DYNAMICAL DEFINITION OF TRANSMISSION RATE

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1. Introduction

In this note we justify the definition of the transmission rate given in [1] in the case of *no* interaction, that is for the linear Schrödinger operator.

2. Review of one dimensional scattering

We follow lecture notes [2] covering scattering by compactly supported potentials but make the exposition self-contained, referring to [2] for detailed proofs only. In our approach to scattering we eventually focus on quantum resonances and hence the resolvent plays a crucial rôle.

Thus let $V \in L^\infty_{\rm comp}(\mathbb{R})$ be real valued, and define

$$H_V \stackrel{\rm def}{=} -\partial_x^2 + V(x)\,.$$

Then the resolvent

$$R_V(\lambda) \stackrel{\rm def}{=} (H_V - \lambda^2)^{-1} \;:\; L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})\,,$$

is meromorphic in $\mathbb{C} \setminus \mathbb{R}$. If we consider

$$R_V(\lambda) \;:\; L^2_{\rm comp}(\mathbb{R}) \longrightarrow L^2_{\rm loc}(\mathbb{R})\,, \quad \operatorname{Im}\lambda > 0\,,$$

then it continues meromorphically to $\mathbb{C}$ and it is analytic in $\mathbb{R}\setminus\{0\}$. Of course, for $\operatorname{Im}\lambda < 0$ this continuation does *not* coincide with the resolvent defined in $\mathbb{C} \setminus \mathbb{R}$. The poles in $\operatorname{Im}\lambda < 0$ are called quantum resonances. We recalled the following important fact: for any $\chi \in \mathcal{C}^\infty_{\rm c}(\mathbb{R})$, there exists a constant $C > 0$ such that

$$\|\chi R_V(\lambda)\chi\|_{\mathcal{L}(L^2,L^2)} \leq C\frac{\exp(-\operatorname{Im}\lambda/C)}{|\lambda|}\,, \quad \operatorname{Im}\lambda > -\log(1+|\operatorname{Re}\lambda|)/C\,. \tag{2.1}$$

Also, the pole of $R_V(\lambda)$ at $\lambda = 0$ has at most multiplicity one, that is, $\lambda R_V(\lambda)$ is always analytic near 0.

We define special solutions, $e_\pm(x,\lambda)$, to $(H_V - \lambda^2)e_\pm = 0$, by requiring that

$$e_\pm(x,\lambda) = \begin{cases} T_\pm(\lambda)e^{\pm i\lambda x} & \text{for} \quad \pm x \gg 0 \\ e^{\pm i\lambda x} + R_\pm(\lambda)e^{\mp i\lambda x} & \text{for} \quad \pm x \ll 0 \end{cases} \tag{2.2}$$

It is not obvious that such solutions have to exist and in fact for complex values of λ they do not, precisely at resonant energies. To see that $e_\pm$ exist when $\lambda \in \mathbb{R}$ we introduce the important notion of *quantum flux* of a (possibly time dependent) wave function $u(x)$:

$$F_u(x) \stackrel{\text{def}}{=} 2\,\mathrm{Im}(\partial_x u \overline{u(x)})\,. \tag{2.3}$$

We have the following simple but important

Lemma 1. *Suppose that* $(H_V - \lambda^2)u = 0$ *and* $\lambda^2 \in \mathbb{R}$. *Then* $\partial_x F_u(x) \equiv 0$, *that is the* quantum flux *is constant.*

Proof. This is the standard calculation:

$$\partial_x F_u(x) = 2\,\mathrm{Im}(\partial_x^2 u(x)\overline{u(x)} + |\partial_x u(x)|^2) = 2\,\mathrm{Im}((V - \lambda^2)|u(x)|^2) = 0\,.$$

□

We can now conclude that the solutions $e_\pm(x, \lambda)$ exist for $\lambda \in \mathbb{R} \setminus \{0\}$. In fact, we can always find a solution $u_\pm$ equal to $e^{\pm i\lambda x}$ for $\pm x \gg 0$. That solution has to equal to $a_\pm e^{\pm i\lambda x} + b_\pm e^{\mp i\lambda x}$ for $\pm x \ll 0$, and we need to show that $a_\pm \neq 0$. If not than

$$F_{u_\pm}(x)\restriction_{\pm x\gg 0} = \pm 2\lambda \neq \mp 2|b_\pm|^2\lambda = F_{u_\pm}(x)\restriction_{\pm x \ll 0}\,,$$

contradicting Lemma 1.

The coefficient $T(\lambda)$ in (2.2) is called the transmission coefficient and, $R_\pm(\lambda)$ the reflection coefficients.

We can write down the expression for $R_V(\lambda)$ in terms of $e_\pm$. For $\lambda \in \mathbb{R}\setminus\{0\}$, we have

$$R_V(\lambda)(x, y) = \frac{1}{2i\lambda T(\lambda)} \left(e_+(x, \lambda)e_-(y, \lambda)(x - y)_+^0 + e_+(y, \lambda)e_-(x, \lambda)(x - y)_-^0 \right)\,. \tag{2.4}$$

In particular we have the following *far field* expression for the resolvent:

$$R_V(\lambda)(\pm r, y) = \frac{1}{2i\lambda}\, e^{\pm i\lambda r}\, e_\mp(y, \lambda) \qquad \text{for } \ r \gg 0\,. \tag{2.5}$$

The following two lemmas are special cases of more precise results allowing for very general potentials and not requiring cut-offs. We present simple proofs in our special setting.

Lemma 2. *Suppose that* $\chi \in \mathcal{C}_{\rm c}^\infty(\mathbb{R})$. *Then*

$$\|\chi e^{-itH_V}\chi\|_{\mathcal{L}(L^1, L^\infty)} \leq C t^{-\frac{1}{2}}\,, \quad t > 0\,.$$

Proof. We first consider t small. We then apply Duhamel's formulæ,

$$\begin{aligned} e^{-itH_V} &= e^{-itH_0} + i\int_0^t e^{-i(t-s)H_0} V e^{-isH_V} ds \\ &= e^{-itH_0} - i\int_0^t e^{-i(t-s)H_V} V e^{-isH_0} ds\,, \end{aligned}$$

to write

$$e^{-itH_V} = e^{-itH_0} + i\int_0^t e^{-i(t-s)H_0} V e^{-isH_0} ds + \int_0^t \int_0^s e^{-i(t-s)H_0} V e^{-i(s-r)H_V} V e^{-irH_0} ds\, dr\,.$$

This gives,

$$\|\chi e^{-itH_V}\chi\|_{\mathcal{L}(L^1,L^\infty)} \le \|e^{-itH_0}\|_{\mathcal{L}(L^1,L^\infty)} + \int_0^t \|e^{-i(t-s)H_0} V\|_{\mathcal{L}(L^\infty,L^\infty)} \|e^{-isH_0}\|_{\mathcal{L}(L^1,L^\infty)} ds$$
$$+ \int_0^t \int_0^s \|\chi e^{-i(t-s)H_0} V\|_{\mathcal{L}(L^2,L^\infty)} \|e^{-i(s-r)H_V}\|_{\mathcal{L}(L^2,L^2)} \|V e^{-irH_0}\chi\|_{\mathcal{L}(L^1,L^2)} ds\, dr\,.$$

Hence, for $t \le C$,

$$\begin{aligned}\|\chi e^{-itH_V}\chi\|_{\mathcal{L}(L^1,L^\infty)} &\le C_0 t^{-\frac{1}{2}} + \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} ds + \int_0^t \int_0^s (t-s)^{-\frac{1}{2}} r^{-\frac{1}{2}} ds dr \\ &\le C_1 t^{-\frac{1}{2}}\,.\end{aligned}$$

For large t we use the representation of the propagator using the resolvent:

$$e^{-itH_V} = \int_0^\infty \frac{\lambda}{\pi i} (R_V(\lambda) - R_V(-\lambda)) e^{-i\lambda^2 t} d\lambda\,. \tag{2.6}$$

Now, we can deform the contour and use (2.1). The contribution from large λ's gives exponential decay in t, and the norm of the contribution from a neighbourhood of $\lambda = 0$ is bounded by

$$\int_0^1 e^{-\lambda^2 t} d\lambda = \mathcal{O}(t^{-\frac{1}{2}})\,.$$

We note that we can insert H_V as λ^2 into the integrand and hence the bound holds between any Sobolev spaces. In particular that shows a bounds between L^1 and L^∞ completing the proof. □

When the resolvent is analytic near 0 we easily obtain an improvement:

Lemma 3. *Suppose that $R_V(\lambda)$ is regular at $\lambda = 0$, that is V does not have a* zero resonance. *Then*

$$\|\chi e^{-itH_V}\chi\|_{\mathcal{L}(L^1,L^\infty)} \le C t^{-\frac{3}{2}}\,, \quad t > 0\,.$$

Proof. In (2.6) we now observe that analyticity of $R_V(\lambda)$ near 0 implies that $\lambda(R_V(\lambda) - R_V(-\lambda)) = \lambda^2 A(\lambda)$ where $A(\lambda)$ is analytic near 0. The contour deformation argument in the proof of Lemma 2 still produces exponential decay in t for λ large, but for λ small we now have the estimate

$$\int_0^1 e^{-\lambda^2 t} \lambda^2 d\lambda = \mathcal{O}(t^{-\frac{3}{2}})\,,$$

which proves the lemma. □

3. Transmission rate using time dependent Schrödinger equation

Following [1] we consider the following scattering problem: at some point left to the support of the potential we add a source term emitting waves and we consider the resulting time evolution. The transmission rate is naturally defined as the ratio of fluxes left to the source point with and without the potential. As we show, at least for potentials without a zero resonance, that definition coincides with the standard stationary definition. This is the content of the following

Theorem. *Suppose that* $V \in L^\infty_{\rm comp}(\mathbb{R})$ *does not have a* zero resonance *and consider the solution of*

$$iu_t = -u_{xx} + Vu + e^{-it\lambda^2}\delta_{x_0}(x)\,, \quad u(x,0) = 0\,. \tag{3.1}$$

Then for any $x \in \mathbb{R}$,

$$u(t,x) = e^{-it\lambda^2} R_V(x,x_0) + \mathcal{O}(t^{-\frac12})\,. \tag{3.2}$$

In particular, if $x_0 < \min_{\operatorname{supp} V} x$, *and* u_0 *denotes the solution of the problem with* $V = 0$, *then for* $x > x_0$,

$$\frac{F_u(x,t)}{F_{u_0}(x,t)} = |T(\lambda)|^2 + \mathcal{O}(t^{-\frac12})\,. \tag{3.3}$$

We start with the following

Lemma 4. *Let* $v_\pm = v_\pm(x,t)$ *be the solutions of the following initial value problems,*

$$i\partial_t v_\pm = -\partial_x^2 v_\pm\,, \quad v_\pm(x,0) = 1\!\!1_{\mathbb{R}_\pm}(x) e^{i\lambda|x|}\,, \quad \lambda > 0\,. \tag{3.4}$$

Then, there exists $C > 0$ *such that for any* $x \in \mathbb{R}$, *there exist* $t_0(x) > 0$, *so that*

$$|v_\pm(x,t)| \leq Ct^{-\frac12}\,, \quad t > t_0(x) \tag{3.5}$$

Proof. This is a direct computation based on the explicit formula for the solution. We consinder the case of v_+ as the other case is identical. Thus,

$$v_+(x,t) = \lim_{\epsilon \to 0+} \frac{1}{(4\pi t)^{\frac12}} \int_0^\infty e^{i\frac{|x-y|^2}{4t}} e^{iy\lambda - \epsilon y} dy\,.$$

We will drop ϵ in the subsequent computations noting that inserting it justifies the integration procedures. Completing the square and changing variables we obtain

$$\begin{aligned} v_+(x,t) &= e^{-i\lambda^2 t + i\lambda x} \frac{1}{(4\pi t)^{\frac12}} \int_0^\infty e^{i(y-x+2t\lambda)^2/(4t)} dy \\ &= \mathcal{O}((2\lambda t^{\frac12} - x/t^{\frac12})_+^{-1})\,, \end{aligned}$$

since

$$\int_0^\infty e^{i(r+s)^2} dr = \int_s^\infty e^{ir^2} dr = \mathcal{O}(s_+^{-1}) .$$

□

We note that the lemma is false if $e^{i\lambda|x|}$ is replaced by $e^{-i\lambda|x|}$ in (3.4), that is *outgoing* initial condition is replaced by an *incoming* one.

Proof of Theorem: Let $u_1(x,t) \stackrel{\text{def}}{=} e^{-it\lambda^2} R_V(\lambda)(x,x_0)$. Then u_1 solves (3.1) but violates the boundary condition. Hence we need to show that the solution to

$$iw_t = -w_{xx} + Vw \,, \quad w(x,0) = R_V(x,x_0) \,,$$

satisfies $w(x,t) = \mathcal{O}(t^{-\frac{1}{2}})$ for every x, that is that for any $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$,

$$\|\chi \exp(-itH_V) R_V(\bullet, x_0)\|_{L^\infty} = \mathcal{O}(t^{-\frac{1}{2}}) . \tag{3.6}$$

We use Duhamel's formula,

$$e^{-itH_V} = e^{-itH_0} - i \int_0^t e^{-i(t-s)H_V} V e^{-isH_0} ds .$$

Now, for some coefficients $a_\pm$,

$$R_V(x,x_0) = a_+ e^{i\lambda|x|} \mathbb{1}_{\mathbb{R}_+}(x) + a_- e^{i\lambda|x|} \mathbb{1}_{\mathbb{R}_-}(x) + r(x) \,, \quad r \in L^\infty_{\text{comp}}(\mathbb{R}) \,,$$

and hence, Lemmas 2 and 4 show that

$$\|\chi_0 \exp(-itH_0)(R_V(\bullet,x_0))\|_{L^\infty} = \mathcal{O}(t^{-\frac{1}{2}}) \,, \quad \chi_0 \in \mathcal{C}_c^\infty(\mathbb{R}) .$$

Taking χ_0 such that $\chi_0 V = V$ this shows that

$$\begin{aligned} \|\chi \exp(-itH_V) R_V(\bullet, x_0)\|_{L^\infty} &\leq Ct^{-1/2} + C \int_0^t \|\chi \exp(-i(t-s)H_V)\chi_0\|_{\mathcal{L}(L^\infty, L^\infty)} s^{-\frac{1}{2}} ds \\ &\leq Ct^{-1/2} + C \int_0^t \min((t-s)^{-\frac{3}{2}}, (t-s)^{-\frac{1}{2}}) s^{-\frac{1}{2}} ds \,, \end{aligned}$$

where we used Lemmas 2 and 3 to estimate the propagator $\exp(-i(t-s)H_V)$. The last integral is bounded by

$$\int_0^{t-1} (t-s)^{-\frac{3}{2}} s^{-\frac{1}{2}} ds + \int_{t-1}^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} ds \leq$$
$$t^{-1} \int_0^{1/t} \sigma^{-\frac{3}{2}} (1-\sigma)^{-\frac{1}{2}} d\sigma + (t-1)^{-\frac{1}{2}} \int_0^1 \sigma^{-1/2} d\sigma = \mathcal{O}(t^{-1/2}) \,,$$

and this gives (3.6) completing the proof of (3.2).

To prove (3.2) we simply combine (2.2) with the far field asymptotics (2.5). □

4. A relevant estimate?

Here we establish some simple estimates needed for the non-linear propagation result.

Lemma 5. *Suppose that* $iv_t(x,t)+v_{xx}(x,t)=u(x,t)$*,* $v(x,0)=0$*. Then, for any* $s\in\mathbb{R}$*,*

$$\begin{aligned}\|v(\bullet,t)\|_{H^{s+2}(\mathbb{R})} &\leq \|u(\bullet,t)\|_{H^s(\mathbb{R})}+\|u(\bullet,0)\|_{H^s(\mathbb{R})}\\ &\quad + C\sqrt{t}\left(\int_0^t(\|u(\bullet,t')\|^2_{H^s(\mathbb{R})}+\|\partial_t u(\bullet,t')\|^2_{H^s(\mathbb{R})})dt'\right)^{\frac12}.\end{aligned}$$

Proof. This follows immediately from Duhamel's formula applied on the Fourier transform side:

$$\hat v(\xi,t)=i\int_0^t e^{-i(t-t')\xi^2}\hat u(\xi,t')dt',$$

which implies that

$$\|v(\bullet,t)\|_{H^s(\mathbb{R})}\leq C\sqrt{t}\left(\int_0^t\|u(\bullet,t')\|^2_{H^s(\mathbb{R})}dt'\right)^{\frac12}.$$

But we also have

$$\begin{aligned}\xi^2\hat v(\xi,t)&=\int_0^t\partial_{t'}\left(e^{-i(t-t')\xi^2}\right)\hat u(\bullet,t')dt'\\ &=\hat u(\xi,t)+\hat u(\xi,0)e^{-it\xi^2}-\hat{\int}_0^t e^{-i(t-t')\xi^2}\partial_{t'}\hat u(\bullet,t')dt',\end{aligned}$$

which completes the proof. ☐

For instance when $u(x,t)=\theta(t)\delta_0(x)$, $\theta,\theta'\in L^\infty(\mathbb{R})$, then $u(\bullet,t)\in H^{-\frac12-\epsilon}(\mathbb{R})$ and consequently

$$\|v(\bullet,t)\|_{H^{\frac32-\epsilon}(\mathbb{R})}\leq C_\epsilon\langle t\rangle. \tag{4.1}$$

We will now consider that case but with a potential:

Lemma 6. *Suppose that* $V\in L^\infty_{\rm comp}(\mathbb{R})$ *has no zero resonance, and that*

$$iu_t=-u_{xx}+V(x)u+\theta(t)\delta_0(x),\quad u(x,0)=0,$$

where $\theta,\theta'\in L^\infty(\mathbb{R})$*. Then for any* $\epsilon>0$*,*

$$\|u(\bullet,t)\|_{H^{\frac32-\epsilon}(\mathbb{R})}\leq C_\epsilon\langle t\rangle.$$

Proof. We write

$$u(x,t)=\int_0^t e^{-i(t-t')H_0}(\theta(t')\delta_0(x))dt'+\int_0^t\int_0^{t'}e^{-i(t-t'-t'')H_0}Ve^{-it''H_V}(\theta(t')\delta_0(x))dt''dt'. \tag{4.2}$$

By (4.1) the first term satisfies the conclusion of the lemma. We write the second term as

$$\int_0^t e^{-(t-t')H_0} v(x,t')dt' \,, \quad v(x,t) \stackrel{\text{def}}{=} \int_0^t e^{isH_0} V e^{-isH_0}(\theta(t)\delta_0(x))dt'' \,.$$

If we show that

$$\|\partial_t^k v(\bullet, t)\|_{L^2(\mathbb{R})} \leq C \,, \quad k = 0, 1 \,, \tag{4.3}$$

then Lemma 5 will conclude the proof by showing that the second term in (4.2) is bounded in H^2 with norm $\mathcal{O}(\langle t \rangle)$.

Lemmas 2 and 3 and the regularity of the kernel (see [2]), show that

$$\|V e^{-isH_V}(\theta(t)\delta_0(x))\|_{L^2} \leq C \|e^{-isH_V}(\theta(t)\delta_0(x))\|_{L^\infty} \leq C' \min(s^{-\frac{1}{2}}, s^{-\frac{3}{2}})$$

and this gives (4.3). □

References

[1] T. Paul, K. Richter, and P. Schlagheck, *Nonlinear Resonant Transport of Bose-Einstein Condensates,* Phys. Rev. Lett. **94**(2005), 020404.

[2] S.H. Tang and M. Zworski, *Potential scattering on the real line,* Lecture notes, `http://www.math.berkeley.edu/∼zworski/tz1.pdf`