POINTWISE BOUNDS ON QUASIMODES OF SEMICLASSICAL SCHRÖDINGER OPERATORS IN DIMENSION TWO

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Abstract. We prove optimal pointwise bounds on quasimodes of semiclassical Schrödinger operators with arbitrary smooth real potentials in dimension two. This end-point estimate was left open in the general study of semiclassical L^p bounds conducted by Koch-Tataru-Zworski [2]. However, we show that the results of [2] imply the two dimensional end-point estimate by scaling and localization.

1. Introduction

Let $\mathrm{g}_{ij}(x)$ be a positive definite Riemannian metric on $\mathbb{R}^2$ with the corresponding Laplace-Beltrami operator,

$$\Delta_{\mathrm{g}} u := \frac{1}{\sqrt{\bar{\mathrm{g}}}} \sum_{i,j} \partial_{x_j} \left(\mathrm{g}^{ij} \sqrt{\bar{\mathrm{g}}}\, \partial_{x_j} u \right), \quad (\mathrm{g}^{ij}) := (\mathrm{g}_{ij})^{-1}, \quad \bar{\mathrm{g}} := \det(\mathrm{g}_{ij}),$$

and let $V \in C^\infty(\mathbb{R}^2)$ be real valued. We prove the following general bound which was already established (under an additional necessary condition) in higher dimensions in [2], but which was open in dimension two:

Theorem 1.1. *Suppose that $h \leq 1$, and $u \in H^2_{\mathrm{loc}}(\mathbb{R}^2)$. Suppose that u satisfies*

$$\big\| -h^2 \Delta_{\mathrm{g}} u + V u \big\|_{L^2} \leq h\,, \qquad \|u\|_{L^2} \leq 1\,. \tag{1.1}$$

Then for all $K \Subset \mathbb{R}^2$,

$$\sup_{x \in K} |u(x)| \leq C_K\, h^{-\frac{1}{2}}\,, \tag{1.2}$$

where the constant C_K depends only on g, V, and K.

A function u satisfying (1.1) is sometimes called a weak quasimode. It is a local object in the sense that if u is a weak quasimode then ψu, $\psi \in C_c^\infty(\mathbb{R}^2)$ is also one, so the theorem is easily reformulated with g, V, and u defined on an open subset of $\mathbb{R}^2$. The localization is also valid in phase space: for instance if $\chi \in C_c^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ then $\chi^w(x, hD)u$ is also a weak

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quasimode – see [1, Chapter 7] or [4, Chapter 4] for the review of the Weyl quantization $\chi \mapsto \chi^w$.

If $\liminf_{|x|\to\infty} V > 0$, then $-h^2\Delta + V$ (defined on $C_c^\infty(\mathbb{R}^2)$) is essentially self-adjoint and the spectrum of $-h^2\Delta+V$ is discrete in a neighbourhood of 0 – see for instance [1, Chapter 4]. In this case weak quasimodes arise as *spectral clusters*:

$$w = \sum_{|E_j|\leq Ch} c_j w_j , \quad (-h^2\Delta + V)w_j = E_j w_j , \quad \langle w_j, w_k\rangle_{L^2} = \delta_{jk}, \quad \sum_j |c_j|^2 \leq 1. \tag{1.3}$$

Then $u = \chi w$, for any $\chi \in C_c^\infty(\mathbb{R}^2)$, is a weak quasimode in the sense of (1.1). Since $V(x) \geq c > 0$ for $|x| \geq R$, Agmon estimates (see for instance [1, Chapter 6]) and Sobolev embedding show that $|u(x)| \leq e^{-c/h}$, $c > 0$, for $|x| \geq R$. Hence we get global bounds

$$|w(x)| \leq Ch^{-\frac{1}{2}}, \quad x \in \mathbb{R}^2.$$

It should be stressed however that a weak quasimode is a more general notion than a spectral cluster.

The result also holds when $\mathbb{R}^2$ is replaced by a two dimensional manifold and, as in the example above, gives global bounds on spectral clusters (1.3) when the manifold is compact. If $V < 0$ this is also a by-product of of the Avakumovic-Levitan-Hörmander bound on the spectral function – see [3], and for a simple proof of a semiclassical generalization see [2, §3] or [4, §7.4].

In higher dimensions the theorem requires an additional phase space localization assumption and is a special case of [2, Theorem 6]: Suppose $p(x, \xi)$ is a function on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying $\partial_x^\alpha \partial_\xi^\beta p(x,\xi) = \mathcal{O}(\langle \xi \rangle^m)$ for some m. Suppose that $K \Subset \mathbb{R}^n$ and $\chi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, and that for $(x, \xi) \in \operatorname{supp} \chi$

$$p(x,\xi) = 0, \quad d_\xi p(x,\xi) = 0 \implies d_\xi^2 p(x,\xi) \ \text{ is nondegenerate.}$$

Then for $u(h)$ such that

$$\operatorname{supp} u(h) \subset K \,, \qquad u(h) = \chi^w(x, hD)u + \mathcal{O}_{\mathscr{S}}(h^\infty) \,, \tag{1.4}$$

we have

$$\|u(h)\|_\infty \leq C\, h^{-\frac{n-1}{2}} \Big(\, \|u(h)\|_{L^2} + \frac{1}{h} \|p^w(x,hD)u\|_{L^2} \Big), \qquad n \geq 3. \tag{1.5}$$

When $n = 2$ the bound holds with $(\log(1/h)/h)^{\frac{1}{2}}$, which is optimal in general if $d_\xi^2 p$ is not positive definite – see [2, §3, §6] and §3 below for examples.

A small bonus for Schrödinger operators in dimension two is the fact that the frequency localization condition in (1.4) required for (1.5) is not necessary – see (2.6) below. And as noted already, in all dimensions the compact support condition on u is easily dropped when working with local estimates on u.

The proof of Theorem 1.1 is reduced to a local result presented in Proposition 2.1. That result follows in turn from a rescaling argument involving several cases, some of which use the following result that forms part of [2, Corollary 1].

Theorem 1.2. *Suppose that* $u = u(h)$ *satisfies* (1.1), *and that* (1.4) *holds. If* $V(x) \neq 0$ *for* $x \in \operatorname{supp} u$, *or if* g^{ij} *is positive definite and* $dV(x) \neq 0$ *for* $x \in \operatorname{supp} u$, *then*

$$\|u\|_{L^\infty} = \mathcal{O}\big(h^{-\frac{n-1}{2}}\big)\,, \quad n \geq 2\,.$$

This result is the basis for Propositions 2.2 and 2.3 used in our proof. The case of Theorem 1.2 with $dV \neq 0$ is the most technically involved result of [2]. We do not know of any simpler way to obtain (1.2).

2. Proof of Theorem 1.1

By compactness of K, it suffices to prove uniform L^∞ bounds on u over a small ball about each point in K, where in our case the diameter of the ball can be taken to depend only on $\mathcal{C}^N$ estimates for g and V over a unit sized neighborhood of K, for some large N. Without loss of generality we consider a ball centered at the origin in $\mathbb{R}^2$. Let

$$B = \{x \in \mathbb{R}^2 \,:\, |x| < 1\}\,, \qquad B^* = \{x \in \mathbb{R}^2 \,:\, |x| < 2\}\,.$$

After a linear change of coordinates, we may assume that

$$\mathrm{g}^{ij}(0) = \delta^{ij}\,. \tag{2.1}$$

Next, by replacing $V(x)$ by $cV(cx)$ and $\mathrm{g}^{ij}(x)$ by $\mathrm{g}^{ij}(cx)$, for some constant $c \leq 1$ depending on the $\mathcal{C}^2$ norm of g and V over a unit neighborhood of K, we may assume that

$$\sup_{x \in B^*} |V(x)| + |dV(x)| \leq 2\,, \quad \sup_{x\in B^*} |d^2V(x)| + \sum_{i,j=1}^{2} |d\mathrm{g}^{ij}(x)| \leq .01\,. \tag{2.2}$$

This has the effect of multiplying h by a constant in the equation (1.1), which can be absorbed into the constant C_K in (1.2).

In general, we let

$$C_N = \sup_{x\in B^*} \sup_{|\alpha| \leq N} \Big(|\partial^\alpha V(x)| + \sum_{i,j=1}^{2} |\partial^\alpha \mathrm{g}^{ij}(x)| \Big)\,, \tag{2.3}$$

and will deduce Theorem 1.1 as a corollary of the following

Proposition 2.1. *Suppose* $h \leq 1$, *that* g, V *satisfy* (2.1) *and* (2.2), *and that* u *satisfies*

$$\big\| -h^2 \Delta_{\mathrm{g}} u + Vu \|_{L^2(B^*)} \leq h\,, \qquad \|u\|_{L^2(B^*)} \leq 1\,. \tag{2.4}$$

Then

$$\|u\|_{L^\infty(B)} \leq C\, h^{-\frac12}\,, \tag{2.5}$$

where the constant C *depends only on* C_N *in* (2.3) *for some fixed* N.

We start the proof of Proposition 2.1 by recording the following two propositions, which are consequences of Theorem 1.2.

Proposition 2.2. *Suppose that* (2.1)-(2.2) *hold, and that* $\frac{1}{2} \le |V(x)| \le 2$ *for* $|x| \le 2$*. If the following holds, and* $h \le 1$*,*

$$\big\| -h^2 \Delta_{\mathrm{g}} u + V u \big\|_{L^2(B^*)} \le h \,, \qquad \|u\|_{L^2(B^*)} \le 1 \,,$$

then $\|u\|_{L^\infty(B)} \le C\, h^{-\frac{1}{2}}$*, where* C *depends only on* C_N *in* (2.3) *for some fixed* N*.*

Proposition 2.3. *Suppose that* (2.1)-(2.2) *hold, and that* $V(0) = 0$ *and* $|dV(0)| = 1$*. If the following holds, and* $h \le 1$*,*

$$\big\| -h^2 \Delta_{\mathrm{g}} u + V u \big\|_{L^2(B^*)} \le h \,, \qquad \|u\|_{L^2(B^*)} \le 1 \,,$$

then $\|u\|_{L^\infty(B)} \le C\, h^{-\frac{1}{2}}$*, where* C *depends only on* C_N *in* (2.3) *for some fixed* N*.*

To see that these follow from Theorem 1.2, we first may assume that u is compactly supported in $|x| < \frac{3}{2}$. Indeed, the assumptions imply $\|du\|_{L^2(|x|<3/2)} \lesssim h^{-1}$, so that one may cut off u by a smooth function which is supported in $|x| < \frac{3}{2}$ and equals 1 for $|x| < 1$ without affecting the hypotheses. We may then modify g and V outside B^* so that (2.2)-(2.3) are global bounds.

In Proposition 2.3 above, since $|d^2 V| \le .01$, we have $.98 \le |dV(x)| \le 1.02$ for $|x| \le 2$, so since g is positive definite the conditions on g and V in Theorem 1.2 are met. We remark that the conditions of Proposition 2.3 guarantee that the zero set of V is a nearly-flat curve through the origin, although this is not strictly needed to apply the results of [2]. That the resulting constant C depends only on C_N for some fixed finite N follows from the proofs in [2].

Finally, the condition (1.4) that $u - \chi^w(x, hD)u = \mathcal{O}_{\mathscr{S}}(h^\infty)$ for some $\chi \in \mathcal{C}_c^\infty$ is not needed for Theorem 1.2 to hold for positive definite g^{ij} in dimension two. To see this, we note that if $|V| < 2$ and $|\mathrm{g}^{ij}(x) - \delta_{ij}| \le .02$ on the ball $|x| < 2$, then if u is supported in $|x| < \frac{3}{2}$ and $\varphi(\xi) = 1$ for $|\xi| < 4$, condition (1.1) implies that

$$\|(hD)^2 (u - \varphi(hD)u)\|_{L^2} = \mathcal{O}(h)\,.$$

This follows by the semiclassical pseudodifferential calculus (see [4, Theorem 4.29]), since for $\varphi_0 \in C_c^\infty(\mathbb{R}^2)$ with $\operatorname{supp} \varphi_0 \subset B^*$, $\varphi_0(x)(1 - \varphi(\xi))|\xi|^2/(|\xi|^2_{\mathrm{g}} + V(x)) \in S(\mathbb{R}^2 \times \mathbb{R}^2)$.

Hence, writing $\hat{u}(\xi)$ for the standard Fourier transform of u,

$$
\begin{aligned}
\|u-\varphi(hD)u\|_{L^\infty} &\leq \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} |1-\varphi(h\xi)||\hat{u}(\xi)|\,d\xi \\
&\leq C\int |h\xi|^2|1-\varphi(h\xi)||\hat{u}(\xi)|(1+|h\xi|^2)^{-1}\,d\xi \\
&\leq C\|(hD)^2(u-\varphi(hD)u)\|_{L^2}\left(\int_{\mathbb{R}^2}(1+|h\xi|^2)^{-2}\,d\xi\right)^{\frac12} \\
&\leq C h\, h^{-1} = C\,,
\end{aligned}
\tag{2.6}
$$

an even better estimate than required. Hence we are reduced to proving estimates on $\varphi(hD)u$, which by compact support of u satisfies (1.4).

We supplement Propositions 2.2 and 2.3 with the following two lemmas.

Lemma 2.4. *Suppose that* (2.1)-(2.2) *hold, and that* $|V(x)|\leq 99\,h$ *for* $|x|\leq 2h^{\frac12}$. *If the following holds, and* $h\leq 1$,

$$\big\|-h^2\Delta_{\rm g}u+Vu\big\|_{L^2(|x|<2h^{1/2})}\leq h\,,\qquad \|u\|_{L^2(|x|<2h^{1/2})}\leq 1\,,$$

then $\|u\|_{L^\infty(|x|<h^{1/2})}\leq C\,h^{-\frac12}$, *where* C *depends only on* C_N *in* (2.3) *for some fixed* N.

Proof. Consider the function $\tilde{u}(x)=h^{\frac12}u(h^{\frac12}x)$, and $\tilde{\rm g}^{ij}(x)={\rm g}^{ij}(h^{\frac12}x)$. Then, since $\|Vu\|_{L^2(|x|<2h^{1/2})}\leq 99h$, we have

$$\|\Delta_{\tilde{\rm g}}\tilde{u}\|_{L^2(|x|<2)}\leq 100\,,\qquad \|\tilde{u}\|_{L^2(|x|<2)}\leq 1\,.$$

Since the spatial dimension equals 2, interior Sobolev estimates yield $\|\tilde{u}\|_{L^\infty(|x|<1)}\leq C$, where we note that the conditions (2.1) and (2.2) hold for $\tilde{\rm g}$ since $h^{\frac12}\leq 1$. □

Lemma 2.5. *Suppose that* (2.1)-(2.2) *hold, and that* $\frac12 c\leq |V(x)|\leq 2c$ *for* $|x|\leq 2c^{\frac12}$. *If the following holds, and* $h\leq c\leq 1$,

$$\big\|-h^2\Delta_{\rm g}u+Vu\big\|_{L^2(|x|<2c^{1/2})}\leq h\,,\qquad \|u\|_{L^2(|x|<2c^{1/2})}\leq 1\,,$$

then $\|u\|_{L^\infty(|x|<c^{1/2})}\leq C\,h^{-\frac12}$, *where* C *depends only* C_N *in* (2.3) *for some fixed* N.

Proof. Let $\tilde{u}(x)=c^{\frac12}u(c^{\frac12}x)$, $\tilde{\rm g}^{ij}(x)={\rm g}^{ij}(c^{\frac12}x)$, and $\tilde{V}(x)=c^{-1}V(c^{\frac12}x)$. Note that the assumptions on $V(x)$ in the statement and in (2.2) imply that $|dV(x)|\leq c^{\frac12}$ for $|x|<2c^{1/2}$, so that $\tilde{V}$ satisfies (2.2), and the constants C_N in (2.3) can only decrease for $c\leq 1$. Then with $\tilde{h}=c^{-1}h\leq 1$,

$$\|-\tilde{h}^2\Delta_{\tilde{\rm g}}\tilde{u}+\tilde{V}\tilde{u}\|_{L^2(|x|<2)}\leq \tilde{h}\,,\qquad \|\tilde{u}\|_{L^2(|x|<2)}\leq 1\,.$$

By Proposition 2.2, we have $\|\tilde{u}\|_{L^\infty(|x|<1)}\leq C\tilde{h}^{-\frac12}$, giving the desired result. □

Proof of Proposition 2.1. It suffices to prove that for each $|x_0| < 1$ there is some $\frac{1}{2} \geq r > 0$ so that $\|u\|_{L^\infty(|x-x_0|<r)} \leq C\, h^{-\frac{1}{2}}$, with a global constant C. Without loss of generality we take $x_0 = 0$.

We will split consideration up into four cases, depending on the relative size of $|V(0)|$ and $|dV(0)|$. Since for h bounded away from 0 the result follows by elliptic estimates, we will assume $h \leq \frac{1}{4}$ so that $h^{\frac{1}{2}}$ below is at most $\frac{1}{2}$.

Case 1: $|V(0)| \leq h\,,\ |dV(0)| \leq 8h^{\frac{1}{2}}$. Since $|d^2V(x)| \leq .01$, then Lemma 2.4 applies to give the result with $r = h^{\frac{1}{2}}$.

Case 2: $|V(0)| \leq h\,,\ |dV(0)| \geq 8h^{\frac{1}{2}}$. Since we may add a constant of size h to V without affecting (2.4), we may assume $V(0) = 0$. By rotating we may then assume

$$V(x) = \beta x_1 + f_{ij}(x) x_i x_j\,,$$

where $\beta = |dV(0)| \geq 8h^{\frac{1}{2}}$. Dividing V by 4 if necessary we may assume $\beta \leq \frac{1}{2}$. Let $\tilde{u} = \beta u(\beta x)$, $\tilde{\mathrm{g}}^{ij}(x) = \mathrm{g}^{ij}(\beta x)$, and

$$\tilde{V}(x) = \beta^{-2} V(\beta x) = x_1 + f_{ij}(\beta x) x_i x_j\,.$$

With $\tilde{h} = \beta^{-2} h < 1$ we have

$$\| - \tilde{h}^2 \Delta_{\tilde{\mathrm{g}}} \tilde{u} + \tilde{V}\tilde{u}\|_{L^2(|x|<2)} \leq \tilde{h}\,, \qquad \|\tilde{u}\|_{L^2(|x|<2)} \leq 1\,.$$

Proposition 2.3 applies, since $\tilde{g}$ and $\tilde{V}$ satisfy (2.1)-(2.2), and the constants C_N in (2.3) for $\tilde{g}$ and $\tilde{V}$ are bounded by those for g and V. Thus $\|\tilde{u}\|_{L^\infty(|x|<1)} \leq C\tilde{h}^{-\frac{1}{2}}$, giving the desired result on u with $r = |dV(0)|$.

Case 3: $|V(0)| \geq h\,,\ |dV(0)| \leq 9|V(0)|^{\frac{1}{2}}$. In this case, with $c = |V(0)|$, it follows that $\frac{1}{2}c \leq |V(x)| \leq 2c$ for $|x| \leq \frac{1}{20}c^{\frac{1}{2}}$. We may apply Lemma 2.5 with V replaced by $\frac{1}{1600}V$ to get the desired result with $r = \frac{1}{40}|V(0)|^{\frac{1}{2}}$.

Case 4: $|V(0)| \geq h\,,\ |dV(0)| \geq 9|V(0)|^{\frac{1}{2}}$. Since $|d^2V(x)| \leq .01$, it follows that there is a point x_0 with $|x_0| \leq \frac{1}{8}|V(0)|^{\frac{1}{2}}$ where $V(x_0) = 0$. Since $|dV(x_0)| \geq 8|V(0)|^{\frac{1}{2}} \geq 8h^{\frac{1}{2}}$, we may translate and apply Case 2 to get L^∞ bounds on u over a neighborhood of radius $|dV(x_0)|$ about x_0. This neighborhood contains the neighborhood about 0 of radius $r = .9998\,|dV(0)|$. □

3. A counter-example for indefinite g.

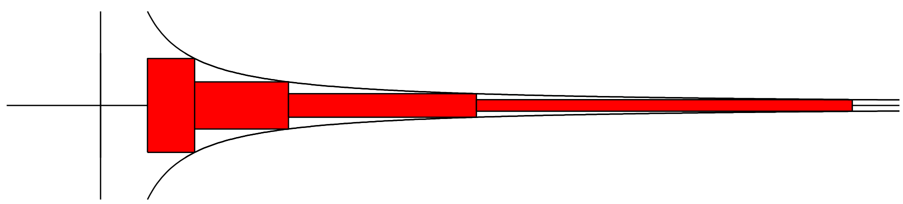
In [2, Section 5], it was shown that there exist u_h for which

$$\| - h^2(\partial_{x_1}^2 - \partial_{x_2}^2) u_h + (x_1^2 - x_2^2) u_h\|_{L^2} \leq h\,, \qquad \|u_h\|_{L^2} \leq 1\,, \tag{3.1}$$

for which $\|u_h\|_{L^\infty} \approx |\log h|^{\frac{1}{2}} h^{-\frac{1}{2}}$, showing that the assumption of definiteness of g cannot be relaxed to non-degeneracy in the main theorem. In [2, Theorem 6] the positive result was established showing that this growth of $\|u_h\|_{L^\infty}$ for indefinite, non-degenerate g in two dimensions is in fact worst case.

The example of [2] was produced using harmonic oscillator eigenstates. Here we present a different construction of such a u_h with similar L^∞ growth to help illustrate the role played by the degeneracy of g. The idea is to produce a collection $u_{h,j}$ of functions satisfying (3.1) (or equivalent), for which $u_{h,j}(0) = h^{-\frac{1}{2}}$, and where j runs over $\approx |\log h|$ different values. The examples will have disjoint frequency support, hence are orthogonal in L^2. Upon summation over j the L^2 norm then grows as $|\log h|^{\frac{1}{2}}$, whereas the L^∞ norm grows as $|\log h|\, h^{-\frac{1}{2}}$, yielding an example with worst case growth after normalization.

We start by considering the form $\xi_1\xi_2$ with $V = 0$. To assure that $\|h^2\partial_{x_1}\partial_{x_2}u_h\|_{L^2} \leq h$, we will take the Fourier transform of u_h to be contained in the set $|\xi_1\xi_2| \leq 2h^{-1}$, as well as $|\xi| \leq 2h^{-1}$ to satisfy the frequency localization condition [2, (1.4)]. Our example is then based on the fact that one can find $\approx |\log h|$ disjoint rectangles, each of volume h^{-1}, within this region, as illustrated in the diagram. Each $u_{h,j}$ will be an appropriately scaled Schwartz function with Fourier transform localized to one of the rectangles.



We now fix $\psi, \chi \in \mathcal{C}_c^\infty(\mathbb{R})$, with $0 \leq \psi(x) \leq 2$ and $0 \leq \chi(x) \leq 1$, with $\int \psi = \int \chi = 1$, and where

$$\operatorname{supp}\psi \subset [1,2]\,, \qquad \operatorname{supp}\chi \subset [-1,1]\,.$$

We additionally assume $\chi(0) = 1$.

Let

$$u_{h,j}(x) = h^{\frac{1}{2}} \int e^{ix_1\xi_1 + ix_2\xi_2}\chi(2^j h\,\xi_1)\psi(2^{-j}\xi_2)\,d\xi_1\,d\xi_2 = h^{-\frac{1}{2}}\check{\chi}(2^{-j}h^{-1}x_1)\check{\psi}(2^j x_2)\,.$$

By the Plancherel theorem, $\|u_{h,j}\|_{L^2} \approx 1$ and $\|h^2 D_1 D_2 u_{h,j}\|_{L^2} \lesssim h$. Furthermore, $u_{h,j}(0) = h^{-\frac{1}{2}}$. By disjointness of the Fourier transforms, for $i \neq j$ we have $\langle u_{h,i}, u_{h,j}\rangle = 0$, and similarly $\langle \partial_{x_1}\partial_{x_2}u_{h,i}, \partial_{x_1}\partial_{x_2}u_{h,j}\rangle = 0$.

We then form

$$u_h(x) = |\log h|^{-\frac{1}{2}} \sum_{1 \leq 2^j \leq h^{-1}} u_{h,j}(x)\,.$$

Since there are $\approx |\log h|$ terms in the sum, and the terms are orthogonal in L^2, it follows that

$$\|u_h\|_{L^2} \approx 1\,, \qquad \|h^2 \partial_{x_1}\partial_{x_2} u_h\|_{L^2(\mathbb{R}^2)} \lesssim h\,, \qquad u_h(0) \approx |\log h|^{\frac12} h^{-\frac12}\,.$$

Although the example is not compactly supported, it is rapidly decreasing (uniformly so for $h < 1$), and one may smoothly cutoff to a bounded set without changing the estimates.

We observe that for this example it also holds that

$$\|x_1 x_2 u_h\|_{L^2} \lesssim h\,.$$

Hence, u_h is also a counterexample for the form $\xi_1\xi_2 \pm x_1x_2$. Rotating by $\pi/4$ gives the form $\xi_1^2 - \xi_2^2 \pm (x_1^2 - x_2^2)$, including in particular the form considered in [2, Section 6].

We also observe that $x_1^2 u_h$ will be $\mathcal{O}_{L^2}(h)$ if one restricts the sum in u_h to $1 \le 2^j \le h^{-\frac12}$, which still has $\approx |\log h|$ values of j, and thus exhibits the same L^∞ growth as u_h. This idea does not however work to yield a counterexample for the form $\xi_1\xi_2 + x_1^2 + x_2^2$.

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