MATH 54 PRACTICE FINAL

Problem 1.
a) Write down a solution of the wave equation
\[ \partial_t^2 u - \partial_x^2 u = 0 , \]
obtained using separation of variables.
b) Solve the wave equation with the following initial condition:
\[ u(0, x) = \cos x + 5 \cos 3x + 8 \sin 4x , \quad u_t(0, x) = \sin x + 10 \sin 16x . \]

Problem 2.
a) Consider the function \( f(x) = \sin ax \) for \( |x| < \pi \) and extend it by periodicity with period \( T = 2\pi \). Assume that \( a \neq 0, \pm 1, \pm 2, \cdots \), and find the Fourier series of \( f(x) \). You can use the formula
\[ \int \sin au \sin bu \, du = \frac{\sin(a - b)u}{2(a - b)} - \frac{\sin(a + b)u}{2(a + b)} + C . \]
b) Show that for a sin series of a function \( g(x) \),
\[ \frac{2}{\pi} \int_0^\pi g(x)^2 \, dx = \sum_{m=1}^\infty b_m^2 . \]
c) Find the value of
\[ \sum_{m=1}^\infty \frac{m^2}{(m^2 - a^2)^2} , \]
as a function of \( a \).

Problem 3. Which of the direction fields (if any!) in Fig.1-4 corresponds to the equation
\[ x' = Ax , \quad A = \begin{bmatrix} -3 & -2 \\ 2 & 2 \end{bmatrix} . \]

Problem 4.
a) Define the Fourier Series of a function \( f(x) \).
b) Clearly state a condition which guarantees that the Fourier series converges to \( f(x) \).

Problem 5.
a) Let \( A \in M_{nn} \) and define \( V \) to be the set of all functions \( y = y(t) \in \mathbb{R}^n \) satisfying \( y' = Ay \). Show that \( V \) is a vector space.
b) Find a basis for $V$ when $n = 4$ and

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 6 & -2 & 0 \\ -5 & 0 & 0 & -2 \end{bmatrix}.$$ 

**Problem 6.** Which of the following statements is not equivalent to the remaining ones, $A \in M_{nn}$.

a) $NS(A) = \{0\}$.

b) $AX = 0 \implies X = 0$.

c) rank$A + \dim NS(A) = n$.

d) dim$CS(A) = n$.

**Problem 7.** Suppose that $V = \text{span} \{\cos x, \sin x, \cos 2x, \sin 2x\}$. Determine the solvability of the following equation: given $g \in V$, find $f \in V$ such that

$$f + f' + f'' = g,$$

where $f', f''$ are first and second derivatives with respect to $f$. (By determining solvability, one means finding if, given $g$, the solution, $f$, exists, and if is unique).

**Problem 8.**

a) State clearly what it means for a matrix to be diagonalizable.

b) How can you invert a matrix using diagonalization?

c) Express the determinant using eigenvalues.

**Problem 9.**

a) Define the determinant of a matrix

b) Find the determinant of

$$\begin{bmatrix} 4 & 1 & 2 & 1 \\ 3 & 0 & 0 & 0 \\ -1 & 0 & 2 & 1 \\ 4 & 5 & 0 & 2 \end{bmatrix}.$$ 

**Problem 10.**

a) Diagonalize

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$ 

b) What is the rank of the matrix above? What is its null space?
Problem 11.

a) Solve the following initial value problem:
\[ y'' - y' - 2y = 0, \quad y(0) = \alpha, \quad y'(0) = 2. \]
b) Find \( \alpha \) such that the solution approaches 0 as \( t \to \infty \).
c) Find two independent solutions of the equation above and compute their Wronskian. State Abel's Theorem and verify its validity in this case.

Problem 12.

a) Define the orthogonal projection, \( P \), into \( W \subset \mathbb{R}^n \).
b) If \( W \subset \mathbb{R}^n \) has dimension \( n - 1 \) we define the reflection of \( y \in \mathbb{R}^n \) with respect to \( W \) as
\[ R_y = 2Py - y. \]
Show that \( R : \mathbb{R}^n \to \mathbb{R}^n \) is a linear transformation.
c) Show that \( R(R_y) = y \).
d) Find \( R \) if \( W \subset \mathbb{R}^3 \) is the set of \( (x,y,z) \) satisfying \( x + y + z = 0 \).
Figure 2. Direction field (b)

Figure 3. Direction field (c)

Figure 4. Direction field (d)