Soliton Home Movies

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Apologies to my colleagues who may not quite agree...
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But in fact they are interested in \textit{numbers}.
Here is one equation for which we know a lot

\[ i \frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + u|u|^2 = 0, \]
Here is one equation for which we know a lot (but of course we want to know more)

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\text{Nonlinear Schrödinger Equation:} \quad i \frac{\partial}{\partial t} u + \frac{1}{2} \frac{\partial^2}{\partial x^2} u + u |u|^2 = 0
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**Nonlinear Schrödinger Equation**

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This equation has traveling wave solutions:

\[ u(x, t) = e^{i(\gamma(t)x - \alpha - vt)} \mu \text{sech}\left(\mu(x - a - vt)\right), \quad \mu > 0, \quad v, a, \gamma \in \mathbb{R}, \]

\[ \gamma(t) = \gamma + vx + \left(\mu^2 - v^2\right)\frac{t}{2}. \]
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$$\gamma(t) = \gamma + vx + (\mu^2 - v^2)t/2.$$
\[ \mu = 1, \quad v = 1, \quad a = -7. \]
One of the amazing features in the stability of solitary waves in interaction.
One of the amazing features in the stability of solitary waves in interaction. Collision of $\mu = 1$ and $\mu = 0.75$: 
\[ iu_t = -\frac{u_{xx}}{2} - |u|^2 u, \quad u(x, 0) = 2\text{sech}x. \]

\[ u(x, t) = 2e^{it/2}\text{sech}x \left( (4 + 3\text{sech}^2(e^{4it} - 1))/(4 - 3\text{sech}^4x \sin^2 2t) \right) \]

This solution is obtained using the inverse scattering method.
Suppose now that we consider a perturbed NLS, that is, the Gross-Pitaevskii equation, by adding an external potential:

\[ i \frac{\partial}{\partial t} u + \frac{1}{2} \frac{\partial^2}{\partial x^2} u - q \delta_0(x) u + u |u|^2 = 0 \]

Here \( \delta_0 \) is the famous Dirac delta function:

\[ \delta_0(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}, \quad \int_{-\infty}^{\infty} \delta_0(x) \, dx = 1. \]
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\begin{aligned}
  i \partial_t u + \frac{1}{2} \partial_x^2 u - q \delta_0(x) u + u|u|^2 &= 0 \\
  u(x, 0) &= u_0(x)
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\( q = 3, \ v = 3, \ x_0 = -3. \)
\( q = -0.02, \ v_0 = 0, \ a_0 = -3. \)
\[ V(x) = -\text{sech}^2\left(\frac{x}{5}\right), \quad u_0(x) = \text{sech}(x + 3). \]
\[ V(x) = -\text{sech}^2\left(\frac{x + 5}{4}\right) - \text{sech}^2\left(\frac{x - 5}{4}\right) - 0.1\text{sech}^2\left(\frac{x}{4}\right), \]

\[ u_0(x) = e^{ix/10}\text{sech}(x + 8). \]
Conclusions

The NLS models many phenomena such as the Bose-Einstein condensate, fiberoptics, impurities in DNA...

Many phenomena hard to see numerically can be explained analytically

And vice versa, many things easy to see numerically are hard analytically

There are many open problems: long time behaviour, radiation and "breathing" patterns, multiple solitons interacting with impurities...
Conclusions

- The NLS models many phenomena such as the Bose-Einstein condensate, fiberoptics, impurities in DNA...
- Many phenomena hard to see numerically can be explained analytically
- And vice versa, many things easy to see numerically are hard analytically
- There are many open problems: long time behaviour, radiation and “breathing” patterns, multiple solitons interacting with impurities...