-(z)M(z,h)v(\pi)\,, \tag{2.4}$$

and we immediately see that

$$M(z,h) = \exp\frac{2\pi i z}{h}\,.$$

We use $I_\pm(z)$ and the point π to work with objects defined on $\mathbb{S}^1$ rather than on its cover: a more intuitive definition of $M(z,h)$ can be given by looking at a value of the solution after going around the circle but that has some technical disadvantages.

Let $\chi \in \mathcal{C}^\infty(\mathbb{S}^1, [0,1])$ have the properties

$$\chi(x) \equiv 1\,, \quad -\epsilon < x < \pi + \epsilon\,, \quad \chi(x) \equiv 0\,, \quad -\pi + 2\epsilon < x < -2\epsilon\,,$$

and put

$$E_+(z) = \chi I_+(z) + (1-\chi)I_-(z)\,.$$

We see that

$$(P-z)E_+ = [P,\chi]I_+(z) - [P,\chi]I_-(z) = [P,\chi]_- I_+(z) - [P,\chi]_- I_-(z)\,,$$

where $[P,\chi]_-$ denotes the part of the commutator supported near π. This can be simplified using (2.4):

$$(P-z)E_+ + [P,\chi]_- I_-(z)(I - M(z,h)) = 0\,,$$

which suggests putting

$$R_-(z) = [P,\chi]_- I_-(z)\,,$$

so that the problem

$$\begin{cases} (P-z)u + R_-(z)u_- = 0 \\ R_+(z)u = v \end{cases}$$

has a solution:

$$\begin{cases} u = E_+(z)v \\ u_- = E_{-+}(z)v \end{cases}\,,$$

with $E_{-+}(z) = I - M(z,h)\,.$

One can show[1] that with this choice of $R_\pm(z)$, (2.3) is invertible and then

$$\mathcal{P}(z)^{-1} = \mathcal{E}(z) = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix}\,,$$

where all the entries are holomorphic in z, and $E_+(z)$, $E_{-+}(z)$, are as above. The operator $E_{-+}(z)$ is the effective Hamiltonian in the sense that its invertibility controls the existence of the resolvent:

$$(P-z)^{-1} = E(z) - E_+(z)E_{-+}(z)^{-1}E_-(z)\,. \tag{2.5}$$

Inserting this in (2.1) and using the holomorphy of $E(z)$ gives

$$\begin{aligned} \operatorname{tr} f\left(\frac{P}{h}\right) &= -\frac{1}{2\pi i}\int_\Gamma f\left(\frac{z}{h}\right) \operatorname{tr} E_+(z)E_{-+}(z)^{-1}E_-(z)dz \\ &= -\frac{1}{2\pi i}\int_\Gamma f\left(\frac{z}{h}\right) \operatorname{tr} E_-(z)E_+(z)E_{-+}(z)^{-1}dz\,, \end{aligned}$$

[1] In this situation it is quite easy but it will be done in greater generality in Sect.5.

where we used the cyclicity of the trace. Differentiating $\mathcal{E}(z)\mathcal{P}(z) = Id$ shows

$$E_-(z)E_+(z) = \partial_z E_{-+}(z) + E_-(z)\partial_z R_-(z)E_{-+}(z)\,,$$

which inserted in the previous identity gives

$$\operatorname{tr} f\left(\frac{P}{h}\right) = -\frac{1}{2\pi i}\int_\Gamma f\left(\frac{z}{h}\right)\operatorname{tr}\partial_z E_{-+}(z)E_{-+}(z)^{-1}dz\,,$$

where we eliminated the other term using countour deformation.

We now use the expression for E_{-+} to write

$$\begin{aligned}\operatorname{tr} f\left(\frac{P}{h}\right) =& \frac{1}{2\pi i}\int_{\Gamma_+} f\left(\frac{z}{h}\right)\operatorname{tr}\partial_z M(z,h)(I-M(z,h))^{-1}dz \\ &+ \frac{1}{2\pi i}\int_{\Gamma_-} f\left(\frac{z}{h}\right)\operatorname{tr}\partial_z M(z,h)M(z,h)^{-1}(I-M(z,h)^{-1})^{-1}dz\,.\end{aligned}$$

The assumption (2.2) and the Paley-Wiener theorem give

$$|\hat{f}(z/h)| \leq e^{2\pi N|\operatorname{Im} z|/h}\langle \operatorname{Re} z/h\rangle^{-\infty}\,.$$

Writing

$$(I-M(z,h))^{-1} = \sum_{k=0}^{N-1} M(z,h)^k + M(z,h)^N(I-M(z,h))^{-1}\,,$$

for Γ_+, and

$$M(z,h)^{-1}(I-M(z,h)^{-1})^{-1} = \sum_{k=1}^{N} M(z,h)^{-k} + M(z,h)^{-N-1}(I-M(z,h))^{-1}\,,$$

for Γ_-, we can eliminate the last terms by deforming the contours to imaginary infinities ($R \to \infty$), and this gives (1.1).

In the general situation we proceed similarly but now *microlocally* in a neighbourhood of the closed orbit described in Theorem 1 – see Sect.3 for a precise definition of microlocalization. The formula (2.1) has to be replaced by

$$\operatorname{tr} f\left(\frac{P}{h}\right)\chi(P)A = -\frac{1}{\pi}\int_{\mathbb{C}} f\left(\frac{z}{h}\right)\bar\partial_z\tilde\chi(z)(P-z)^{-1}A\,\mathcal{L}(dz)\,, \tag{2.6}$$

where $\tilde\chi$ is an *almost analytic extension* of χ, that is an extension satisfying $\bar\partial_z\chi(z) = \mathcal{O}(|\operatorname{Im} z|^\infty)$ – see Sect.3, and we want to procceed with a similar reduction to the effective Hamiltonian given in terms of the *monodromy operator*.

To construct the monodromy operator we fix two different points on γ, m_0, m_1 (corresponding to 0 and π in the example), and their disjoint neighbourhoods, W_+ and W_- respectively. We then consider local kernels of $P-z$ near m_0 and m_1 (that is, sets of disctributions satisfying $(P-z)u = 0$ near m_i's), $\ker_{m_j}(P-z)$, $j = 0,1$, with elements microlocally defined in $W_\pm$. and the forward and backward solutions:

$$I_\pm(z)\ :\ \ker_{m_0}(P-z)\ \longrightarrow\ \ker_{m_1}(P-z)\,.$$

We then define the *quantum monodromy operator*, $\mathcal{M}(z)$ by

$$I_-(z)\mathcal{M}(z) = I_+(z)\,,\qquad \mathcal{M}(z)\ :\ \ker_{m_0}(P-z)\ \longrightarrow\ \ker_{m_0}(P-z)\,.$$

The operator P is assumed to be self-adjoint with respect to some inner product $\langle\bullet,\bullet\rangle$, and we define the *quantum flux* norm on $\ker_{m_0}(P-z)$ as follows[2]: let χ be a microlocal cut-off function,

[2]See [6] for an earlier mathematical development of this basic quantum mechanical idea.

with basic properties of the function χ in the example. Roughly speaking χ should supported near γ and be equal to one near the part of γ between W_+ and W_-. We denote by $[P,\chi]_{W_+}$ the part of the commutator supported in W_+, and put

$$\langle u, v\rangle_{\rm QF} \stackrel{\rm def}{=} \langle [(h/i)P,\chi]_{W_+} u, v\rangle\,, \quad u,v\in \ker_{m_0}(P-z)\,.$$

It is easy to check that this norm is independent of the choice of χ – see the proof of Lemma 4.4. This independence leads to the unitarity of $\mathcal{M}(z)$:

$$\langle \mathcal{M}(z)u, \mathcal{M}(z)u\rangle_{\rm QF} = \langle u,u\rangle_{\rm QF}\,, \quad u\in\ker_{m_0}(P-z)\,.$$

For practical reasons we identify $\ker_{m_0}(P-z)$ with $\mathcal{D}'(\mathbb{R}^{n-1})$, microlocally near $(0,0)$, and choose the idenfification so that the corresponding monodromy map is unitary (microlocally near $(0,0)$ where $(0,0)$ corresponds to the closed orbit intersecting a transversal identified with $T^*\mathbb{R}^{n-1}$). This gives

$$M(z,h)\;:\;\mathcal{D}'(\mathbb{R}^{n-1})\;\longrightarrow\;\mathcal{D}'(\mathbb{R}^{n-1})\,,$$

microlocally defined near $(0,0)$ (see Sect.3 for a precise definition of this notion) and unitary there. This is the operator appearing in Theorem 1 and it shares many properties with its simple version $\exp(2\pi iz/h)$ appearing for $\mathbb{S}^1$.

As shown in the example of the Poisson formula, traces can be expressed in terms of traces of effective Hamiltonians ($E_{-+}(z)$ there). Hence in our final formula, we replace $P-z$ by a more general operator $P(z)$, for which we do not demand holomorphy z but only that $P(z)$ is self-adjoint for z real and that it is an almost analytic family of operators. In Theorem 2 in Sect.6 we will compute the trace of

$$-\frac{1}{\pi}\mathrm{tr}\int f(z/h)\,\bar\partial_z\left[\tilde\chi(z)\,\partial_z P(z)\;P(z)^{-1}\right]A\,\mathcal{L}(dz)\,,$$

which for $P(z)=P-z$ reduces to (2.6).

The only prerequisite to reading the paper is the basic calculus of semi-classical pseudodifferential operators (see [3]). In Sect.3 we review various aspects of semi-classical microlocal analysis needed here. In Sect.4 we define the *quantum time* and *quantum monodromy.* Then in Sect.5 we follow the procedure described for $\mathbb{S}^1$ to solve a Grushin problem allowing us to represent $P(z)^{-1}$ near a closed orbit. That is applied in the proof of the trace formula in Sect.6, and in Sect.7 we derive the more standard trace formula in the case of a non-degenerate orbit.

3. Semi-classical operators and their almost analytic extensions

Let X be a compact $\mathcal{C}^\infty$ manifold. We introduce the usual class of semi-classical symbols on X:

$$S^{m,k}(T^*X)=\{a\in\mathcal{C}^\infty(T^*X\times(0,1]) : |\partial_x^\alpha\partial_\xi^\beta a(x,\xi;h)|\leq C_{\alpha,\beta}h^{-m}\langle\xi\rangle^{k-|\beta|}\}\,,$$

and the class corresponding pseudodifferential operators, $\Psi_h^{m,k}(X)$, with the quantization and symbol maps:

$$\begin{aligned}\mathrm{Op}_h^w\;&:\;S^{m,k}(T^*X)\;\longrightarrow\Psi_h^{m,k}(X)\\ \sigma_h\;&:\;\Psi_h^{m,k}(X)\;\longrightarrow S^{m,k}(T^*X)/S^{m-1,k-1}(T^*X)\,,\end{aligned}$$

with both maps surjective, and the usual properties

$$\sigma_h(A\circ B)=\sigma_h(A)\sigma_h(B)\,,$$

$$0\to\Psi^{m-1,k-1}(X)\hookrightarrow\Psi^{m,k}(X)\stackrel{\sigma_h}{\to}S^{m,k}(T^*X)/S^{m-1,k-1}(T^*X)\to 0\,,$$

a short exact sequence, and

$$\sigma_h \circ \mathrm{Op}_h^w : S^{m,k}(T^*X) \longrightarrow S^{m,k}(T^*X)/S^{m-1,k-1}(T^*X)\,,$$

the natural projection map. The class of operators and the quantization map are defined locally using the definition on $\mathbb{R}^n$:

$$\mathrm{Op}_h^w(a)u(x) = \frac{1}{(2\pi h)^n}\int\int a\left(\frac{x+y}{2},\xi\right) e^{i\langle x-y,\xi\rangle/h}u(y)dyd\xi\,, \tag{3.1}$$

and we refer to [3] or [13] for a detailed discussion. We remark only that unlike the invariantly defined symbol map, σ_h, the quantization map Op_h^w can be chosen in many different ways.

In this paper we consider pseudo-differential operators as acting on half-densities and consequently the symbols will also be considered as half-densities – see [8, Sect.18.1] for a general introduction and the Appendix to this paper for a semi-classical discussion. For notational simplicity we supress the half-density notation. The only result we will need here is that in Weyl quantization, the symbol is well defined up to terms of order $\mathcal{O}(h^2)$ – see Appendix.

For $a \in S^{m,k}(T^*X)$ we define

$$\text{ess-supp}_h\, a \subset T^*X \sqcup S^*X\,, \quad S^*X \stackrel{\text{def}}{=} (T^*X\setminus 0)/\mathbb{R}_+\,,$$

where the usual $\mathbb{R}_+$ action is given by multiplication on the fibers: $(x,\xi)\mapsto(x,t\xi)$, as

$$\begin{aligned}&\text{ess-supp}_h\, a = \\ &\quad \complement\{(x,\xi)\in T^*X \;:\; \exists\,\epsilon>0\ \partial_x^\alpha\partial_\xi^\beta a(x',\xi') = \mathcal{O}(h^\infty)\,,\ d(x,x')+|\xi-\xi'|<\epsilon\} \\ &\quad \cup \complement\{(x,\xi)\in T^*X\setminus 0 \;:\; \exists\,\epsilon>0\ \partial_x^\alpha\partial_\xi^\beta a(x',\xi') = \mathcal{O}(h^\infty\langle\xi'\rangle^{-\infty})\,, \\ &\qquad d(x,x')+1/|\xi'|+|\xi/|\xi|-\xi'/|\xi'||<\epsilon\}/\mathbb{R}_+\end{aligned}\;.$$

For $A \in \Psi_h^{m,k}(X)$, then define

$$WF_h(A) = \text{ess-supp}_h\, a\,, \quad A = \mathrm{Op}_h^w(a)\,,$$

noting that, as usual, the definition does not depend on the choice of Op_h^w. For

$$u \in \mathcal{C}^\infty((0,1]_h;\mathcal{D}'(X))\,, \quad \exists\, N_0\,, \quad h^{-N_0}u \ \text{ is bounded in } \mathcal{D}'(X),$$

we define

$$WF_h(u) = \complement\{(x,\xi) \;:\; \exists\, A\in\Psi_h^{0,0}(X)\ \sigma_h(A)(x,\xi)\neq 0\,,\ Au\in h^\infty\mathcal{C}^\infty((0,1]_h;\mathcal{C}^\infty(X))\}\,.$$

When u is not necessarily smooth we can give a definition analogous to that of $\text{ess-supp}_h\, a$. Since in this note we will be concerned with a purely semi-classical theory and deal only with *compact* subsets of T^*X this definition is sufficient for our purposes (for more general definitions of wave front set which include this usual semi-classical definition, see [11]).

To discuss *almost analytic continuation* of semi-classical pseudodifferential operators let us first recall the scalar case. For $f\in\mathcal{C}^\infty(\mathbb{R})$, an almost analytic extension of f is $\tilde f\in\mathcal{C}^\infty(\mathbb{C})$ such that locally uniformly

$$\bar\partial_z \tilde f(z) = \mathcal{O}(|\operatorname{Im} z|^\infty)\,, \quad \tilde f\restriction_{\mathbb{R}} = f\,.$$

The almost analytic extensions were introduced by Hörmander and are unique up to $\mathcal{O}(|\operatorname{Im} z|^\infty)$ terms (see [3, Sect.8] and references given there).

Suppose now that

$$A(x)\in\mathcal{C}^\infty(\mathbb{R}_x;\Psi_h^{m,k}(X))$$

is a smooth family of pseudodifferential operators. We can then find $a(x) \in \mathcal{C}^\infty(\mathbb{R}_x; S^{m,k}(T^*X))$ such that $A(x) = \mathrm{Op}_h^w(a(x))$. We then define the *almost analytic extension of the family* $A(x)$ as

$$\widetilde{A}(z) = \mathrm{Op}_h^w(\tilde{a}(z)) \,,$$

where $\tilde{a}(z) \in \mathcal{C}^\infty(\mathbb{C}_z; S^{m,k}(T^*X))$ is an almost analytic extension of $a(x)$. To justify this definition we need the following easy

Lemma 3.1. *If $a(x) \in \mathcal{C}^\infty(\mathbb{R}_x; S^{m,k}(T^*X))$ then there exists an almost analytic extension of a satisfying*

$$\tilde{a}(z) \in \mathcal{C}^\infty(\mathbb{C}_z; S^{m,k}(T^*X)) \,, \quad \partial_x^\alpha \partial_\xi^\beta \bar{\partial}_z \tilde{a}(z)(x,\xi;h) = \mathcal{O}(|\operatorname{Im} z|^\infty \langle \xi \rangle^{k-|\beta|}) \,.$$

We will also need certain aspects of the theory of semi-classical Fourier Integral Operators. Rather than review the full theory we will consider a special class, to which the general calculus reduces in local situations. Thus let $A(t)$ be a smooth family of pseudodifferential operators, $A(t) = \mathrm{Op}_h^w(a(t))$, $a(t) \in \mathcal{C}^\infty([-1,1]_t; S^{0,-\infty}(T^*X))$, such that for all t, $WF(A(t)) \Subset T^*X$. We then define a family of operators

$$\begin{gathered} U(t) \ : \ L^2(X) \to L^2(X) \,, \\ hD_t U(t) + U(t)A(t) = 0 \,, \quad U(0) = U_0 \in \Psi_h^{0,0}(X) \,. \end{gathered} \tag{3.2}$$

This is an example of a family of h-Fourier Integral Operators, $U(t)$, associated to canonical transformations $\kappa(t)$, generated by the Hamilton vector fields $H_{a_0(t)}$, where the real valued $a_0(t)$ is the h-principal symbol of $A(t)$,

$$\frac{d}{dt}\kappa(t)(x,\xi) = (\kappa(t))_*(H_{a_0(t)}(x,\xi)) \,, \quad \kappa(0)(x,\xi) = (x,\xi) \,, \quad (x,\xi) \in T^*X \,.$$

All that we will need in this note is the *Egorov theorem* which can be proved directly from this definition: when U_0 in (3.2) is elliptic (that is $|\sigma(U_0)| > c > 0$ on T^*X, then for $B \in \Psi_h^{m,k}(X)$

$$\begin{gathered} \sigma(V(t)BU(t)) = (\kappa(t))^*\sigma(B) \,, \\ V(t)U(t) - I \,, \, U(t)V(t) - I \in \Psi_h^{-\infty,-\infty}(T^*X) \end{gathered} \tag{3.3}$$

where the approximate inverse is constructed by taking

$$hD_t V(t) - A(t)V(t) = 0 \,, \quad V(0) = V_0 \,, \quad V_0 U_0 - I \,, \, U_0 V_0 - I \in \Psi_h^{-\infty,-\infty}(T^*X) \,,$$

the existence of V_0 being guaranteed by the ellipticity of U_0. The proof of (3.3) follows from writing $B(t) = V(t)BU(t)$, so that, in view of the properties of $V(t)$,

$$hD_t B(t) \equiv [A(t), B(t)] \mod \Psi_h^{-\infty,-\infty} \,, \quad B(0) = B_0 \,.$$

Since the symbol of the commutator is given by $(h/i)H_{a_0(t)}\sigma(B(t))$, (3.3) follows directly from the definition of $\kappa(t)$.

If $U = U(1)$, say, and the graph of $\kappa(1)$ is denoted by C, we conform to the usual notation and write

$$U \in I_h^0(X \times X; C') \,, \quad C' = \{(x,\xi;y,-\eta) \ : \ (x,\xi) = \kappa(y,\eta)\} \,,$$

which means that U is an h-Fourier Integral Operator associated to the canonical graphs C. Locally all h-Fourier Integral Operators associated to canonical graphs are of the form $U(1)$ thanks to the following well known

Lemma 3.2. *Suppose that U_1, U_2 are open neighbourhoods of $(0,0) \in T^*\mathbb{R}^n$, and $\kappa : U_1 \to U_2$ is a canonical transformation satisfying $\kappa(0,0) = (0,0)$. Then there exists a smooth family of canonical transformations $\kappa_t : U_1 \to U_2$, $0 \le t \le 1$, satisfying $\kappa_0 = id$, $\kappa_1 = \kappa$, $\kappa_t(0,0) = (0,0)$.*

Proof. Since the symplectic group, $Sp(n,\mathbb{R})$, is connected we can first deform κ so that $d\kappa(0,0) = Id$. Hence, near $(0,0)$, $((x(\kappa(y,\eta)),\xi(\kappa(y,\eta));y,\eta) \mapsto (x,\eta)$ is surjective, and on the graph of κ, y and η can be regarded as functions of x and η. Since the symplectic forms, $-d(\langle y, d\eta\rangle)$ $d(\langle \xi, dx\rangle)$ are equal, their difference can be written locally as a differential:

$$\langle y, d\eta\rangle + \langle \xi, dx\rangle = d\phi\,, \quad \phi = \phi(x,\eta)\,, \quad d\phi(0,0) = 0\,,$$

so that $\kappa \;:\; (\phi'_\eta(x,\eta),\eta) \mapsto (x, \phi'_x(x,\eta))$. We could now take as our family

$$\kappa_t \;:\; (t\phi'_\eta(x,\eta) + (1-t)x, \eta) \mapsto (x, t\phi'_x(x,\eta) + (1-t)\eta)\,.$$

The two steps can be connected smoothly by making the deformations flat at their junction. $\Box$

The almost analytic continuation of a family of h-Fourier Integral Operators defined by (3.2) is obtained by means of the following

Lemma 3.3. *Suppose that $U(t)$ is defined by* (3.2) *and that $\widetilde{A}(z)$ is an almost analytic continuation of the family $A(t)$, as given by Lemma 3.1. Let $\widetilde{U}(z) = \widetilde{U}(t+is)$ be the solution of*

$$\frac{1}{i} hD_s \widetilde{U}(t+is) + \widetilde{U}(t+is)\widetilde{A}(t+is) = 0\,, \quad \widetilde{U}(t+is)\restriction_{s=0} = U(t)\,. \tag{3.4}$$

Then for $|\operatorname{Im} z| \leq h \log h^{-L}$ we have

$$\|\widetilde{U}(z)\|_{L^2\to L^2} \leq C\exp(C|\operatorname{Im} z|/h) \tag{3.5}$$

$$\|\bar{\partial}_z \widetilde{U}(z)\|_{L^2\to L^2} = \mathcal{O}(|\operatorname{Im} z|^\infty) \tag{3.6}$$

$$hD_z\widetilde{U}(z) = \widetilde{A}(z)\widetilde{U}(z) + \mathcal{O}_{L^2\to L^2}(|\operatorname{Im} z|^\infty)\,. \tag{3.7}$$

Proof. To see (3.5) we write

$$h\frac{d}{ds}\|\widetilde{U}(t+is)v\|^2 = 2\operatorname{Re}\langle \widetilde{U}(t+is)\widetilde{A}(t+is)v, \widetilde{U}(t+is)v\rangle) \leq C\|\widetilde{U}(t+is)\|^2\|v\|^2\,, \quad s>0\,.$$

Let us now take v with $\|v\| = 1$ so that, by integration,

$$\|\widetilde{U}(t+is)v\|^2 \leq \|\widetilde{U}(t)\|^2 + \frac{C}{h}\int_0^s \|\widetilde{U}(t+i\sigma)\|^2 d\sigma\,.$$

Since this holds for every v with $\|v\| = 1$ we can replace the left hand side of the inequality by $\|\widetilde{U}(t+is)\|^2$, and the standard Gronwall inequality argument shows that

$$\|\widetilde{U}(t+is)\|^2 \leq Ce^{Cs/h}\,,$$

which is the desired bound. Putting $\widetilde{V}(t+is) = \bar{\partial}_z\widetilde{U}(t+is)$ we have

$$h\partial_s\widetilde{V}(t+is) = \widetilde{U}(t+is)\bar{\partial}_z\widetilde{A}(t+is) + \widetilde{V}(t+is)\widetilde{A}(t+is) = \widetilde{V}(t+is)\widetilde{A}(t+is) + \mathcal{O}(s^\infty)\,,$$
$$|s| < h\log h^{-L}\,,\ \widetilde{V}(t+is)\restriction_{s=0} = 0\,,$$

where the initial condition came from the equation on the real axis: $hD_t\widetilde{U}(t+is)\restriction_{s=0} = U(t)A(t)$. As in the argument for (3.5), this implies (3.6) and (3.7). $\Box$

Our definitions of pseudo-differential operators and of (the special class of) h-Fourier Integral Operators were global. It is useful and natural to consider the operators and their properties microlocally. We consider classes of *tempered* operators:

$$T \;:\; \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)\,,$$

and for any semi-norms $\| \bullet \|_1$ and $\| \bullet \|_2$ on $\mathcal{C}^\infty(X)$ there exists M_0 such that

$$\|Tu\|_1 = \mathcal{O}(h^{-M_0})\|u\|_2 .$$

For open sets, $V \subset T^*X$, $U \subset T^*X$, the operators *defined microlocally* near $V \times U$ are given by equivalence classes of tempered operators given by the relation

$$T \sim T' \iff A(T - T')B = \mathcal{O}(h^\infty) \; : \; \mathcal{D}'(X) \longrightarrow \mathcal{C}^\infty(X) ,$$

for any $A, B \in \Psi_h^{0,0}(X)$ such that

$$\begin{gathered} WF(A) \subset \widetilde{V} , \quad WF(B) \subset \widetilde{U} , \\ \bar{V} \Subset \widetilde{V} \Subset T^*X , \quad \bar{U} \Subset \widetilde{U} \Subset T^*X , \quad \widetilde{U} , \widetilde{V} \quad \text{open} . \end{gathered} \tag{3.8}$$

The equivalence class T, h-Fourier Integral Operator associated to a local canonical graph C if, again for any A and B above

$$ATB \in I^0(X \times X; \widetilde{C}') ,$$

where C needs to be defined only near $U \times V$.

We say that $P = Q$ *microlocally* near $U \times V$ if $APB - AQB = \mathcal{O}_{L^2 \to L^2}(h^\infty)$, where because of the assumed pre-compactness of U and V the L^2 norms can be replaced by any other norms. For operator identities this will be the meaning of equality of operators in this paper, with U, V specified (or clear from the context). Similarly, we say that $B = T^{-1}$ microlocally near $V \times V$, if $BT = I$ microlocally near $U \times U$, and $TB = I$ microlocally near $V \times U$. More generally, we could say that $P = Q$ microlocally on $W \subset T^*X \times T^*X$ (or, say, P is microlocally defined there), if for any U, V, $U \times V \subset W$, $P = Q$ microlocally in $U \times V$. We should stress that "microlocally" is always meant in this semi-classical sense in our paper.

In this terminology we have a characterization of local h-Fourier Integral operators, which is essentially the converse of Egorov's theorem:

Lemma 3.4. *Suppose that $U = \mathcal{O}(1) : L^2(X) \to L^2(X)$, and that for every $A \in \Psi_h^{0,0}(X)$ we have*

$$AU = UB , \quad B \in \Psi_h^{0,0}(X) , \quad \sigma(B) = \kappa^*\sigma(A) ,$$

microlocally near (m_0, m_0) where $\kappa : T^*X \to T^*X$ is a symplectomorphism, defined locally near m_0, $\kappa(m_0) = m_0$. Then

$$\begin{gathered} U \in I_h^0(X \times X; C') , \quad \text{microlocally near } (m_0, m_0), \\ C' = \{(x, \xi; y, -\eta) \; : \; (y, \eta) = \kappa(x, \xi)\} . \end{gathered}$$

Proof. From Lemma 3.2 we know that there exists a family of local symplectomorphisms, κ_t, satisfying $\kappa_t(m_0) = m_0$, and $\kappa_1 = \kappa$, $\kappa_0 = id$. Since we are working locally, there exists a function $a(t)$, such that κ_t is generated by its Hamilton vectorfield $H_{a(t)}$. Let us now consider

$$hD_t U(t) = U(t)A(t) , \quad U(1) = U , \quad 0 \leq t \leq 1 .$$

The same arguments as the one used in the proof of (3.3) shows that $U(0)$ satisfies

$$[U(0), A] = \mathcal{O}(h) , \quad \text{for any } A \in \Psi_h^{0,0}(X). \tag{3.9}$$

In fact, we take $V(t)$ with $V(0) = Id$ microlocally near (m_0, m_0), so that

$$AU(t)V(t) = U(t)V(t)(V(t)^{-1}BV(t)) = U(t)V(t)A + \mathcal{O}(h) ,$$

where we used Egorov's theorem and the assumption that $\sigma(B) = \kappa^*\sigma(A)$. Putting $t = 0$ gives (3.9). By Beals's Lemma [3, Prop.8.3] we conclude that $U(0) \in \Psi_h^{0,0}(X)$, and hence U is a microlocally defined h-Fourier Integral Operator associated to κ. $\square$

If the open sets U or V in (3.8) are small enough, so that they can be identified with neighbourhoods of points in $T^*\mathbb{R}^n$, we can use that identification to state that T is microlocally defined near, say, $(m,(0,0))$, $m \in T^*X$, $(0,0) \in T^*\mathbb{R}^n$. An example useful here is given in the next proposition.

By Darboux's theorem we know that if p is a function with a non-vanishing differential then there exists a local canonical transfomation κ such that $\kappa^* p = \xi_1$ where ξ_1 is part of a coordinate system in which the symplectic form is the canonical one $d(\langle \xi, dx\rangle)$. The quantum version is given in

Proposition 3.5. *Suppose that* $P \in \Psi_h^{0,k}(X)$ *is a semi-classical* real principal type operator*:* $p = \sigma(P)$ *is real, independent of* h*, and*

$$p = 0 \Longrightarrow dp \neq 0 \,.$$

For any $m_0 \in \Sigma_p \stackrel{\text{def}}{=} \{m \in T^*X \;:\; p(m) = 0\}$ *there exists a local canonical transformation* $\kappa : T^*X \to T^*\mathbb{R}^n$ *defined near* $((0,0), m_0)$*, and an* h*-Fourier Integral Operator,* T*, associated to its graph, such that*

$$\kappa^* \xi_1 = p \,,$$
$$TP = hD_{x_1} T \,, \quad \textit{microlocally near } ((0,0), m_0)$$
$$T^{-1} \quad \textit{exists microlocally near } (m_0, (0,0)) \,.$$

For the reader's convenience we outline a self-contained proof of this semi-classical analogue of the standard $\mathcal{C}^\infty$ result [8, Proposition 26.1.3′].

Proof. By assumption $dp(m_0) \neq 0$, and consequently Darboux's theorem gives κ with the desired properties. Lemma 3.2 then gives us a family of symplectic transfomations κ_t. If $T_0 = U(1)$, where $U(1)$ was defined using the family κ_t, then (3.3) shows that $T_0 P - hD_{x_1} = E \in \Psi^{-1,0}$ microlocally near $(0,0)$. Hence we look for A such that $hD_{x_1} + E = A hD_{x_1} A^{-1}$, microlocally near $(0,0)$. That is the same as solving

$$[hD_{x_1}, A] + EA = 0 \,,$$

Since the principal symbol of P is independent of h, same is true for the principal symbol of E, e. Hence we can find $a \in S^{0,0}(T^*\mathbb{R}^n)$, independent of h, $a(0,0) \neq 0$, and such that

$$\frac{1}{i}\{\xi_1, a\} + ea = 0$$

near $(0,0)$. Choosing A_0 with the principal symbol a we can now find $A_j \in \Psi_h^{-j,0}(T^*\mathbb{R}^n)$ so that

$$[hD_{x_1}, A_0 + A_1 + \cdots A_N] + E(A_0 + A_1 + \cdots A_N) \in \Psi_h^{-N,0}(T^*\mathbb{R}^n) \,.$$

We then put $A \sim A_1 + A_2 + \cdots + A_N + \cdots$ which is elliptic near $(0,0)$, and finally $T = A^{-1} T_0$. □

Using the proposition we can transplant objects related to P to the much easier to study objects related to hD_{x_1}. In particular, we can microlocally define

$$\ker_{m_0}(P) \stackrel{\text{def}}{=} T^{-1}(\ker\,(hD_{x_1}))\,, \quad \ker\,(hD_{x_1})) = \{u \in \mathcal{D}'(\mathbb{R}^n) \;:\; hD_{x_1} u = 0\} \,. \tag{3.10}$$

Since $\ker\,(hD_{x_1}))$ can be identified with $\mathcal{D}'(\mathbb{R}^{n-1})$ we can also identify $\ker_{m_0}(P)$ with $\mathcal{D}'(\mathbb{R}^{n-1})$, microlocally near $(m_0, (0,0))$:

$$K \;:\; \mathcal{D}'(\mathbb{R}^{n-1}) \;\longrightarrow \ker_{m_0}(P)\,, \quad K = T^{-1}\pi^* \,, \quad \pi \;:\; x \mapsto (x_2, \cdots x_n) \,. \tag{3.11}$$

4. Quantum time and quantum monodromy

Let $P(z) \in \mathcal{C}^\infty(I_z; \Psi_h^{0,m}(X))$, $I = (-a, a) \subset \mathbb{R}$, be a smooth family of real principal type operators, with principal symbols, $p(z)$, independent of h. We will assume that

$$\Sigma_{p(z)} \stackrel{\text{def}}{=} \{m \in T^*X \; : \; p(z)(m) = 0\} \Subset T^*X \, , \quad \text{for } z \in I$$
$$P(z) \text{ is formally self-adjoint for } z \in I.$$

We assume that $m_0(z)$ is a smooth family of periodic points of $H_{p(z)}$, with the minimal periods $T(z)$ also smooth in z, and the orbits $\gamma(z)$:

$$\exp(T(z)H_{p(z)})(m_0(z)) = m_0(z) \, , \quad \gamma \stackrel{\text{def}}{=} \{\exp(tH_{p(z)})(m_0(z)) \; : \; 0 \le t \le T(z)\} \, .$$

When no confusion is likely to arise we may drop the dependence on z in the notation.

Let Ω be a neighbourhood of $\gamma(0)$ in T^*X,

$$\Omega \simeq \gamma(0) \times \mathbb{B}_{\mathbb{R}^{2n-1}}(0, \epsilon) \, ,$$

and we assume that for $z \in I$, the orbits $\gamma(z)$ are also contained in Ω. We now introduce a covering space of this tubular neighbourhood

$$\widetilde{\Omega} \simeq \mathbb{R} \times \mathbb{B}_{\mathbb{R}^{2n-1}}(0, \epsilon) \, , \quad \pi : \widetilde{\Omega} \longrightarrow \Omega \, ,$$

with the lift of $p(z)$ denoted by $\widetilde{p}(z)$, and we will use the same notation for other objects.

We start with the following

Lemma 4.1. *The tubular neighbourhood, Ω, of $\gamma(0)$, can be chosen small enough, so that the cover $\widetilde{\Omega}$ contains no closed orbits of $H_{\widetilde{p}(z)}$, $z \in [-\delta, \delta] \subset I$, for some small $\delta > 0$.*

Proof. Let $m \mapsto \tilde{t}(m)$ be a smooth function on $\widetilde{\Omega}$ with the property that $\tilde{t}(\exp(tH_{\widetilde{p}(0)})) = t$, and that $d\tilde{=}\pi^* d\hat{t}$, where $d\hat{t}$ is a well defined one form in Ω. Then $H_{\widetilde{p}(0)}\tilde{t} > 0$ on the lift of γ, and by shrinking $\widetilde{\Omega}$ if necessary we conclude that $H_{\widetilde{p}(0)}\tilde{t} > 0$ on $\widetilde{\Omega}$. By the periodicity and and a compactness argument we conclude that this holds for 0 replaced by $z \in [-\delta, \delta]$. Hence there are no closed orbits of $H_{\widetilde{p}(z)}$ in $\widetilde{\Omega}$. $\square$

We will now replace $\widetilde{\Omega}$ by a finite part: $\widetilde{\Omega} \simeq [-L, L] \times \mathbb{B}_{\mathbb{R}^{2n-1}}(0, \epsilon)$, $L \gg T$.

A *classical time function*, $\tilde{q}(z) \in \mathcal{C}^\infty(\widetilde{\Omega}; \mathbb{R})$, on $\widetilde{\Omega}$ is defined as a solution of

$$\partial_z \tilde{p}(z) = -\{\tilde{p}(z), \tilde{q}(z)\} \, . \tag{4.1}$$

In view of Lemma 4.1 this equation can be solved (strictly speaking that may involve shrinking $\widetilde{\Omega}$ further depending on the initial data, but for simplicity of exposition we will ignore this point), and we can in particular consider solutions satisfying $\widetilde{q}(\tilde{m}_0(z)) = 0$. In a neighbourhood of $m_0 = m_0(0) \in \Omega$ we can define $q(z) \in \mathcal{C}^\infty$ such that

$$\tilde{q}(z) = \pi^* q(z) \, , \quad \text{near } \tilde{m}_0, \quad q(z)(m_0(z)) = 0 \quad \pi : \widetilde{\Omega} \to \Omega \, .$$

We clearly have $\partial_z p = \{p, q\}$ near m_0. This defines the *local classical time* near m_0. We also define the *first return classical time* near m_0 by demanding that

$$\tilde{q}(z) = \pi^* q_{\circlearrowright}(z) \, , \quad \text{near } \exp T(0) H_{\tilde{p}(0)} \tilde{m}_0, \;\; q_{\circlearrowright}(z)(m_0(z)) = \tilde{q}(\exp T(z) H_{\tilde{p}(z)}(\tilde{m}_0(z)) \, .$$

An iteration procedure similar to the one recalled in the proof of Proposition 3.5 gives the quantum analogues microlocally defined near m_0:

$$\partial_z P(z) = -\frac{i}{h}[P(z), Q(z)], \quad \sigma(Q(z)) = q(z), \tag{4.2}$$

$$\partial_z P(z) = -\frac{i}{h}[P(z), Q_{\circlearrowleft}(z)], \quad \sigma(Q_{\circlearrowleft}(z)) = q_{\circlearrowleft}(z), \tag{4.3}$$

Replacing $Q(z)$ by $(Q(z)+Q(z)^*)/2$, we can assume that $Q(z)$ is formally self-adjoint. We clearly have

$$Q_{\circlearrowleft}(z) - Q(z) \;:\; \ker_{m_0} P(z) \longrightarrow \ker_{m_0} P(z).$$

Then $Q(z)$ is the *quantum time* near m_0, and $Q_{\circlearrowleft}$ is the *first return quantum time* near m_0. See the proof of Lemma 7.4 for further discussion of these objects in the classical context.

For (z, w) near $(0, 0)$, and microlocally near m_0, we can solve the following system of equations

$$\begin{aligned} (hD_z - Q(z)_L)U(z,w) &\stackrel{\text{def}}{=} hD_z U(z,w) - Q(z)U(z,w) = 0 \\ (hD_w + Q(w)_R)U(z,w) &\stackrel{\text{def}}{=} hD_z U(z,w) + U(z,w)Q(w) = 0 \end{aligned} \tag{4.4}$$

with the initial condition $U(0,0) = Id$, and with $U(z,w)$ bounded on L^2 (microlocally near (m_0, m_0)): the solvability of the system follows from the fact that

$$[hD_z - Q(z)_L, hD_w + Q(w)_R] = 0.$$

We easily check that (as always, microlocally)

$$U(z,z) = Id, \quad U(z,w)U(w,v) = U(z,v), \tag{4.5}$$

and that $U(z,w)$ is unitary. In fact,

$$hD_z(U(z,z)) = Q(z)U(z,z) - U(z,z)Q(z) = -[U(z,z), Q(z)], \quad U(0,0) = Id,$$

and $U(z,z) = Id$ is the unique solution. The other property is derived similarly:

$$\begin{gathered} hD_w(U(z,w)U(w,v)) = -U(z,w)Q(w)U(w,v) + U(z,w)Q(w)U(w,v) = 0, \\ U(z,w)U(w,v)\restriction_{w=z} = U(z,v). \end{gathered}$$

By varying m_0 along the orbit of $H_{p(0)}$, and by extending $Q(z)$ maximally forward $(+)$ and backward $(-)$, we can define semi-global versions of $U(z,w)$:

$$\begin{aligned} &U_+(z,w) \text{ microlocally on a neighbourhood of the diagonal over} \\ &\qquad \{\exp tH_p(m_0) \;:\; -\epsilon < t < T(0) - 2\epsilon\}, \\ &U_-(z,w) \text{ microlocally on a neighbourhood of the diagonal over} \\ &\qquad \{\exp tH_p(m_0) \;:\; -\epsilon < -t < T(0) - 2\epsilon\}. \end{aligned}$$

The operators have the following intertwining property:

Proposition 4.2. *Microlocally near the diagonal over*

$$\{\exp tH_p(m_0) \;:\; -\epsilon < \pm t < T(0) - 2\epsilon\},$$

and for z, w close to 0, we have

$$P(z)U_\pm(z,w) = U_\pm(z,w)P(w).$$

Proof. We define $P^\sharp(w) = U(w,z)P(z)U(z,w)$ and differentiate with respect to w:

$$hD_w P^\sharp(w) = Q(w)U(w,z)P(z)U(z,w) - U(w,z)P(z)U(z,w)Q(w) = -[P^\sharp(w), Q(w)]\,,$$
$$P^\sharp(w)\restriction_{w=z} = P(z)\,,$$

that is, $P^\sharp(w)$ satisfies (4.2) and consequently $P^\sharp(w) = P(w)$. $\square$

By replacing the local quantum time, $Q(z)$, by the first return quantum time, $Q_\circlearrowright(z)$ (see (4.2), (4.3)), we also define $U_\circlearrowright(z,w)$,

$$U_\circlearrowright(z,z) = Id\,, \quad U_\circlearrowright(z,w)U_\circlearrowright(w,v) = U_\circlearrowright(z,v) \ \text{ microlocally near } m_0.$$

This definition will be useful when we study the *quantum monodromy operator*. To introduce it, we first define the forward and backward propagators:

$$\begin{gathered} I_\pm(z) \,:\, \ker_{m_0(z)}(P(z)) \longrightarrow \mathcal{D}'(X) \\ I_\pm(z) = Id\restriction_{\ker_{m_0(z)}(P(z))}\,, \quad \text{microlocally near } m_0(z), \\ P(z)I_\pm(z) = 0\,, \quad \text{microlocally near } \{\exp(tH_{p(z)})m_0(z) \,:\, -\epsilon < \pm t < T(z) - 2\epsilon\}. \end{gathered} \tag{4.6}$$

That the operators $I_\pm(z)$ are microlocally well defined follows from Proposition 3.5, and "microlocally" is meant via the identification of $\ker_{m_0(z)}(P(z))$ with $\mathcal{D}'(\mathbb{R}^{n-1})$ as in (3.11). We fix $m_0 = m_0(0)$ as in the definition of $\widetilde{\Omega}$ above, and define

$$\begin{aligned} W_+ = \ & \text{a neighbourhood of } m_0 \text{ in } T^*X, \\ W_- = \ & \text{a neighbourhood of } \exp((T(0)/2)H_{p(0)})m_0 \text{ in } T^*X, \\ W_- \subset & \bigcup_{|t+T(0)/2|<\epsilon} \exp tH_{p(0)}(W_+)\,, \end{aligned} \tag{4.7}$$

noting that for z small enough, we can replace m_0, $T(0)$, $p(0)$, by $m_0(z)$, $T(z)$, $p(z)$ in this definition. This shows that $I_-(z)$ maps $\ker_{m_0(z)}(P(z))$ onto $\ker_{\exp((T(z)/2)H_{p(z)})(m_0(z))}(P(z))$, microlocally near $W_- \times W_+$. This means that the left microlocal inverse exists and we can give the following

Definition. *The (absolute) quantum monodromy operator*

$$\mathcal{M}(z) \,:\, \ker_{m_0(z)}(P(z)) \longrightarrow \ker_{m_0(z)}(P(z))\,,$$

is microlocally defined near W_+ *by*

$$I_+(z)f = I_-(z)\mathcal{M}(z)f\,, \quad f \in \ker_{m_0(z)}(P(z))\,, \quad \textit{microlocally near } W_-. \tag{4.8}$$

The quantum monodromy operator,

$$M(z) \,:\, \mathcal{D}'(\mathbb{R}^{n-1}) \to \mathcal{D}'(\mathbb{R}^{n-1})\,,$$

is microlocally defined near $(0,0) \in T^*\mathbb{R}^{n-1}$*, by*

$$M(z) = K(z)^{-1}\mathcal{M}(z)K(z)\,, \tag{4.9}$$

where $K(z)$ *is as in* (3.11).

The basic properties are given in

Proposition 4.3. *Let $U(z,w)$ and $U_{\circlearrowleft}(z,w)$ be given by (4.2), (4.3), and (4.4). Then the following diagram commutes (microlocally near m_0):*

$$
\begin{array}{ccc}
\ker_{m_0(w)}(P(w)) & \xrightarrow{\mathcal{M}(w)} & \ker_{m_0(w)}(P(w)) \\
\Big\downarrow U(z,w) & & \Big\downarrow U_{\circlearrowleft}(z,w) \\
\ker_{m_0(z)}(P(z)) & \xrightarrow{\mathcal{M}(z)} & \ker_{m_0(z)}(P(z))
\end{array}
\tag{4.10}
$$

Choosing $K(z)$, so that $K(z)=U(z,w)K(w)$, we also have

$$
hD_z M(z) = [K(z)^{-1}(Q_{\circlearrowleft}(z)-Q(z))K(z)]M(z)\,,
\tag{4.11}
$$

where we recall that by (4.2) and (4.3), $Q_{\circlearrowleft}(z)-Q(z) \;:\; \ker_{\tilde m_0(z)}(\widetilde P(z)) \to \ker_{\tilde m_0(z)}(\widetilde P(z))$, and hence $K(z)^{-1}$ is well defined.

Proof. We need to show that $U_{\circlearrowleft}(z,w)\mathcal{M}(w)=\mathcal{M}(z)U(z,w)$, and since $U_{\circlearrowleft}$ is naturally defined using the covering space, we will translate this into a statement there. We can microlocally define $\widetilde P(w)$ on $\widetilde\Omega$ and then,

$$
\widetilde I_+(w) \;:\; \ker_{\tilde m_0(w)}(\widetilde P(w)) \longrightarrow \ker(\widetilde P(w))\,,
$$

and we define

$$
\widetilde{\mathcal{M}}(w) \;:\; \ker_{\tilde m_0(w)}(\widetilde P(w)) \longrightarrow \ker_{\exp(T(w)H_{\tilde p(w)})(\tilde m_0(w))}(\widetilde P(w))\,,
$$

by restricting $\widetilde I_+(w)$ to a neighbourhood of $\exp(T(w)H_{\tilde p(w)})(\tilde m_0(w))$. Since for $\pi : \widetilde\Omega \to \Omega$, we microlocally have

$$
\pi_* \;:\; \ker_{\exp(T(w)H_{\tilde p(w)})(\tilde m_0(w))}(\widetilde P(w)) \longrightarrow \ker_{m_0(w)} P(w)\,,
$$
$$
\pi_* \widetilde{\mathcal{M}}(w)\pi^* = \mathcal{M}(w)\,.
$$

Using the quantized version of $\tilde q$ in (4.1), we also define $\widetilde U(z,w)$, so that $\widetilde U(z,w)\widetilde P(w) = \widetilde P(z)\widetilde U(z,w)$. In particular we have

$$
\widetilde U(z,w)\widetilde I_+(w) = \widetilde I_+(z)\widetilde U(z,w)\,.
$$

Restricting (microlocally) to a neighbourhood of

$$
(\exp(T(z)H_{\tilde p(z)})(\tilde m_0(z)), \tilde m_0(z)) \in \widetilde\Omega \times \widetilde\Omega\,,
$$

and projecting to $\Omega\times\Omega$, we obtain

$$
U_{\circlearrowleft}(z,w)\mathcal{M}(w) = \mathcal{M}(z)U(z,w)\,.
$$

To see (4.11) we first note that differentiation of $K(z)=U(z,w)K(w)$ and the definition of $U(z,w)$ gives

$$
hD_z K(z) = Q(z)K(z)\,.
$$

We then use the commutative diagram to see that

$$
K(z)M(z) = U_{\circlearrowleft}(z,w)\mathcal{M}(w)U(w,z)K(z)\,.
$$

Differentiating this with respect to z and using the previous equation gives

$$
K(z)hD_z M(z) = (Q_{\circlearrowleft}(z)-Q(z))K(z)M(z)\,.
$$

We then recall that by (4.2) and (4.3), $Q(z)-Q_{\circlearrowleft}(z) \;:\; \ker_{\tilde m_0(z)}(\widetilde P(z)) \to \ker_{\tilde m_0(z)}(\widetilde P(z))$, and hence $K(z)^{-1}$ can be applied to both sides. □

We can define the *Poincaré map* for γ with primitive period T:

$$C : T^*\mathbb{R}^{n-1} \longrightarrow T^*\mathbb{R}^{n-1}\,, \quad \text{defined near } (0,0)\,, \quad C(0,0) = (0,0)\,,$$

as follows: for a neighborhood of $m_0 \in \gamma$, U_0, $U_0/\exp(tH_p)$ can be identified with a neighbourhood of $(0,0) \in T^*\mathbb{R}^{n-1}$ (using the local identification of p with ξ_1, as in the proof of Proposition 3.5), with $[m_0]$ corresponding to $(0,0)$. The Poincaré map is then given by

$$\begin{gathered} C : \kappa^{-1}([m]) \longmapsto \kappa^{-1}([\exp(TH_p)m])\,, \\ [m] \in U_0/\exp(tH_p)\,, \kappa : T^*\mathbb{R}^{n-1} \to U_0/\exp(tH_p)\,. \end{gathered} \tag{4.12}$$

It will always be undestood that κ chosen here is the symplectic transfomation corresponding to $K = K(z)$ in (3.11) and (4.9).

To study quantum properties of the monodromy operator it is convenient to introduce $\chi \in \mathcal{C}_{\rm c}^\infty(T^*X)$ satisfying

$$\begin{gathered} \chi \equiv \begin{cases} 1 & \text{near } \{\exp(tH_{p(0)}(m_0)) : \epsilon < t < T(0)/2 - \epsilon\} \\ 0 & \text{near } \{\exp(tH_{p(0)}(m_0)) : \epsilon < -t < T(0)/2 - \epsilon\} \end{cases} \\ \Omega \cap \{m \ : \ \chi(m) \neq 1\} \cap \{m \ : \ \chi(m) \neq 0\} \subset W_+ \cup W_-\,, \end{gathered} \tag{4.13}$$

where $W_\pm$ are as in (4.7), and Ω is a small neighbourhood of γ. If $\rho_\pm \equiv 1$ microlocally near $W_\pm$, and $\rho_\pm \equiv 0$ near $W_\mp$, we define

$$[P, \chi]_{W_\pm} = \rho_\pm [P, \chi]\,,$$

where we use the same notation for χ and $\mathrm{Op}_h(\chi)$. We then have the basic property of the *quantum flux* (see [6]):

Lemma 4.4. *Let $K(z)$ be in* (3.11)*. Then*

$$U(z) \stackrel{\text{def}}{=} K(z)^*[(i/h)P(z), \chi]_{W_+} K(z) \ : \ \mathcal{D}'(\mathbb{R}^{n-1}) \ \longrightarrow \ \mathcal{D}'(\mathbb{R}^{n-1})$$

is microlocally positive near $(0,0) \in T^*\mathbb{R}^{n-1}$ and independent of χ with the properties (4.13)*.*

If we replace $K(z)$ by $K(z)U(z)^{-\frac{1}{2}}$ then

$$K(z)^*[(i/h)P(z), \chi]_{W_+} K(z) = Id \quad \textit{microlocally near } (0,0) \in T^*\mathbb{R}^{n-1}. \tag{4.14}$$

Proof. We note that if $P(z)u = 0$ near W_+, and $\tilde\chi$ is another function satisfying (4.13), then

$$K(z)^*[(i/h)P(z), \chi - \tilde\chi]u = K(z)^* P(z)(\chi - \tilde\chi)u - K(z)^*(\chi - \tilde\chi)P(z)u = 0\,,$$

since $P(z)u = 0$, and $K(z)^* P(z) = (P(z)K(z))^* = 0$. The positivity also comes from expanding the commutator and using Proposition 3.5:

$$\langle \tilde K(z)^*[(i/h)hD_{x_1}, \chi]_{W_+} \tilde K(z)u, u\rangle = \langle \partial_{x_1}\chi\rho_+ \tilde K(z)u, \tilde K(z)u\rangle \geq \langle \tilde\rho u, \tilde\rho u\rangle\,,$$

where $\tilde K(z)$ is the composition of $K(z)$ and T of Proposition 3.5, and $\tilde\rho \equiv 1$ in a neighbourhood of $(0,0) \in T^*\mathbb{R}^{n-1}$ (we again use the same notation for the function and its quantization). $\Box$

From now on, our choice of $K(z)$ in (3.11) is made so that (4.14) holds. We only need to check that we still have

$$K(z) = U(z, w)K(w)\,.$$

In fact, we have in general, in the microlocal sense,

$$\begin{aligned} & K(z)^*[(i/h)P(z), \chi]_{W_+} K(z) = \\ & K(w)^* U(w, z)[(i/h)P(z), \chi]_{W_+} U(z, w)K(w) = \\ & K(w)^*[(i/h)P(w), \tilde\chi]_{W_+} K(w)\,, \end{aligned}$$

and the last expression is unchanged if we replace $\tilde{\chi}$ by χ (the quantum flux property used before). We also used the unitarity of $U(z,w)$.

With this choice of $K(z)$ we have the following important and well known

Proposition 4.5. *The monodromy operator, $M(z)$, defined by* (4.9) *with $K(z)$ satisfying* (4.14) *is microlocally unitary:*

$$M(z)^* = M(z)^{-1} \quad \textit{microlocally near } (0,0) \in T^*\mathbb{R}^{n-1}\,,$$

and it is an h-Fourier Integral Operator:

$$M(z) \in I^0(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; C(z)')\,,$$

where $C(z)$ is the Poincaré map (4.12).

Proof. We need to show that for $v \in \mathcal{D}'(\mathbb{R}^{n-1})$ with $WF_h(v)$ in a neighbourhood of $(0,0)$, we have

$$\langle M(z)v, M(z)v\rangle = \langle v, v\rangle + \mathcal{O}(h^\infty)\|v\|^2\,. \tag{4.15}$$

If we put $u = K(z)v$, use (4.14), and the defintion of $M(z)$, (4.9), then the left hand side of (4.15) becomes:

$$\langle K(z)^*[P(z),\chi]_{W_+}\mathcal{M}(z)u, K(z)^{-1}\mathcal{M}(z)u\rangle = \langle [P(z),\chi]_{W_+}\mathcal{M}(z)u, \mathcal{M}(z)u\rangle,\,.$$

As in the proof of Lemma 4.4, we see that for $0 < t < T(0)/2 + \epsilon$, the right hand side of the previous expression is equal to (modulo $\mathcal{O}(h^\infty)$)

$$\langle [(i/h)P(z), (\exp(tH_{p(0)})^*\chi]_{\exp(-tH_{p(0)})W_+} I_-(z)\mathcal{M}(z)u, I_-(z)\mathcal{M}(z)u\rangle\,,$$

which corresponds to moving the support of χ in the direction opposite to the flow of $H_{p(0)}$, and simultaneously moving W_+ so that (4.13) holds.

Similarly, for $-T(0)/2 - \epsilon < t < 0$, the right hand side of (4.15) is equal to

$$\langle [(i/h)P(z), (\exp(-tH_{p(0)})^*\chi]_{\exp(tH_{p(0)})W_+} I_+(z)u, I_+(z)\rangle\,.$$

For $t \sim T(0)/2$, $\exp(\pm tH_{p(0)}(W_+) \subset W_-$, and hence

$$\begin{gathered}\langle [(i/h)P(z), (\exp(-tH_{p(0)})^*\chi]_{\exp(tH_{p(0)})W_+} I_+(z)u, I_+(z)u\rangle = \\ \langle [(i/h)P(z), (\exp(tH_{p(0)})^*\chi]_{\exp(-tH_{p(0)})W_+} U_-(z)\mathcal{M}(z)u, I_-(z)\mathcal{M}(z)u\rangle\,, \quad t \sim T(0)/2\,,\end{gathered}$$

from the definition of $\mathcal{M}(z)$, (4.8). But this shows (4.15) proving the first part of the proposition.

To see the second part we use use Lemma 3.4, and the obvious conjugation properties of the solution in the model case discussed in Proposition 3.5: going around the closed orbit we obtain that the underlying symplectomorphism is given by the Poincaré map. □

So far we have discussed only the case of $z \in \mathbb{R}$. We can now consider almost analytic extensions of the operators $Q(z)$, $Q_\circlearrowright(z)$, $U_\pm(z,w)$, $I_\pm(z)$, and $M(z)$. For that we consider a complex neighbourhood of $I \subset \mathbb{R}$:

$$I_{h,L} = \{z \;:\; \operatorname{Re} z \in I\,,\ |\operatorname{Im} z| \leq Lh\log(1/h)\}\,.$$

The families of pseudo-differential operators $P(z)$, $Q(z)$, and $Q_\circlearrowright(z)$ have almost analytic extensions given by Lemma 3.1, and we use the same notation for them. We then use Lemma 3.3 and (4.4) to extend $U(z,w)$, $U_\circlearrowright(z,w)$, and $U_\pm(z,w)$ to $(z,w) \in I_{h,L} \times I_{h,L}$. We then have

$$P(z)U_\bullet(z,w) = U_\bullet(z,w)P(w)\,, \quad (z,w) \in I_{h,L} \times I_{h,L}\,,$$

microlocally (that is, in particular modulo $\mathcal{O}(h^\infty)$). Indeed, for $x, w \in I \times I$, and $|y| \leq Lh\log(1/h)$, we have, as in the proof of Lemma 3.3,

$$\partial_y[P(x+iy)U(x+iy,w) - U(x+iy,w)P(w)] = \mathcal{O}(y^\infty)\,,$$
$$[P(x+iy)U(x+iy,w) - U(x+iy)P(w)]\restriction_{y=0} = 0\,.$$

Hence we can define

$$I_\pm(z) = U_\pm(z,w)I_\pm(w)\,,\ (z,w) \in I_{h,L} \times I\,,$$

so that $P(z)I_\pm(z) = 0$.

To define an almost analytic extension of $M(z)$ we first almost analytically extend the pseudodifferential operator $K(z)^{-1}(Q(z) - Q_\circlearrowleft(z))K(z)$, and then use (4.11) and Lemma 3.2. In particular, Proposition 4.5 gives,

$$M(z)^{-1} = M(\bar{z})^*\,,\quad |\operatorname{Im} z| \leq Lh\log(1/h)\,.$$

5. Grushin problem near a closed trajectory

As in the previous section we assume that $P(z)$ is *self-adjoint* for $z \in \mathbb{R}$, and denote by the same symbol the almost analytic continuation of $P(z)$. Although the inverse of $P(z)$ does not normally exist near $\gamma = \gamma(0)$ for all $z \in I$ we will describe $P(z)^{-1}$ in terms of the inverse of a microlocal effective Hamiltonian $E_{-+}(z) = I - M(z)$. We will do it first for z real and then use the extensions of operators $U_\pm(z,w)$ described at the end of the last section to transplant the results to complex values of z.

To do that we follow the now standard Grushin reduction [6], and consider the system

$$\mathcal{P}(z) = \begin{pmatrix} (i/h)P(z) & R_-(z) \\ R_+(z) & 0 \end{pmatrix} : \mathcal{D}'(X) \times \mathcal{D}'(\mathbb{R}^{n-1}) \longrightarrow \mathcal{D}'(X) \times \mathcal{D}'(\mathbb{R}^{n-1})\,, \tag{5.1}$$

defined microlocally near $\gamma \times (0,0)$, and where the operators $R_\pm$ need to be suitably chosen.

We will successively build the operator $\mathcal{P}(z)$ and its inverse. We start by putting

$$R_+(z) = K(z)^*[(i/h)P(z), \chi]_{W_+}\,, \tag{5.2}$$

and u with $Pu = 0$ near W_+, $R_+(z)u$, is its Cauchy data. Hence $u = K(z)v$ provides a *local* solution to the microlocal Cauchy problem:

$$\begin{cases} P(z)u = 0 \\ R_+(z)u = v \end{cases} \tag{5.3}$$

To obtain a global Cauchy problem we need to introduce $R_-(z)$. To do that we define

$$K_f(z) = I_+(z)K(z)\,,\quad K_b(z) = I_-(z)K(z)\,,$$

where the operators $I_\pm(z)$ are defined in (4.6). We recall the definition of the monodromy operator:

$$K_f(z) = K_b(z)M(z) \text{ microlocally near } W_- \times (0,0). \tag{5.4}$$

We can build a solution of (5.3) in $\Omega \setminus W_-$ by putting

$$E_+(z)v = \chi K_f(z)v + (1-\chi)K_b(z)v\,, \tag{5.5}$$

so that in particular, $E_+(z)v = K(z)v$, in W_+, and consequently

$$R_+(z)E_+(z) = Id \text{ microlocally near } (0,0) \in T^*\mathbb{R}^{n-1}. \tag{5.6}$$

Applying the operator, and using (5.4) we obtain

$$\begin{aligned}\frac{i}{h}P(z)E_+(z)v &= [(i/h)P(z),\chi]_{W_-}K_f(z)v - [(i/h)P(z),\chi]_{W_-}K_b(z)v \\ &= [(i/h)P(z),\chi]_{W_-}K_b(z)(M(z)-I)v\,.\end{aligned}$$

Hence we obtain a globally (near γ) solvable Cauchy problem by putting

$$\left\{\begin{array}{l} \frac{i}{h}P(z)u + R_-(z)u_- = 0 \\ R_+(z)u = v \end{array}\right. , \tag{5.7}$$

with

$$R_-(z) = [(i/h)P(z),\chi]_{W_-}K_b(z)\,. \tag{5.8}$$

The problem (5.7) is solved by putting

$$u = E_+(z)v\,, \quad u_- = E_{-+}(z)v\,, \quad E_{-+}(z) \stackrel{\text{def}}{=} I - M(z)\,, \tag{5.9}$$

where $E_+(z)$ was given by (5.5).

The definitions (5.2) and (5.8) give $\mathcal{P}(z)$ in (5.1). If the microlocal inverse, $\mathcal{E}(z)$, exists, it is necessarily given by

$$\mathcal{E}(z) = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix} \tag{5.10}$$

where $E_+(z)$ and $E_{-+}(z)$ have already been constructed.

It remains to find $E(z)\,, E_-(z)$, and to show that the resulting operator $\mathcal{E}(z)$ is the right and left microlocal inverse of $\mathcal{P}(z)$. For the right inverse, this means solving

$$\left\{\begin{array}{l} \frac{i}{h}P(z)u + R_-(z)u_- = v \\ R_+(z)u = v_+ \end{array}\right. . \tag{5.11}$$

We first introduce the forward and backward fundamental solutions of $(i/h)P(z)$:

$$\begin{aligned} &L_f(z) \quad \text{microlocally defined near } (\Omega\times_\epsilon\Omega)_+\ , \\ &L_b(z) \quad \text{microlocally defined near } (\Omega\times_\epsilon\Omega)_-\,, \end{aligned}$$

where $(\Omega\times_\epsilon\Omega)_\pm$ is given by

$$(\Omega\times_\epsilon\Omega)_\pm \stackrel{\text{def}}{=} \left(\bigcup_{m\in\Omega}\{(\exp(tH_{p(0)})m,m)\right)\cap\Omega\times\Omega\ :\ -\epsilon<\pm t<T(0)-2\epsilon\}\,.$$

To do that we use Proposition 3.5 and the corresponding local forward and backward fundamental solutions:

$$\begin{aligned} L_f^0v(x) &= \int_{-\infty}^{x_1} v(t,x')dt\,, \\ L_b^0v(x) &= -\int_{x_1}^{\infty} v(t,x')dt\,, \end{aligned}$$

$v\in\mathcal{E}'(\mathbb{R}^n)$.

We will now try to build an approximate solution of $(i/h)P(z)u=v$ using $L_\bullet(z)$. For that let us put

$$\tilde{u} = L_f(z)(1-\chi)v\,.$$

Let us also define χ_b, χ_f satisfying (4.13) and in addition,

$$\chi_b = 1 \text{ on supp }\chi\cap W_+,\ \chi = 1 \text{ on supp }\chi_f\cap W_+.$$

We now put

$$\tilde{u} = L_f(z)(1-\chi)v\,,$$

where we can think of $\tilde{u}$ as being microlocally defined on the covering space of Ω, $\widetilde{\Omega}$ (see the proof of Proposition 4.3). Hence, $P(z)\tilde{u} = 0$ to the right of the support of $1-\chi$ (in the direction of the flow), in particular on the support of χ_f. Hence, to the right of the support of $1-\chi$,

$$\begin{aligned}\tilde{u} &= K(z)K(z)^*[(i/h)P(z), \chi_f]_{W_+}\tilde{u} \\ &= K(z)K(z)^*[(i/h)P(z), \chi_f]_{W_+}L_f(z)(1-\chi)v\,.\end{aligned}$$

If we use the notation from the proof of Proposition 4.3 and put $\widetilde{K}_f(z) = \widetilde{I}_+(z)K(z)$, then in the forward direction of propagation past the support of $1-\chi$, we have in $\widetilde{\Omega}$,

$$\tilde{u} = \widetilde{K}_f(z)K(z)^*[(i/h)P(z), \chi_f]_{W_+}L_f(z)(1-\chi)v\,. \tag{5.12}$$

Similarly, if $\hat{u} = L_b(z)\chi v$ then, left to the support of χ, we have $P(z)\hat{u} = 0$, and we can extend $\hat{u}$ father left, microlocally in $\widetilde{\Omega}$:

$$\hat{u} = \widetilde{K}_b(z)K(z)^*[(i/h)P(z), \chi_b]_{W_+}L_b(z)\chi v\,. \tag{5.13}$$

We can think of $\tilde{u}$ and $\hat{u}$ as multivalued in Ω and we will define, near W_-,

$$\begin{aligned}L_{ff}v &= \text{ second branch of } \tilde{u} \text{ near } W_-\,, \\ L_{bb}v &= \text{ second branch of } \hat{u} \text{ near } W_-\,.\end{aligned}$$

With this notation we put

$$u_0 = E_0(z)v \stackrel{\text{def}}{=} \begin{cases} L_b(z)\chi v + L_f(z)(1-\chi)v & \text{outside } W_- \\ L_b(z)\chi v + (1-\chi)L_{bb}(z)\chi v + L_f(z)(1-\chi)v + \chi L_{ff}(1-\chi)v & \text{in } W_- \end{cases}. \tag{5.14}$$

An application of $(i/h)P(z)$ gives

$$(i/h)P(z)u_0 = v - [(i/h)P(z), \chi]_{W_-}L_{bb}(z)\chi v + [(i/h)P(z), \chi]_{W_-}L_{ff}(z)(1-\chi)v\,,$$

and using (5.12) and (5.13) (where we now drop the $\tilde{}$ as we are taking the second branch of $\tilde{u}$ and $\hat{u}$, and the definition of $M(z)$, we get

$$\begin{aligned}\frac{i}{h}P(z)u_0 &= v - [(i/h)P(z), \chi]_{W_-}K_b(z)K(z)^*[(i/h)P(z), \chi_b]_{W_+}L_b(z)\chi v \\ &\quad + [(i/h)P(z), \chi]_{W_-}K_f(z)K(z)^*[(i/h)P(z), \chi_f]_{W_+}L_f(z)(1-\chi)v \\ &= v - [(i/h)P(z), \chi]_{W_-}K_b(z)\left(K(z)^*[(i/h)P(z), \chi_b]_{W_+}L_b(z)\chi v\right. \\ &\quad \left. - M(z)K(z)^*[(i/h)P(z), \chi_f]_{W_+}L_f(z)(1-\chi)v\right)\,.\end{aligned}$$

In other terms,

$$\frac{i}{h}P(z)E_0(z)v + R_-(z)E_{0,-}(z)v = v\,, \tag{5.15}$$

where we defined $E_0(z)$ by (5.14) and

$$E_{0,-}(z) = K(z)^*[(i/h)P(z), \chi_b]L_b(z)\chi - M(z)K(z)^*[(i/h)P(z), \chi_f]_{W_+}L_f(z)(1-\chi)v\,. \tag{5.16}$$

If we now put

$$E(z) \stackrel{\text{def}}{=} E_0(z) - E_+(z)R_+(z)E_0(z)\,, \quad E_-(z) \stackrel{\text{def}}{=} E_{0,-}(z)v - E_{-+}(z)R_+(z)E_0(z)\,,$$

then $\mathcal{E}(z)$ given by (5.10) is a right microlocal inverse of $\mathcal{P}(z)$.

To show that it is also a left inverse, we observe that

$$\mathcal{P}(z)^* = \begin{pmatrix} -(i/h)P(z) & R_+(z)^* \\ R_-(z)^* & 0 \end{pmatrix} \; : \; \mathcal{D}'(X) \times \mathcal{D}'(\mathbb{R}^{n-1}) \longrightarrow \mathcal{D}'(X) \times \mathcal{D}'(\mathbb{R}^{n-1}) \,,$$

is microlocally defined in the same region as $\mathcal{P}(z)$ and is essentially of the same form but with W_+ replaced by W_- and χ by $1-\chi$:

$$R_+(z)^* = [(i/h)P(z), \chi]_{W_+} K(z) \,,$$
$$R_-(z)^* = K_b(z)^* [(i/h)P(z), \chi]_{W_-} \,.$$

To see this we first note that

$$K_b(z)^* [(i/h)P(z), \chi]_{W_-} K_b(z) = -Id \,.$$

In fact, as in the proof of Proposition 4.5, (4.14) is invariant under the change of χ and $W_\pm$, as long as (4.13) hold. In particular, for $0 < t < T(0) - \epsilon$,

$$K_b(z)^* [(i/h)P(z), (\exp(tH_{p(0)})^* \chi]_{\exp(-tH_{p(0)}) W_+} K_b(z) = Id \,.$$

For $t \sim T(0)/2$, W_+ is moved to W_-, and $(\exp(-tH_{p(0)})^* \chi$ satisfies the properties of $1-\chi$. Hence, using the idependence of χ,

$$K^b(z)^* [(i/h)P(z), (1-\chi)]_{W_-} K_b(z) = Id_{\mathcal{D}'(\mathbb{R}^{n-1})} \,, \quad \text{microlocally near } (0,0) \in T^*\mathbb{R}^{n-1}.$$

If we now replace $K(z)$ by $K_b(z)$, then $K(z)$ plays the rôle of $K_b(z)$, and this proves that $R_+(z)^*$ is the same as $-R_-(z)$ with W_+ and W_- switched and χ replaced by $1-\chi$.

Hence, a similar argument to the one used for the construction of $\mathcal{E}(z)$ shows that $\mathcal{P}(z)^*$ has a right inverse,

$$\mathcal{F}(z)^* = \begin{pmatrix} F(z) & F_+(z) \\ F_-(z) & F_{-+}(z) \end{pmatrix}^* \,.$$

Then $\mathcal{F}(z)$ is a left inverse of $\mathcal{P}(z)$, and the usual argument ($\mathcal{F}(z) = \mathcal{F}(z)\mathcal{P}(z)\mathcal{E}(z) = \mathcal{E}(z)$, microlocally) shows that it is equal to our right inverse.

Remark. By constructing part of the left inverse directly we can arrive at a simpler expression for $E_-(z)$:

$$E_-(z) = -(M(z)K_f(z)^*\chi + K_b(z)^*(1-\chi)) \,, \tag{5.17}$$

and it is useful to have it. To obtain it we will directly solve the problem

$$\begin{cases} \widetilde{E}_-(z)(i/h)P(z) + E_{-+}(z)R_+(z) = 0 \\ \widetilde{E}_-(z)R_-(z) = Id_{\mathcal{D}'(\mathbb{R}^{n-1})} \end{cases} \tag{5.18}$$

Motivated by the structure of $E_+(z)$ and the fact that $R_-(z)$ is close to being an adjoint of $R_+(z)$ (if it were, then $E_-(z)$ would simply be the ajoint of $E_+(z)$), we put

$$\widetilde{E}_-(z) = -(M(z)K_f(z)^*\chi + K_b(z)^*(1-\chi)) \,. \tag{5.19}$$

We now compute

$$\begin{aligned} -\widetilde{E}_-(z)R_-(z) &= (M(z)K_f(z)^*\chi + K_b(z)^*(1-\chi)) \, [(i/h)P(z), \chi]_{W_-} K_b(z) \\ &= K^b(z)^* [(i/h)P(z), \chi]_{W_-} K_b(z) \,. \end{aligned}$$

To analyze the last expression, we note that $K(z)$, in the definition of $K_\bullet(z)$ was chosen, in Lemma 4.4, so that $K(z)^*[(i/h)P(z), \chi]_{W_+} K(z) = Id$. As in the proof of Proposition 4.5, this is invariant under the change of χ and $W_\pm$, as long as (4.13) hold: for $0 < t < T(0) - \epsilon$,

$$K_b(z)^* [(i/h)P(z), (\exp(tH_{p(0)})^* \chi]_{\exp(-tH_{p(0)}) W_+} K_b(z) = Id \,.$$

For $t \sim T(0)/2$, W_+ is moved to W_-, and $(\exp(-tH_{p(0)})^*\chi$ satisfies the properties of $1-\chi$. Hence, using the independence of χ,

$$K_b(z)^*[(i/h)P(z),(1-\chi)]_{W_-}K_b(z) = Id_{\mathcal{D}'(\mathbb{R}^{n-1})}\,, \quad \text{microlocally near } (0,0)\in T^*\mathbb{R}^{n-1}.$$

This shows that $\widetilde{E}_-(z)R_-(z) = Id$ and we need to verify the first identity in (5.18). For that we use $K_\bullet(z)^*P(z) = 0$, $M(z)K_f(z)^* = K_b(z)^*$, near $(0,0)\times W_- \subset T^*\mathbb{R}^{n-1}\times T^*X$, to obtain

$$\begin{aligned} -\widetilde{E}_-(z)(i/h)P(z) &= (M(z)K_f(z)^*\chi + K_b(z)^*(1-\chi))(i/h)P(z) \\ &= M(z)K_f(z)^*[\chi,(i/h)P(z)]_{W_+} - K_b(z)^*[\chi,(i/h)P(z)]_{W_+} \\ &= K(z)^*[(i/h)P(z),\chi]_{W_+} - M(z)K(z)^*[(i/h)P(z),\chi]_{W_+} \\ &= (1-M(z))R_+(z) = E_{-+}(z)R_+(z)\,, \end{aligned}$$

and that establishes (5.18), so $\widetilde{E}_-(z) = E_-(z)$ and we have (5.17).

So far we considered only the case of $z\in\mathbb{R}$, and $P(z) = P(z)^*$. Arguing as at the end of Sect.4, we see that all the operators occuring in the construction of $\mathcal{P}(z)$ and $\mathcal{E}(z)$ have almost analytic extensions to $|\operatorname{Im} z| < Lh\log(1/h)$ for any L. It follows that the extention of $\mathcal{E}(z)$ is a microlocal inverse of the extension of $\mathcal{P}(z)$ modulo $|\operatorname{Im} z|^\infty$, which in this neighbourhood of the real axis is $\mathcal{O}(h^\infty)$, that is, it remains a microlocal inverse. The bounds on the continuation of $\mathcal{E}(z)$ follow from (3.5). This gives

Proposition 5.1. *Let $P(z)$ be an almost analytic extension of the self-adjoint family of operators $P(z)\in\mathcal{C}^\infty(I_z;\Psi^{0,k}(X))$, such that*

$$\begin{gathered} \text{The flow of } H_p \text{ has a closed orbit } \gamma, \\ \text{on which } p = \sigma(P(0)) = 0 \text{ and } dp \neq 0. \end{gathered}$$

Then, there exist operators $R_\pm(z)$, defined in $|\operatorname{Im} z| \leq Lh\log(1/h)$, such that

$$\mathcal{P}(z) = \begin{pmatrix} (i/h)P(z) & R_-(z) \\ R_+(z) & 0 \end{pmatrix} : \mathcal{D}'(X)\times\mathcal{D}'(\mathbb{R}^{n-1}) \longrightarrow \mathcal{D}'(X)\times\mathcal{D}'(\mathbb{R}^{n-1})\,,$$

defined microlocally near $\gamma\times(0,0)$, has a microlocal inverse there:

$$\mathcal{E}(z) = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix} = \mathcal{O}(e^{C|\operatorname{Im} z|/h}) : L^2(X)\times L^2(\mathbb{R}^{n-1}) \to L^2(X)\times L^2(\mathbb{R}^{n-1})\,,$$

$$\bar\partial_z\mathcal{P}(z) = \mathcal{O}(|\operatorname{Im} z|^\infty)$$

and $E_{-+}(z) = I - M(z)$, where $M(z)$ is the quantum monodromy operator defined by (4.9).

Remark. The constant C in the estimate of the norm of $\mathcal{E}(z)$ could be described more explicitely if stronger conditions on $P(z)$ were made. If we assumed (6.1) then C could be related to C_p in (6.11).

6. Proof of the trace formula

We can now prove the main result of the paper. We strengthen our assumptions further here by demanding that $P(z)$ is a smooth family of operators, self-adjoint for the real values of the parameter, and elliptic off the real axis.

Theorem 2. *Let $P(z) \in \mathcal{C}^\infty(I_z; \Psi_h^{0,k}(X))$, $I = (-a,a) \subset \mathbb{R}$, be a family of self-adjoint, principal type operators, such that $\Sigma_z = \{m \, : \, \sigma(P(z)) = 0\} \subset T^*X$ is compact. We assume that*

$$\begin{aligned} &\sigma(\partial_z P(z)) \leq -C < 0 \,, \quad \text{near } \Sigma_z . \\ &|\sigma(P(z))| \geq C|\xi|^k \,, \quad \text{for } |\xi| \geq C . \end{aligned} \tag{6.1}$$

We also assume that for z near 0, the Hamilton vector field, $H_{p(z)}$, $p(z) = \sigma(P(z))$, has a simple closed orbit $\gamma(z) \subset \Sigma_0$ with perid $T(z)$, and that $\gamma(z)$ has a neighbourhood Ω such that

$$m \in \Omega \text{ and } \exp tH_{p(z)}(m) = m \,, \; p(m) = 0 \,, \; 0 < |t| \leq T(z)N + \epsilon \,, \; z \in I \,, \implies m \in \gamma(z) \,, \tag{6.2}$$

where $T(z)$ is the period of $\gamma(z)$, assumed to depend smoothly on z. Let $A \in \Psi_h^{0,0}(X)$ be a microlocal cut-off to a sufficiently small neighbourhood of $\gamma(0)$.

Then if $P(z)$ is an almost analytic extension of $P(z)$, $z \in \mathbb{R}$, $\chi \in \mathcal{C}_{\rm c}^\infty(I)$, $\tilde\chi \in \mathcal{C}_{\rm c}^\infty(\mathbb{C})$, its almost analytic extension, $f \in \mathcal{C}^\infty(\mathbb{R})$, and $\operatorname{supp} \hat f \subset (-N(C_p - \epsilon) + C, N(C_p - \epsilon) - C) \setminus \{0\}$, we have,

$$\begin{aligned} &\frac{1}{\pi} \operatorname{tr} \int f(z/h) \bar\partial_z \left[\tilde\chi(z) \, \partial_z P(z) \, P(z)^{-1} \right] A \mathcal{L}(dz) = \\ &-\frac{1}{2\pi i} \sum_{-N-1}^{N-1} \operatorname{tr} \int_{\mathbb{R}} f(z/h) M(z,h)^k \frac{d}{dz} M(z,h) \chi(z) dz + \mathcal{O}(h^\infty) \,, \end{aligned} \tag{6.3}$$

where $M(z,h)$ is the quantum monodromy operator defined in (4.9) *with $K(z)$ satisfying* (4.14)*. The constant $C_p > 0$, in the condition on $\hat f$ depends on $p(z)$ only and is given in* (6.11)*.*

We observe that the left hand side of (6.3) is independent of the choice of the almost analytic extension of χ: if $\tilde\chi^\sharp$ is another extension then, then $\tilde\chi - \tilde\chi^\sharp = \mathcal{O}(|\operatorname{Im} z|^\infty)$. In view of Lemma 6.1 below,

$$(\tilde\chi(z) - \tilde\chi^\sharp(z)) \, \partial_z P(z) \, P(z)^{-1}$$

is smooth in z, and $\mathcal{O}(|\operatorname{Im} z|^\infty)$. By Green's formula and holomorphy of f, the corresponding integral vanishes.

As described in Sect.2 Theorem 1 is an immediate consequence of Theorem 2.

Before proceeding with a proof we remark that we can assume that

$$P(z) \in \Psi_h^{0,0}(X) \,,$$

since $P(z)$ can be multiplied by an z-independent elliptic $B \in \Psi_h^{0,-k}(X)$, without changing (6.3).

We start with a lemma which justifies taking the traces in (6.3):

Lemma 6.1. *Under the assumptions of Theorem 2, $P(z)^{-1}$ exists in $U \setminus I$, where U is a complex neighbourhood of $J \Subset I$, and*

$$\|P(z)^{-1}\| \leq C |\operatorname{Im} z|^{-1} \,, \quad 0 < |\operatorname{Im} z| \leq 1/C \,.$$

Proof. Let $\psi = \psi^w(x, hD; z)$ be a microlocal cut-off to a a small neighbourhood of Σ_z. Let us put $v = P(z)u$, so that (semi-classical) elliptic regularity gives

$$\|(1-\psi)u\| \leq C\|v\| + \mathcal{O}(h^\infty)\|u\| \,. \tag{6.4}$$

For complex values of z we write

$$P(z) = P(\operatorname{Re} z) + \operatorname{Im} z \, Q(z) \,,$$

where $P(\operatorname{Re} z)$ is self-adjoint and $\sigma(Q(z)) > 1/C > 0$ near Σ_z. This shows that

$$\operatorname{Im}\langle P(z)\psi u, \psi u\rangle = \operatorname{Im} z \; \operatorname{Re}\langle Q(z)\psi u, \psi u\rangle \geq \operatorname{Im} z \; \left(\|\psi u\|^2/C - \mathcal{O}(h^\infty)\|u\|^2 \right) \,, \tag{6.5}$$

where we used the semi-classical Gårding inequality.

We also write

$$\begin{aligned}&\operatorname{Im}\langle P(z)u,u\rangle - \operatorname{Im}\langle P(z)\psi u,\psi u\rangle = \operatorname{Im} z\left(\langle Q(z)u,u\rangle - \langle Q(z)\psi u,\psi u\rangle\right) = \\ &\operatorname{Im} z\,\mathcal{O}(1)\|(1-\psi)u\|\|u\| = \operatorname{Im} z\,\mathcal{O}(1)\left(\|v\|\|u\| + \mathcal{O}(h^\infty)\|u\|^2\right)\,,\end{aligned}$$

where we used elliptic regularity (6.4) in the last estimate. Then, applying (6.5),

$$\begin{aligned}\|u\|\|v\| &\geq \operatorname{Im}\langle P(z)\psi u,\psi u\rangle - \operatorname{Im} z\,\mathcal{O}(1)\left(\|v\|\|u\| + \mathcal{O}(h^\infty)\|u\|^2\right)\\ &\geq \operatorname{Im} z\left(\|\psi u\|^2/C - \mathcal{O}(1)\|v\|\|u\| - \mathcal{O}(h^\infty)\|u\|^2\right)\,.\end{aligned}$$

For small $\operatorname{Im} z$ the term $\|v\|\|u\|$ on the left hand side can be absorbed in the right hand side, and by adding $\operatorname{Im} z\|(1-\psi)u\|^2$ to both sides we obtain

$$\operatorname{Im} z\|u\|^2/C \leq \|u\|\|v\| + \mathcal{O}(h^\infty)\operatorname{Im} z\|u\|^2\,,$$

and that gives

$$\|u\| \leq \frac{C}{\operatorname{Im} z}\|v\|\,,$$

proving the estimate for $P(z)^{-1}$. □

Proof of Theorem: Using Proposition 5.1 we can *formally* write

$$P(z)^{-1}A = E(z)A - E_+(z)E_{-+}(z)^{-1}E_-(z)A\,,\quad E_{-+}(z) = I - M(z)\,,$$

microlocally near Ω, and for $0 < |\operatorname{Im} z| \leq Lh\log(1/h)$, with any L. To apply this formal expression rigourously, we rewrite the left hand side of (6.3) as

$$\begin{aligned}&\frac{1}{\pi}\operatorname{tr}\int f(z/h)\bar\partial_z\tilde{[}\chi(z)\,\partial_z P(z)\,P(z)^{-1}]\,A\,\mathcal{L}(dz) = \\ &\frac{1}{\pi}\sum_{\pm}\operatorname{tr}\int_{\mathbb{C}_\pm} f(z/h)\bar\partial_z\tilde{[}\chi(z)\,\partial_z P(z)\,P(z)^{-1}\,]A\,\mathcal{L}(dz)\,.\end{aligned}\tag{6.6}$$

Then, motivated by the formal Neumann series expansion of $(I-M(z))^{-1}$ we define

$$T_N^+(z) \stackrel{\text{def}}{=} E(z)A - E_+(z)\sum_{k=0}^{N} M(z)^k E_-(z)A\,,\tag{6.7}$$

so that

$$P(z)^{-1}A = T_N^+(z) + P(z)^{-1}R_-(z)M(z)^{N+1}E_-(z)A\,,\tag{6.8}$$

microlocally near γ and for $0 < |\operatorname{Im} z| \leq Lh\log(1/h)$, for any L. In fact, from $\mathcal{P}(z)\mathcal{E}(z) = Id$, and $E_\pm(z) = I - M(z)$, we have

$$P(z)E_+(z) = -R_-(z)(I - M(z))\,,\quad P(z)E(z) = I - R_-(z)E_-(z)\,,$$

and hence

$$\begin{aligned}P(z)T_N^+(z) &= P(z)(E(z)A - E_+(z)\sum_{k=0}^{N} M(z)^k E_-(z)A)\\ &= A - R_-(z)E_-(z)A + R_-(z)(I-M(z))\sum_{k=0}^{N} M(z)^k E_-(z)A\\ &= A - R_-(z)M(z)^{N+1}E_-(z)A\,,\end{aligned}$$

which gives (6.8).

To use this in (6.6) we need to have the support of the almost analytic extension of the cut-off function χ to be contained in the region where $|\operatorname{Im} z| \leq Lh\log(1/h)$. To do that we follow the method of [3, Sect.12] by fixing an almost analytic extension of χ, $\chi^{\#}$, and then putting

$$\tilde\chi = \tilde\chi_{L,h} = \chi^{\#}\psi_{L,h}\,, \quad \psi_{L,h}(z) = \psi\left(\frac{\operatorname{Im} z}{Lh\log(1/h)}\right)\,, \quad \psi(t) = \begin{cases} 1\,, & |t| < 1/2 \\ 0\,, & |t| > 1 \end{cases}\,.$$

By the remark after the statement of Theorem 2, (6.3) is independent of the choice of $\tilde\chi$ and hence we can use $\tilde\chi_{L,h}$. Since now $\mathcal{O}(|\operatorname{Im} z|^\infty) = \mathcal{O}(h^\infty)$, the almost analyticity of $P(z)$ also shows that the left hand side of (6.3) can be rewritten as

$$\frac{1}{\pi}\operatorname{tr}\int f(z/h)\bar\partial_z\tilde\chi(z)\,\partial_z P(z)\,P(z)^{-1}\,A\mathcal{L}(dz)\,, \tag{6.9}$$

and this is what we will use from now on.

We claim that with the choice of $\tilde\chi$ above

$$\begin{aligned} &\operatorname{tr}\int_{\mathbb{C}_+} f(z/h)\bar\partial_z\tilde\chi(z)\,\partial_z P(z)\,P(z)^{-1}\,A\,\mathcal{L}(dz) = \\ &\operatorname{tr}\int_{\mathbb{C}_+} f(z/h)\bar\partial_z\tilde\chi(z)\,\partial_z P(z)\,T_N^+(z)\mathcal{L}(dz) + \mathcal{O}(h^{L/C})\,, \end{aligned} \tag{6.10}$$

where C is fixed depending on N and $\operatorname{supp}\hat f$.

To show this we first need the following

Lemma 6.2. *The almost analytic continuation of the monodromy operator satisfies, for z sufficiently close to* 0, *and for any L,*

$$\begin{aligned} &\|M(z)\| \leq e^{-(C_p-\epsilon)\operatorname{Im} z/h} + \mathcal{O}(h^\infty)\,, \quad 0 < \operatorname{Im} z < Lh\log(1/h)\,, \\ &\|M(z)^{-1}\| \leq e^{(C_p-\epsilon)\operatorname{Im} z/h} + \mathcal{O}(h^\infty)\,, \quad -Lh\log(1/h) < \operatorname{Im} z < 0\,, \\ &C_p = -\int_0^{T(0)} \sigma(\partial_z P(z))(\exp(tH_{p(0)})(m_0)dt\,, \end{aligned} \tag{6.11}$$

where $\epsilon > 0$ can be taken arbitrarily small by shrinking the neighbourhood of γ. The constant C_p is positive thanks to (6.1).

Proof. We use the differential equation (4.11) and observe that for z real, and $m_0(z) \in \gamma(z)$,

$$\sigma(K(z)^{-1}(Q_\circlearrowleft(z) - Q(z))K(z))(0,0) = -\int_0^{T(z)} \sigma(\partial_z P(z))(\exp(tH_{p(z)})(m_0(z))dt\,.$$

Hence, writing $z = x + iy$, $0 < y < Lh\log(1/h)$, and $B(z) = K(z)^{-1}(Q(z) - Q_\circlearrowleft(z))K(z)$, we have, for $v \in \mathcal{D}'(\mathbb{R}^{n-1})$, with $WF(v)$ close to $(0,0)$,

$$\begin{aligned} h\frac{d}{dy}\left(\|M(z)v\|^2\right) &= h\frac{d}{dy}\langle M(z)v, M(z)v\rangle \\ &= ih(\partial_z - \partial_{\bar z})\langle M(z)v, M(z)v\rangle \\ &= -\langle B(z)M(z)v, M(z)v\rangle - \langle M(z)v, B(z)M(z)v\rangle + \mathcal{O}(|\operatorname{Im} z|^\infty)\|v\|^2 \\ &= -\langle (B(z) + B(z)^*)M(z)v, M(z)v\rangle + \mathcal{O}(|\operatorname{Im} z|^\infty)\|v\|^2\,. \end{aligned}$$

The Gårding inequality now shows that for x small enough,

$$h\frac{d}{dy}\left(\|M(z)v\|^2\right) \leq -(C_p - \epsilon)\|M(z)v\|^2 + \mathcal{O}(y^\infty)\|v\|^2\,.$$

Since by Proposition 4.5, $\|M(x)v\|^2 = \|v\|^2(1 + \mathcal{O}(h^\infty))$, the lemma follows. $\square$

Proof of (6.10)*:* By (6.6) and (6.8) we need to estimate

$$\operatorname{tr} \int_{\mathbb{C}_+} f(z/h)\bar\partial_z \tilde\chi_{L,h}(z)\, \partial_z P(z)\, P(z)^{-1}\, R_-(z) M(z)^{N+1} E_-(z)\, A\, \mathcal{L}(dz)\,, \tag{6.12}$$

where by Lemmas 6.1 and 6.2 we have

$$\|M(z)^{N+1}\| \leq \|M(z)\|^{N+1} \leq e^{-(C_p-\epsilon)(N+1)\frac{\operatorname{Im} z}{h}}\,, \quad 0 \leq \operatorname{Im} z \leq Lh\log(1/h)\,,$$
$$\|P(z)^{-1}\| \leq 1/|\operatorname{Im} z|\,.$$

All the operators coming from $\mathcal{P}(z)$ and $\mathcal{E}(z)$ are bounded by $\exp(C|\operatorname{Im} z|/h)$, and if $\operatorname{supp}\hat f \subset [-b+C, b-C]$, then

$$|f(z/h)| \leq Ce^{(b-C)\frac{|\operatorname{Im} z|}{h}}\,.$$

Using the definition of $\tilde\chi_{L,h}$, the above estimates, and the characteristic function

$$\rho_{L,h}(t) = 1\!\!1_{Lh\log(1/h)/2 \leq t \leq Lh\log(1/h)}\,,$$

we can bound (6.12) by a constant times

$$\begin{aligned} h^{-n}\int \left(|\bar\partial_z \chi^\#(z)| + (Lh\log(1/h))^{-1}\rho_{L,h}(\operatorname{Im} z)\right)|\operatorname{Im} z|^{-1} e^{\frac{\operatorname{Im} z}{h}(b-(C_p-\epsilon)(N+1))}\mathcal{L}(dz) \leq \\ Ch^{-n}\int_{0\leq \operatorname{Im} z < Lh\log(1/h)} |\operatorname{Im} z|^M e^{\frac{\operatorname{Im} z}{h}(b-(C_p-\epsilon)(N+1))}\mathcal{L}(dz) \\ + Ch^{-n}(Lh\log(1/h))^{-2} h^{((C_p-\epsilon)(N+1)-b)L} \leq C_N' h^{L/C-n}\,, \end{aligned}$$

where $C>0$ is fixed. □

With (6.10) established, we have to study the leading term on its right hand side which we rewrite using the definition (6.7) and the cyclicity of the trace:

$$\begin{aligned} &\frac{1}{\pi}\operatorname{tr} \int_{\mathbb{C}_+} \bar\partial_z(\tilde\chi_{L,h})\partial_z P(z)\, E(z)\, A\, f(z/h)\mathcal{L}(dz) - \\ &\frac{1}{\pi}\operatorname{tr} \int_{\mathbb{C}_+} \bar\partial_z(\tilde\chi_{L,h}) \left(\sum_{k=0}^{N} M(z)^k E_-(z) A \partial_z P(z) E_+(z)\right) f(z/h)\mathcal{L}(dz)\,. \end{aligned} \tag{6.13}$$

Since all the operators are almost analytic (in particular $\bar\partial_z E_\bullet(z) = \mathcal{O}(h^\infty)$ on the support of $\tilde\chi_{L,h}$) and $f(z/h)$ is holomorphic, we can apply Green's formula and reduce the first integral to an integral over the real axis:

$$\frac{1}{2\pi i}\operatorname{tr} \int_{\mathbb{R}} \chi(z) f(z/h)\, \partial_z P(z) E(z) A dz\,.$$

To analyze the second term (with integration still over $\mathbb{C}_+$) we see that the explicit expressions $E_-(z)$ and $E_+(z)$, (5.5) and (5.17), show that

$$E_-(z) A \partial_z P(z) E_+(z) = A_1(z) M(z) + A_2(z)\,, \quad A_j(z) \in \mathcal{C}^\infty(I_z; \Psi^{0,-\infty}(\mathbb{R}^{n-1}))\,. \tag{6.14}$$

To analyze the contributions to the trace we need the following simple

Lemma 6.3. *Suppose that* $A \in \Psi_h^{0,-\infty}(\mathbb{R}^{n-1})$ *and* $U \in I_h^0(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}; C')$, *satisfy*

$$m \in WF_h(A) \implies (m,m) \notin C\,.$$

Then, for $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^{n-1})$,

$$\operatorname{tr} \chi A U = \mathcal{O}(h^\infty)\,.$$

Proof. It is clear that if $\chi_1, \chi_2 \in \mathcal{C}_{\rm c}^\infty(T^*\mathbb{R}^{n-1})$ have disjoint supports then $\operatorname{tr} \chi_1 A U \chi_2 = \mathcal{O}(h^\infty)$, where we denoted the quantizations by the same symbols. Using the hypothesis, we can write $\chi A U$ as a sum of negligible terms ($\mathcal{O}(h^\infty)$), and of terms of that form. $\square$

The assumption (6.2) implies that the main contribution (modulo $\mathcal{O}(h^\infty)$ as usual) to the trace of

$$M(z)^k E_-(z) A \partial_z P(z) E_+(z)\,, \quad 0 < |k| \leq N\,,$$

comes from an arbitrarily small neighbourhood of the fixed point, $(0,0) \in T^*\mathbb{R}^{n-1}$, of $C(z)$, the canonical relation of $M(z)$. Here we used (6.14) to see what the canonical relation was.

We can therefore replace A by 1 and introduce a microlocal cut-off, ρ^w, to a neighbourhood of $(0,0)$:

$$\operatorname{tr} M(z)^k E_-(z) A \partial P(z) E_+(z) = M(z)^k E_-(z) \partial P(z) E_+(z) \rho^w + \mathcal{O}(h^\infty)\,, \quad k > 0\,.$$

For $k=0$ the same discussion is valid for the contribution of $M(z)A_1(z)$ in $E_-(z)\partial P(z) E_+(z)$, but for the pseudo-differential contribution, $A_2(z)$, we need to use the support assumption on $\hat f$: $0 \notin \operatorname{supp} \hat f$. We write

$$\operatorname{tr} \int_{\mathbb{C}_+} f(z/h) \bar\partial_z \tilde\chi(z) A_2(z) \rho^w \mathcal{L}(dz) = \operatorname{tr} \int_{\mathbb{R}} f(z/h) \chi(z) A_2(z) \rho^w dz = \mathcal{O}(h^\infty)\,,$$

by the standard argument: put $g(z) = \operatorname{tr} A_2(z) \rho^w$, so that by Plancherel's theorem

$$\begin{aligned} 2\pi \int_{\mathbb{R}} f\left(\frac{z}{h}\right) g(z) dz &= h \int_{\mathbb{R}} \hat f(h\zeta) \hat g(-\zeta) d\zeta \\ &= h^{M+1} \int_{\mathbb{R}} \hat f(h\zeta)/(h\zeta)^M \zeta^M \hat g(-\zeta) d\zeta \\ &= \mathcal{O}(h^{M+1})\,. \end{aligned} \tag{6.15}$$

Hence the second term in (6.13) becomes

$$-\frac{1}{\pi} \operatorname{tr} \int_{\mathbb{C}_+} \bar\partial_z \tilde\chi \left(\sum_{k=0}^N M(z)^k E_-(z) \partial_z P(z) E_+(z) \right) \rho^w f(z/h) \mathcal{L}(dz)\,, \tag{6.16}$$

where ρ^w is a microlocal cut-off to a neighbourhood of $(0,0)$.

We recall that when $R_\pm$ are independent of z, the following standard formula holds:

$$\partial_z M(z) = -\partial_z E_{-+}(z) = E_-(z)\, \partial_z P(z)\, E_+(z)\,,$$

as is easily seen from $\partial_z \mathcal{E} = -\mathcal{E} \partial_z \mathcal{P} \mathcal{E}$. In the general case, the same argument gives

$$\begin{aligned} \partial_z M(z) &= -\partial_z E_{-+}(z) \\ &= E_-(z)\, \partial_z P(z)\, E_+(z) + E_{-+}(z)\, \partial_z R_+(z)\, E_+(z) + E_-(z)\, \partial_z R_-(z)\, E_{-+}(z)\,. \end{aligned} \tag{6.17}$$

Inserting this we obtain the following expression for (6.16):

$$
\begin{aligned}
&-\frac{1}{\pi}\operatorname{tr}\int_{\mathbb{C}_+} f(z/h)\bar\partial\tilde\chi(z)\sum_{k=0}^{N} M(z)^k\partial_z M(z)\rho^w\mathcal{L}(dz)\\
&+\frac{1}{\pi}\operatorname{tr}\int_{\mathbb{C}_+} f(z/h)\bar\partial\tilde\chi(z)\sum_{k=0}^{N} M(z)^k(1-M(z))^{-1}\partial_z R_+(z)E_+(z)\rho^w\mathcal{L}(dz)\\
&+\frac{1}{\pi}\operatorname{tr}\int_{\mathbb{C}_+} f(z/h)\bar\partial\tilde\chi(z)\sum_{k=0}^{N} M(z)^k E_-(z)\partial_z R_-(z)(1-M(z))^{-1}\rho^w M(z)^k\mathcal{L}(dz)\\
&= J_1+J_2+J_3
\end{aligned}
\tag{6.18}
$$

By Green's formula

$$
J_1 = -\frac{1}{2\pi i}\sum_{k=0}^{N}\operatorname{tr}\int_{\mathbb{R}} f(z/h)\chi(z)M(z)^k\partial_z M(z)\rho^w dx\,,
$$

which is a term appearing in (6.3). We want to show that the remaining two terms, J_2, J_3, are negligible.

To see this we need

Lemma 6.4. *We have*

$$
\partial_z R_+(z)E_+(z)\,,\quad E_-(z)\partial_z R_-(z)\in\mathcal{C}^\infty(I_z;\Psi_h^{1,-\infty}(X))\,.
$$

Proof. From the definitions (5.2), (5.5), and from (4.14) we see that

$$
\partial_z R_+(z)E_+(z) = \partial_z\left(K^*(z)[(i/h)P(z),\chi]_{W_+}\right)K(z) = -K^*(z)[(i/h)P(z),\chi]_{W_+}\partial_z K(z)\,.
$$

From the proof of Proposition 4.3 we recall that

$$
hD_z K(z) = -Q(z)K(z)\,,
$$

and hence

$$
\partial_z R_+(z)E_+(z) = (i/h)K^*(z)[(i/h)P(z),\chi]_{W_+}Q(z)K(z)\,.
$$

This expression is microlocal near $(0,0)$ and as far as $K(z)$ is concerned microlocal near $(m_0,(0,0))$. Hence we can use a model given in Proposition 3.5: $P(z)=hD_{x_1}$ (the microlocal z-dependent conjugation will not affect the uniform pseudo-differential behaviour), and

$$
K(z)u(x_1,x') = \frac{1}{(2\pi h)^{n-1}}\int e^{i(\langle y',\eta'\rangle-\phi_z(x',\eta'))/h}a_z(x',\eta')u(y')dy'd\eta'\,,
$$

where we used local representation of the h-Fourier Integral Operators (see the proof of Proposition 7.3 below for the derivation of a local representation). After composing the operators, and applying the stationary phase method we arrive at the following expression for the kernel of $\partial_z R_+(z)E_+(z)$:

$$
\frac{1}{(2\pi h)^{n-1}}\int e^{i(\phi_z(x',\eta')-\phi_z(y',\eta'))/h}A_z(y',x',\eta')d\eta'\,,\quad A_z\in S^{1,-\infty}\,,
$$

which by a standard "Kuranishi trick" argument (see the appendix) shows that we get a smooth z-dependent family of pseudo-differential operators. ☐

In J_2 we can replace $\sum_{k=0}^{N} M(z)^k(1-M(z))$ by $1-M(z)^{N+1}$. As in the proof of (6.10), we show that the term corresponding to $M(z)^{N+1}$ is negligible. The remaining term is transformed to an integral over $\mathbb{R}$:

$$\frac{1}{2\pi i}\int_{\mathbb{R}} f(z/h)\chi(z)\,\mathrm{tr}(\partial_z R_+(z)E_+(z)\rho^w)dx\,,\quad 0\notin \mathrm{supp}\,\hat{f}$$

which is negligible by Lemma 6.4 and (6.15). Similar arguments then apply to J_3.

To summarize, we have shown that

$$\mathrm{tr}\int_{\mathbb{C}_+} f(z/h)\bar{\partial}_z\tilde{\chi}(z)\;\partial_z P(z)\;P(z)^{-1}\;A\mathcal{L}(dz) =$$

$$\frac{1}{2\pi i}\mathrm{tr}\int_{\mathbb{R}}\chi(z)f(z/h)\;\partial_z P(z)E(z)Adz - \frac{1}{2\pi i}\sum_{k=0}^{N}\mathrm{tr}\int_{\mathbb{R}} f(z/h)\chi(z)M(z)^k\partial_z M(z)\rho^w dx + \mathcal{O}(h^\infty)\,,$$

We proceed in a similar way for the integral over $\mathbb{C}_-$ in (6.6). We write $I-M(z) = -M(z)(I-M(z)^{-1})$, and motivated by the resulting formal Neumann series put

$$T_N^-(z) \stackrel{\mathrm{def}}{=} E(z)A + E_+(z)\sum_{k=1}^{N}(M(z)^{-1})^k E_-(z)A\,.$$

The same arguments apply and Green's formula gives

$$-\frac{1}{2\pi i}\mathrm{tr}\int_{\mathbb{R}}\chi(z)f(z/h)E(z)Adz - \frac{1}{2\pi i}\mathrm{tr}\int_{\mathbb{R}}\chi(z)f(z/h)\sum_{k=1}^{N+1}M(z)^{-k}\partial_z M(z)dz + \mathcal{O}(h^\infty)\,.$$

When we now add the contributions from the integrations over $\mathbb{C}_\pm$ we see that the integrals involving $E(z)$ cancel and the remaning terms give (6.3)

□

7. Trace formula for non-degenerate closed trajectories

We say that a closed trajectory $\gamma(z)$ of $P(z)$ *N-fold non-degenerate* if

$$\det(I-(dC(z)_{m_0(z)})^k)\neq 0\,,\quad 0\neq |k|\leq N\,, \tag{7.1}$$

where $C(z)$ is the Poincaré map of $\gamma(z)$, (4.12). When this holds our theorem translates into the standard semi-classical trace formula, generalizing the Gutzwiller, Balian-Bloch, and Duistermaat-Guillemin trace formulæ.

We start by a general discussion of traces of Fourier Integral Operators.

Lemma 7.1. *Suppose that B, microlocally defined near $(0,0)\in T^*\mathbb{R}^n$ is given by*

$$Bu(x) = \frac{1}{(2\pi h)^n}\iint e^{i(\phi(x,\eta)-y\eta)/h}b(x,\eta;h)u(y)dyd\eta \tag{7.2}$$

where $\phi(x,\eta)$ is defined near $(0,0)$, a is a classical symbol of order 0, supported near $(0,0)$, $\phi'(0,0)=0$, and $\phi''_{x\eta}(0,0)\neq 0$. The corresponding canonical transformation is given by

$$\kappa : (\phi'_\eta(x,\eta),\eta)\mapsto (x,\phi'_x(x,\eta))\,. \tag{7.3}$$

It is defined between two neighbourhoods of $(0,0)$ and we assume that $(0,0)$ is its only fixed point there, and

$$\det(d\kappa(0,0)-1)\neq 0\,. \tag{7.4}$$

Under these assumptions

$$\operatorname{tr} B = i^{\frac{1}{2}s} \frac{(b_0(0,0) + \mathcal{O}(h))e^{i\phi(0,0)/h}}{|\det \phi''_{\eta x} \det(d\kappa(0,0) - 1)|^{\frac{1}{2}}}\,, \quad s = \operatorname{sgn} \begin{pmatrix} \phi''_{xx} & \phi''_{x\eta} - 1 \\ \phi''_{\eta x} - 1 & \phi''_{\eta\eta} \end{pmatrix}, \tag{7.5}$$

where the signature of a symmetric matrix A, sgn A*, is the difference between the number of positive and negative eigenvalues.*

Proof. The fact that (7.3) defines a smooth map is equivalent to the assumption that

$$\det \phi''_{x\eta} \neq 0\,. \tag{7.6}$$

Here and in the following, second derivatives of ϕ are computed at (0,0) if nothing else is specified. The differential, $d\kappa(0,0)$, is the map $(\delta_y, \delta_\eta) \mapsto (\delta_x, \delta_\xi)$, where

$$\begin{aligned} \delta_y &= \phi''_{\eta x}\delta_x + \phi''_{\eta\eta}\delta_\eta \\ \delta_\xi &= \phi''_{x\eta}\delta_\eta + \phi''_{xx}\delta_x\,. \end{aligned}$$

Here we can express δ_x and δ_ξ in terms of δ_y, δ_η and it follows that $d\kappa(0,0)$ is given by the matrix:

$$d\kappa(0,0) = \begin{pmatrix} (\phi''_{\eta x})^{-1} & -(\phi''_{\eta x})^{-1}\phi''_{\eta\eta} \\ \phi''_{xx}(\phi''_{\eta x})^{-1} & \phi''_{x\eta} - \phi''_{xx}(\phi''_{\eta x})^{-1}\phi''_{\eta\eta} \end{pmatrix}. \tag{7.7}$$

We find the following factorization:

$$\begin{aligned} &d\kappa(0,0) - 1 = \\ &\begin{pmatrix} -(\phi''_{\eta x})^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\phi''_{xx}(\phi''_{\eta x})^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi''_{xx} & \phi''_{x\eta} - 1 \\ \phi''_{\eta x} - 1 & \phi''_{\eta\eta}\,. \end{pmatrix}. \end{aligned} \tag{7.8}$$

In particular,

$$\det(d\kappa(0,0) - 1) = \frac{1}{\det \phi''_{\eta x}} \det \begin{pmatrix} \phi''_{xx} & \phi''_{x\eta} - 1 \\ \phi''_{\eta x} - 1 & \phi''_{\eta\eta} \end{pmatrix}. \tag{7.9}$$

Here

$$\begin{pmatrix} \phi''_{xx} & \phi''_{x\eta} - 1 \\ \phi''_{\eta x} - 1 & \phi''_{\eta\eta} \end{pmatrix}$$

is the Hessian of $\phi(x,\eta) - x\eta$. The stationary phase method applied to the trace of (7.2) gives

$$\operatorname{tr} B = \left(\left(\det \frac{1}{i} \begin{pmatrix} \phi''_{xx} & \phi''_{x\eta} - 1 \\ \phi''_{\eta x} - 1 & \phi''_{\eta\eta} \end{pmatrix} \right)^{-1/2} b_0(0,0) + \mathcal{O}(h) \right) e^{i\phi(0,0)/h}.$$

Here we choose the branch of the square root of the determinant on the set of non-degenerate symmetric matrices with non-negative real part which is equal to 1 for the identity. Using (7.9), we get (7.6). $\square$

To give a geometric meaning to the signature s appearing in (7.5) in terms of a *Maslov index* we first recall the definition of the Hörmander-Kashiwara index of a Lagrangian triple: let $\lambda_1, \lambda_2, \lambda_3$ be Lagrangian planes in a symplectic vector space (V, ω), and put

$$s(\lambda_1, \lambda_2, \lambda_3) = \operatorname{sgn} Q(\lambda_1, \lambda_2, \lambda_3)\,, \tag{7.10}$$

where $Q(\lambda_1, \lambda_2, \lambda_3)$ is a quadratic form on $\lambda_1 \oplus \lambda_2 \oplus \lambda_3$ given by

$$Q(\lambda_1, \lambda_2, \lambda_3)(v_1 \oplus v_2 \oplus v_3) = \omega(v_1, v_2) + \omega(v_2, v_3) + \omega(v_3, v_1)\,,$$

see [9] for a comprehensive introduction. Here we only mention that if λ_i's are mutually transversal, then $s(\lambda_1, \lambda_2, \lambda_3)$ is the only symplectic invariant of such three Lagrangian planes. It is antisymmetric and satisfies the cocycle condition.

We then have

Lemma 7.2. *Let $V = T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ with the symplectic form $\omega = \omega_1 - \omega_2$, where ω_1 and ω_2 are the canonical forms on the factors. In the notation of Lemma 7.1, let $\Gamma_{d\kappa}$ be the graph of $d\kappa(0,0)$, $\Delta \subset T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ be the diagonal, and $M = \{0\} \oplus \mathbb{R}^n \oplus \mathbb{R}^n \oplus \{0\} \subset T^*\mathbb{R}^n \times T^*\mathbb{R}^n$. Then*

$$s(\Gamma_{d\kappa}, \Delta, M) = -\mathrm{sgn} \begin{pmatrix} \phi''_{xx} & \phi''_{x\eta} - 1 \\ \phi''_{\eta x} - 1 & \phi''_{\eta\eta} \end{pmatrix} .$$

Proof. Let us write

$$\phi''(0,0) = \begin{pmatrix} \alpha & \beta \\ \beta^t & \gamma \end{pmatrix} .$$

Then $\Gamma_{d\kappa} = \{(x, \alpha x + \beta\eta; \beta^t x + \gamma\eta, \eta) \ : \ (x,\eta) \in \mathbb{R}^n \times \mathbb{R}^n\}$. Since Δ and M are transversal, [9, Lemma 1.5.4] says that

$$s(\Gamma_{d\kappa}, \Delta, M) = -\mathrm{sgn}\, \omega(\pi\bullet, \bullet)|_{\Gamma_{d\kappa}} ,$$

where $\pi : T^*\mathbb{R}^n \times T^*\mathbb{R}^n \to M$ is the projection along Δ: $\pi(x,\xi; y,\eta) = (0, \xi - \eta; y - x, 0)$. In the (x,η) coordinates on $\Gamma_{d\kappa}$, $\omega(\pi\bullet,\bullet)|_{\Gamma_{d\kappa}}$ is then given by $\langle \Omega\bullet, \bullet\rangle$, where

$$\Omega = \begin{pmatrix} \alpha & \beta - 1 \\ \beta^t - 1 & \gamma \end{pmatrix} ,$$

which proves the lemma. □

As is well known, and as will be seen in the proof of the next proposition, *any* locally defined Fourier Integral Operator can be represented by (7.2). To compute its trace in terms of invariantly defined objects we also have to recall the definition of the *Maslov index of a curve of linear symplectomorphisms* – see [2] for more details and references.

Thus let $\Gamma(t) \subset T^*\mathbb{R}^n \times T^*\mathbb{R}^n$, $a \le t \le b$, be a curve of graphs of symplectomorphisms. Choose a subdivision $a = t_0 < t_1 < \cdots < t_k = b$, such that, for all $j = 1, \cdots, k$, there is a Lagrangian subspace M^j transversal to $\Gamma(t)$ and the diagonal, Δ, for $t \in [t_{j-1}, t_j]$. We now follow [2] and define the *Maslov index* of a curve of linear symplectomorphisms as

$$\mu \stackrel{\text{def}}{=} \frac{1}{2} \sum_{j=1}^{k} \big(s(\Gamma(t_{j-1}), \Delta, M^j) - s(\Gamma(t_j, \Delta, M^j)) \big) . \tag{7.11}$$

It is independent of the choice of the transversal Lagrangians, M^j, and of the subdivision.

We can now prove

Proposition 7.3. *Suppose that $U(t)$ is a family of Fourier Integral Operators defined using a family of pseudodifferential operators, $A(t) \in \Psi^{0,0}(T^*X)$, as in (3.2):*

$$hD_t U(t) + A(t)U(t) = 0 , \quad U(0) = U_0 \in \Psi_h^{0,0}(T^*X) .$$

Let us also assume that $a_t = \sigma(A(t))$, the Weyl symbol (with a possible dependence on h in the subprincipal symbol part) of $A(t)$, is real and generates a family of canonical transformations:

$$\frac{d}{dt}\kappa_t(x,\xi) = (\kappa_t)_*(H_{a_t}(x,\xi)) , \quad \kappa_0(x,\xi) = (x,\xi) , \quad (x,\xi) \in T^*X$$
$$\kappa_t(0,0) = (0,0) .$$

If

$$\det(1 - d\kappa_T(0,0)) \neq 0 , \tag{7.12}$$

then

$$\operatorname{tr} U(T) = i^{\mu(T)} \frac{(1+\mathcal{O}(h)) e^{-i\int_0^T a_t(0,0)dt/h}}{|\det(d\kappa_T(0,0)-1)|^{\frac{1}{2}}} \sigma(U_0)(0,0)\,, \tag{7.13}$$

where $\mu(T)$ is the Maslov index of the curve of linear symplectic transformations $d\kappa_t(0,0)$, $0 \leq t \leq T$.

Proof. Let us first assume that for $0 \leq t \leq T$

$$(\kappa_t(y,\eta);(y,\eta)) \mapsto (x(\kappa_t(y,\eta)),x) \text{ is surjective near } (0,0)\,. \tag{7.14}$$

We follow the presentation from [6, Appendix a]. Let a_t be the Weyl-symbol of A_t defined modulo $\mathcal{O}(h^2)$ (if there is a subprincipal symbol we include it in the principal one and obtain an h dependent symbol). Consequently the influence of the subprincipal symbol will be accounted for as an $\mathcal{O}(h)$-dependence in the canonical transformation κ_t.) Let κ_t be the canonical transformation generated by H_{a_t} as described in the statement of the proposition. We can then view κ_t as the canonical transformation associated to $U(t)$ (defined modulo $\mathcal{O}(h^2)$) and we claim that

$$U(t)u(x) = \frac{1}{(2\pi h)^n} \iint e^{\frac{i}{h}(\phi(t,x,\eta)-y\cdot\eta)} b(t,x,\eta;h)u(y)dyd\eta\,, \tag{7.15}$$

where

$$\partial_t\phi(t,x,\eta) + a_t(x,\partial_x\phi) = 0, \quad \phi(0,x,\eta) = x\cdot\eta\,, \tag{7.16}$$

so that $\kappa_t : (\partial_\eta\phi,\eta) \mapsto (x,\partial_x\phi)$. The amplitude b has to satisfy

$$\begin{gathered}(hD_t + a_t^w(x,hD))(e^{i\phi(t,x,\eta)/h} b(t,x,\eta;h)) = 0\,, \\ (\partial_t\phi + hD_t + e^{-i\phi/h} a_t^w(x,hD) e^{i\phi/h})(b) = 0\,.\end{gathered} \tag{7.17}$$

Here the Weyl symbol of $e^{-i\phi/h} a_t^w e^{i\phi/h}$ is $q_t(x,\xi) = a_t(x,\phi'_x+\xi) + \mathcal{O}(h^2)$, and using that $\partial_t\phi = -a_t(x,\partial_x\phi)$, we get

$$(hD_t + \operatorname{Op}(f_t(x,\xi)))b = \mathcal{O}(h^2),$$

where $f_t(x,\xi) = a_t(x,\phi'_x+\xi) - a_t(x,\phi'_x)$ (and η is just a parameter). This gives

$$(hD_t + \frac{1}{2}\sum_1^n((\partial_{\xi_j}a_t)hD_{x_j} + hD_{x_j}\circ(\partial_{\xi_j}a_t)))b_0 = 0\,, \tag{7.18}$$

for the leading part of b. With $\nu_t = \sum(\partial_{\xi_j}a_t)\partial_{x_j}$, this can also be written

$$(\partial_t + \nu_t + \frac{1}{2}\operatorname{div}\nu_t)b_0 = 0,$$

or

$$(\partial_t + \mathcal{L}_{\nu_t})(b_0(t,x,\eta)(dx_1\wedge\ldots\wedge dx_n)^{1/2}) = 0\,, \tag{7.19}$$

where $\mathcal{L}_{\nu_t}$ denotes the Lie derivative. If we consider $b_0(t,x,\eta)(dx_1\wedge\ldots\wedge dx_n)^{1/2}$ as a half-density on $\Lambda_{\phi_{t,\eta}} = \{(x,\partial_x\phi_t(x,\eta))\}$, then (7.19) means that

$$\kappa_t^*(b_0(t,x,\eta)(dx_1\wedge\ldots\wedge dx_n)^{1/2}|_{\Lambda_{\phi_{t,\eta}}}) = (dx_1\wedge\ldots\wedge dx_n)^{1/2}|_{\Lambda_{\phi_{0,\eta}}}\,. \tag{7.20}$$

From (7.7) it follows that the restriction of the differential of κ_t to $T\Lambda_{\phi_{0,\eta}}$ followed by the x-space projection is given by

$$\delta_y \mapsto (\phi''_{\eta x})^{-1}(\delta_y),$$

so (7.20) says that

$$\frac{b_0(t,x,\eta)}{(\det \phi''_{\eta x})^{1/2}} = 1\,. \tag{7.21}$$

We note that $\det \phi''_{\eta x} > 0$ for $0 \leq t \leq T$.

From (7.16) and $d\phi_t(0,0) = 0$ (since $\kappa_t(0,0) = (0,0)$) we see that

$$\phi_T(0,0) = -\int_0^T a_t(0,0)dt\,.$$

Applying Lemma 7.1 and Lemma 7.2 we obtain (7.13): under the assumption (7.14) we only need one transversal Lagrangian in (7.11), and we can take M from Lemma 7.2. Then

$$\begin{aligned}
\mu(T) &= \frac{1}{2}\left(s(\Gamma_{d\kappa_0}, \Delta, M) - s(\Gamma_{d\kappa_T}, \Delta, M)\right) \\
&= \frac{1}{2}\left(s(\Delta, \Delta, M) - s(\Gamma_{d\kappa_T}, \Delta, M)\right) \\
&= -\frac{1}{2}s(\Gamma_{d\kappa_T}, \Delta, M) = \frac{1}{2}\mathrm{sgn}\begin{pmatrix} \phi''_{xx} & \phi''_{x\eta} - 1 \\ \phi''_{\eta x} - 1 & \phi''_{\eta\eta} \end{pmatrix}.
\end{aligned}$$

In the case (7.14) does not hold for $0 \leq t \leq T$, we have to choose different coordinates in which (7.14) holds for $t_{j-1} \leq t \leq t_j$. That gives corresponding Lagrangians M^j (defined as M was) and the phase shifts add up precisely to give (7.11). In fact, we can conjugate $U(t)$ by an h-Fourier Integral Operator (so without affecting the trace), so that for $t_1 - \delta < t < t_2 + \delta$ the resulting operator is given by

$$\widetilde{U}(t)u(x) = \frac{i^\nu}{(2\pi h)^n}\iint e^{\frac{i}{h}(\tilde{\phi}(t,x,\eta) - y\cdot\eta)}\tilde{b}(t,x,\eta;h)u(y)dyd\eta\,,$$

where we can arrange that $\tilde{b} > 0$, and that (7.12) holds with $T = t_1, t_2$. We then use Lemma 7.1 and the geometric discussion above to compute the trace:

$$\mathrm{tr}\,U(t_1) = i^\nu i^{-\frac{1}{2}\tilde{s}}\frac{(1+\mathcal{O}(h))e^{-i\int_0^{t_1} a_{t_1}(0,0)dt/h}}{|\det(d\kappa_{t_1}(0,0) - 1)|^{\frac{1}{2}}}\sigma(U_0)(0,0)\,,$$

where

$$\tilde{s} = -\mathrm{sgn}\begin{pmatrix} \tilde{\phi}''_{xx} & \tilde{\phi}''_{x\eta} - 1 \\ \tilde{\phi}''_{\eta x} - 1 & \tilde{\phi}''_{\eta\eta} \end{pmatrix}.$$

As in Lemma 7.2 we interpret $\tilde{s}$ as

$$\tilde{s} = s(\Gamma_{d\kappa_{t_1}}, \Delta, M^2)\,,$$

where M_2 was chosen in new coordinates. Comparing it with the previous expression for the trace (where we put $T = t_1$ and $M = M^1$), we see that

$$\nu = -s(\Gamma_{d\kappa_{t_1}}, \Delta, M^1) + s(\Gamma_{d\kappa_{t_1}}, \Delta, M^2)\,.$$

We now use $\widetilde{U}(t)$ to compute the trace at $t = t_2$ which in view of the expression for ν, and the fact that $s(\Gamma_{d\kappa_0}, \Delta, M^1) = 0$ is

$$\begin{aligned}
\mathrm{tr}\,U(t_2) =& i^{\frac{1}{2}(s(\Gamma_{d\kappa_0},\Delta,M^1) - s(\Gamma_{d\kappa_{t_1}},\Delta,M^1) + s(\Gamma_{d\kappa_{t_1}},\Delta,M^2) - s(\Gamma_{d\kappa_{t_2}},\Delta,M^2))}\times \\
&\frac{(1+\mathcal{O}(h))e^{-i\int_0^{t_1} a_{t_1}(0,0)dt/h}}{|\det(d\kappa_{t_1}(0,0) - 1)|^{\frac{1}{2}}}\sigma(U_0)(0,0)\,,
\end{aligned}$$

and comparing with (7.11) see that the power of i is given by the Maslov index for the curve $d\kappa_t(0,0)$, $0 \leq t \leq t_2$. We can continue in the same way which gives us the final index $\mu(T)$.

□

We now want to evaluate the trace of $M(z,h)^k M'(z,h)$, and for this we need to identify the Maslov factor and the phase. For this we recall the definition of the classical action:

$$I(z) \stackrel{\text{def}}{=} \int_{\gamma(z)} \xi dx \,. \tag{7.22}$$

The well known relation with the periods is given in

Lemma 7.4. *Let $q(z)$ and $q_{\circlearrowleft}(z)$ be the local time and the first return local time defined in* (4.2) *and* (4.3). *Then*

$$(q_{\circlearrowleft}(z) - q(z))|_{\gamma(z)} = -\int_0^{T(z)} \sigma(\partial_z P(z))(\exp(tH_{p(z)})(m_0(z))dt \,,$$

$$(q_{\circlearrowleft}(z) - q(z))|_{\gamma(z)} = \frac{dI}{dz}(z) \,.$$

Proof. The first identity follows directly from the definition and was already used in the proof of Lemma 6.2.

Since $\partial_z p(z) \neq 0$, we can write $p(z) = c(z)(z - \tilde{p})$. Hence on $p(z) = 0$, the equations for q and $q_{\circlearrowleft}$ are

$$H_{\tilde{p}} q = -1 \,, \quad H_{\tilde{p}} q_{\circlearrowleft} = -1 \,,$$

and consequently $(q_{\circlearrowleft}(z) - q(z))|_{\gamma(z)}$ is the period of $\gamma(z)$, thought of as an orbit of $H_{\tilde{p}}$ on $\tilde{p} = z$. We now introduce an isotropic submanifold, Γ, of $T^*(X \times \mathbb{R})$, where the new variable (on $\mathbb{R}$) is denoted by ζ with z its dual variable:

$$\Gamma = \{(m; (\zeta, z)) \in T^*(M \times \mathbb{R}) \;:\; m \in \gamma(z) \,,\; \zeta = q(z)(m) \,, z_0 \leq z \leq z_1\} \,.$$

The symplectic form $d\xi \wedge dx + dz \wedge d\zeta$ vanishes on Γ, and hence we obtain from Stokes's theorem:

$$\begin{aligned} I(z_1) - I(z_0) &= \int_{\gamma(z_1)} \xi dx - \int_{\gamma(z_2)} \xi dx \\ &= \int_{z_0}^{z_1} (q(z)(m_0(z)) - q_{\circlearrowleft}(z)(m_0(z)))\, dz = \int_{z_0}^{z_1} T(z) dz \,, \end{aligned}$$

which proves the lemma. □

Using this lemma we will be able to identify the phase in the trace of the monodromy operator. For that let $\mathcal{T}(z)$ be the quantum time appearing in (4.11):

$$\mathcal{T}(z) = K(z)^{-1}(Q(z) - Q_{\circlearrowleft}(z))K(z) \,,$$

so that that formula becomes

$$hD_z M(z) = \mathcal{T}(z) M(z) \,. \tag{7.23}$$

This and Proposition 7.3 show that the phase factor in $\operatorname{tr} M(z,h)^k \mathcal{T}(z)$ satisfies

$$J_k'(z) = k(q_{\circlearrowleft}(z) - q(z))|_{\gamma(z)} \,.$$

In fact, for any family of $P(z,h)$ satisfying the assumptions of Proposition 5.1, we can associate to $M(z,h)^k$ (not necessarily satisfying the non-degeneracy condition) a phase factor, $J_k(z)$ which has to satisfy

$$J_k(z) = kI(z) + C_k \,,$$

We want to show that $C_k = 0$. For that we note that if we put $P_\epsilon(z,h) = P(z,h/\epsilon)$ then the corresponding $J_k(z)$ is given by $J_k(z)\epsilon$. On the other hand, the action corresponding to P_ϵ is $kI(z)\epsilon$. Since we can consider P_ϵ as another deformation of our operator we must have

$$\forall\ \epsilon > 0\,, \quad J_k(z)\epsilon = kI(z)\epsilon + C_k\,,$$

and for that we need that $C_k = 0$.

To obtain the Maslov factor we need to find a family of symplectic transformations interpolating between the identity and the Poincaré map. For that let us fix z and supress dependence on z in the subsequent formulæ. We want to define a family $M(t) : \mathcal{D}'(\mathbb{R}^{n-1}) \to \mathcal{D}'(\mathbb{R}^{n-1})$, of h-Fourier Integral Operators such that $M(0) = Id$ and $M(T) = M$. To do this we modify the definition of I_+ in (4.6) to

$$I_+(t) \ : \ \ker_{m_0} P \ \longrightarrow \ \ker_{\exp tH_p m_0}(P)$$

We also generalize the definition of K to

$$K(t) \ : \ \mathcal{D}'(\mathbb{R}^{n-1}) \ \longrightarrow \ \ker_{\exp tH_p m_0}(P)\,,$$

defined using Proposition 3.5 as in (3.11). We can now define

$$M(t) \stackrel{\text{def}}{=} K(t)^{-1} I_+(t) K(0) \ : \ \mathcal{D}'(\mathbb{R}^{n-1}) \ \longrightarrow \mathcal{D}'(\mathbb{R}^{n-1})\,, \tag{7.24}$$

microlocally near $(0,0)$. This family has desired properties and quantizes a curve of local symplectomorphism κ_t:

$$\kappa_t = (\Phi_t)^{-1}\Psi_t\Phi_0 \ : \ T^*\mathbb{R}^{n-1} \ \longrightarrow \ T^*\mathbb{R}^{n-1}\,, \quad \kappa_t(0,0) = (0,0)\,,$$

where Φ_t symplectically identifies a neighbourhood of $(0,0)$ in $T^*\mathbb{R}^{n-1}$ with a submanifold of $p = 0$, S_t, transversal to γ at $\exp tH_p(m_0)$, and $\Psi_t : S_0 \longrightarrow S_t$ is the restriction of the flow $\exp(sH_p)$ to S_t. The construction above allows an arbitrary choice of S_t and Φ_t.

We can summarize this discussion in

Proposition 7.5. *Suppose that the orbit $\gamma(z)$ is primitive and N-fold non-degenerate in the sense that* (7.1) *holds. Let $I(z)$ be the classical actions defined by* (7.22)*, and $T(z)$ the periods of $\gamma(z)$.*

If t, $0 \leq t \leq T(z)$ parametrizes $\gamma(z)$, let S_t be a family of submanifolds of $p(z) = 0$, transversal to $\gamma(z)$ at t, Φ_t a symplectic identification of S_t with a neighbourhood of $(0,0)$ *in $T^*\mathbb{R}^{n-1}$, and $\Psi_t : S_0 \to S_t$ the restriction of the flow to S_t. Then for $0 < |k| \leq N$,*

$$\operatorname{tr} M(z,h)^{k-1} hD_z M(z,h) = \frac{e^{ikI(z)} e^{i\nu_k(z)\frac{\pi}{2}} (q_{\circlearrowleft}(z) - q(z))|_{\gamma(z)}}{|(dC(z)_{m_0(z)})^k - 1|^{\frac{1}{2}}} (1 + \mathcal{O}(h))\,,$$

where $\nu_k(z)$ is the Maslov index of the curve of linear symplectic transformations:

$$d(\Phi_t^{-1}\Psi_t\Phi_0)_{(0,0)}\,, \quad 0 \leq t \leq kT(z)\,.$$

Remark. The Maslov index $\nu_k(z)$ is a locally constant function of z: it does not change as long as (7.1) holds. Its value may depend on the non-unique choices of the identifications Φ_t, and the transversals S_t. Since $\exp(i\nu_k\pi/2)$ is determined uniquely (as it appears in the trace!), ν is termined only modulo 4. In the case when $\gamma(z) \to \pi(\gamma(z))$ is a diffeomorphism, with $\pi : T^*X \to X$, the natural projection, a choice of transversals submanifolds in the base gives natural S_t's in $\{p = 0\} \subset T^*X$. Thus in the case of the geodesic flow on a Riemannian manifold ν is the index of a closed geodesic.

The usual semi-classical trace formula for non-degenerate orbits follows from Theorem 1 and the following

Proposition 7.6. *Suppose that the assumptions of Theorem 1 are satisfied, $M(z)$ is the quantum monodromy defined in* (4.9)*, and in addition the closed orbit $\gamma = \gamma(0)$ is N-fold non-degenerate* (7.1)*. Then, for $k \neq 0$, $|k| \leq N$, and $g \in \mathcal{C}^\infty_c(\mathbb{R})$, we have*

$$\frac{1}{2\pi i}\operatorname{tr}\int_{\mathbb{R}} \widehat{g}(z/h)M(z,h)^{k-1}\frac{d}{dz}M(z,h)\chi(z)dz = \frac{e^{ikS_\gamma/h + i\nu_{\gamma,k}\frac{\pi}{2}}T_\gamma g(kT_\gamma)}{|\det((dC_\gamma)^k - I)|^{\frac12}} + \mathcal{O}(h)\,, \tag{7.25}$$

where T_γ is the primitive period of γ, dC_γ is the linear Poincaré map, S_γ, the classical action of γ, and $\nu_{\gamma,k}$ the Maslov index of $k\gamma$.

Proof. Since P is assumed to be self-adjoint, the subprincipal symbold of P is zero. Let κ_z be the Poincaré map and assume that $(d\kappa_0(0,0))^k - 1$ is non-degenerate. Let $\mathcal{T}(z)$ be the quantum time appearing in (7.23) above. Using the cyclicity of the trace, we can write the left hand side of (7.25) as

$$\frac{1}{2\pi}\operatorname{tr}\int_{\mathbf{R}} \chi(z)\widehat{g}(\frac{z}{h})M(z)^k\mathcal{T}(z)\frac{dz}{h}. \tag{7.26}$$

The leading symbol of $\mathcal{T}(z)$ at the fixed point is the period $T(z) = dI(z)/dz$, where $I(z)$ is the action along the closed trajectory. Then to leading order, (7.26) becomes

$$\frac{i^\mu}{2\pi}\int \chi(z)\hat{g}(\frac{z}{h})\frac{e^{ikI(z)/h}}{|\det((d\kappa_z(0,0)^k - 1)|^{\frac12}}T(z)\frac{dz}{h}, \tag{7.27}$$

where μ is the Maslov index. Write $E = z/h$, so that

$$\frac{I(z)}{h} = \frac{I(0)}{h} + I'(0)E + \mathcal{O}(h).$$

Then (7.27) becomes, again to leading order,

$$\frac{i^\mu}{2\pi}\int \hat{g}(E)e^{ikI'(0)E}dE\frac{e^{ikI(0)/h}T(0)\chi(0)}{|\det((d\kappa_0(0,0))^k - 1)|^{\frac12}} = i^\mu\frac{e^{ikI(0)/h}T(0)\chi(0)g(kT(0))}{|\det((d\kappa_0(0,0))^k - 1)|^{\frac12}}. \tag{7.28}$$

□

The usual Gutzwiller trace formula for a more general class of operators is given in

Theorem 3. *Suppose that the assumptions of Theorem 1 hold and that in addition γ is an N-fold non-degenerate orbit in the sense that* (7.1) *holds. Then in the notation of Proposition 7.6 we have*

$$\operatorname{tr} f(P/h)\chi(P)A = \frac{1}{2\pi}\sum_{k=-N}^{N}\frac{e^{ikS_\gamma/h + i\nu_{\gamma,k}\frac{\pi}{2}}T_\gamma\hat{f}(-kT_\gamma)}{|\det((dC_\gamma)^k - I)|^{\frac12}} + \mathcal{O}(h)\,.$$

Appendix

In the classical treatment of pseudo-differential operators, the subprincipal symbol is invariant under coordinate changes when the pseudo-differential operators are considered as acting on half-densities. This invariance is particularly nice in the Weyl calculus, where the subprincipal symbol is contained in the leading symbol – see [8, Sect.18.5].

For the reader's convenience we present here a self-contained discussion of the analogous result in the semiclassical setting.

We use the informal notation for sections of the half-density bundles:

$$u \in \mathcal{C}^\infty(X, \Omega_X^{\frac12}) \iff u = u(x)|dx|^{\frac12}\,,$$
$$a \in S^{0,0}(T^*X, \Omega_{T^*X}^{\frac12}) \iff a = a(x,\xi)|dx|^{\frac12}|d\xi|^{\frac12}\,,$$

which captures the transformation laws under changes of coordinates:

$$u(x)|dx|^{\frac{1}{2}} = \tilde{u}(\tilde{x})|d\tilde{x}|^{\frac{1}{2}} \,,\ \tilde{x} = \kappa(x) \iff \tilde{u}(\kappa(x))|\kappa'(x)|^{\frac{1}{2}} = u(x)\,,$$

where for a linear tranformation A we denote its determinant by $|A|$.

We observe that the half-density sections over T^*X are identified with functions if we consider symplectic changes of variables, and in particular

$$(x,\xi) \longmapsto (\tilde{x},\tilde{\xi}) = (\kappa(x)\,, (\kappa'(x))^t\xi)\,. \tag{A.1}$$

As stated after (3.1) in this paper we considered pseudodifferential operators acting on half-densities:

$$\Psi_h^{m,k}(X) = \Psi_h^{m,k}(X, \Omega_X^{\frac{1}{2}})\,,$$

with distributional kernels given by

$$K_a(x,y)|dx|^{\frac{1}{2}}|dy|^{\frac{1}{2}} = \frac{1}{(2\pi h)^n}\int a\left(\frac{x+y}{2},\xi\right)e^{i\langle x-y,\xi\rangle/h}d\xi|dx|^{\frac{1}{2}}|dy|^{\frac{1}{2}}\,. \tag{A.2}$$

We will show that

$$K_a(x,y)|dx|^{\frac{1}{2}}|dy|^{\frac{1}{2}} = K_{\tilde{a}}(\tilde{x},\tilde{y})|d\tilde{x}|^{\frac{1}{2}}|d\tilde{y}|^{\frac{1}{2}} \quad \text{with } \tilde{a}(\tilde{x},\tilde{\xi}) = a(x,\xi) + \mathcal{O}(h^2\langle\xi\rangle^{-2}), \tag{A.3}$$

where $(\tilde{x},\tilde{\xi})$ is given by (A.1).

To establish (A.3) we start with its right hand side using the coordinates $\tilde{x} = \kappa(x)$ and $\tilde{y} = \kappa(y)$:

$$\frac{1}{(2\pi h)^n}\int \tilde{a}\left(\frac{\tilde{x}+\tilde{y}}{2},\tilde{\xi}\right)e^{i\langle\tilde{x}-\tilde{y},\tilde{\xi}\rangle/h}d\tilde{\xi}|d\tilde{x}|^{\frac{1}{2}}|d\tilde{y}|^{\frac{1}{2}}\,.$$

Making a substition we obtain

$$\frac{1}{(2\pi h)^n}\int \tilde{a}\left(\frac{\kappa(x)+\kappa(y)}{2},\tilde{\xi}\right)e^{i\langle\kappa(x)-\kappa(y),\tilde{\xi}\rangle/h}d\tilde{\xi}|\kappa'(x)|^{\frac{1}{2}}|\kappa'(y)|^{\frac{1}{2}}|dx|^{\frac{1}{2}}|dy|^{\frac{1}{2}}\,.$$

We now apply the "Kuranishi trick" and for that write

$$\begin{gathered}\kappa(x)-\kappa(y) = F(x,y)(x-y)\,,\quad F(x,y) = \kappa'\left(\frac{x+y}{2}\right) + \mathcal{O}((x-y)^2)\,,\\ \kappa(x)+\kappa(y) = \kappa\left(\frac{x+y}{2}\right) + \mathcal{O}((x-y)^2)\,.\end{gathered} \tag{A.4}$$

We put $\xi = F(x,y)^t\tilde{\xi}$ and rewrite the expression above as

$$\begin{aligned}&\frac{1}{(2\pi h)^n}\int\left(\tilde{a}\left(\kappa\left(\frac{x+y}{2}\right),(K(x,y)^t)^{-1}\xi\right)+\mathcal{O}(x-y)^2\right)e^{i\langle x-y,\xi\rangle/h}|K(x,y)^t|^{-1}\\ &\qquad d\xi|\kappa'(x)|^{\frac{1}{2}}|\kappa'(y)|^{\frac{1}{2}}|dx|^{\frac{1}{2}}|dy|^{\frac{1}{2}} =\\ &\frac{1}{(2\pi h)^n}\int\left(\tilde{a}\left(\kappa\left((x+y)/2\right),\left(\kappa'\left((x+y)/2\right)^t\right)^{-1}\xi\right)+\mathcal{O}(x-y)^2\right)e^{i\langle x-y,\xi\rangle/h}|\kappa'((x+y)/2)|^{-1}\\ &\qquad d\xi|\kappa'(x)|^{\frac{1}{2}}|\kappa'(y)|^{\frac{1}{2}}|dx|^{\frac{1}{2}}|dy|^{\frac{1}{2}}\,,\end{aligned}$$

and the terms $\mathcal{O}((x-y)^2)$ will influence the symbol only modulo $\mathcal{O}(\langle\xi\rangle^{-2}h^2)$ (by integration by parts based on $(x-y)\exp(\langle x-y,\xi\rangle/h) = hD_\xi\exp(\langle x-y,\xi\rangle/h)$), and hence can be neglected.

We now observe that

$$|\kappa'((x+y)/2)|^2 = |\kappa'(y)||\kappa'(x)| + \mathcal{O}((x-y)^2)\,,$$

and consequently

$$K_{\tilde{a}}(\tilde{x},\tilde{y}) = \frac{1}{(2\pi h)^n}\int \left(\tilde{a}\left(\kappa\left((x+y)/2\right), \left(\kappa'\left((x+y)/2\right)^t\right)^{-1}\xi\right) + \mathcal{O}(h^2\langle\xi\rangle^{-2})\right) e^{i\langle x-y,\xi\rangle/h}|d\xi||dx|^{\frac{1}{2}}|dy|^{\frac{1}{2}},$$

which is the same as $K_a(x,y)|dx|^{\frac{1}{2}}|dy|^{\frac{1}{2}}$, if

$$a(x,\xi) = \tilde{a}\left(\kappa\left(x\right), \left(\kappa'\left(x\right)^t\right)^{-1}\xi\right) + \mathcal{O}(h^2\langle\xi\rangle^{-2}).$$

This proves (A.3) completing the appendix.

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