Fractal Weyl laws for chaotic open systems

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Abstract

We present a result relating the density of resonances for an open chaotic system to the dimension of the classical fractal repeller of the system. The result is supported by numerical computation of the resonances of the system of n disks on a plane. The result generalizes the Weyl law for the density of states of a closed system to chaotic open systems.

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1 Introduction

The celebrated Weyl law concerning the density of eigenvalues of bound states is a central result in the spectroscopy of quantum systems [1]. The Weyl formula states that the asymptotic level number $N(k)$, defined as the number of levels with $k_n < k$ (where $k \to \infty$) is given after smoothing by $N(k) \stackrel{\text{def}}{=} \#\{k_n : k_n \leq k\} = Vk^D/(D/2)!(4\pi)^{D/2} + \cdots$, for a quantum system bounded in a region R of D-dimensional space whose volume is V. For closed systems with smooth boundaries, the Weyl formula is well-established, and although primarily valid in the semi-classical limit, nevertheless can be applied with astonishing accuracy to very low energies extending almost down to the ground state of integrable and chaotic closed systems. Generalizations of the Weyl law to other situations have long been sought. The most notable example is the conjecture by Berry [2] for the density of states of closed systems with fractal boundaries, *i.e.* "fractal drums".

Open systems are characterized by resonances defined by complex wavevector $k_n = \text{Re}(k_n) + i\text{Im}(k_n)$, corresponding to states with finite life times arising from escape to infinity. Open chaotic systems, which occur in a variety of physical situations, are generically characterized by a classical phase space repeller that is fractal. In this letter we present a result relating the density of resonances for an open chaotic system to the fractal dimension of the associated *classical* repeller. The central result of the present work can be stated as:

$$N_\gamma(k) \stackrel{\text{def}}{=} \#\{k_n : \text{Im}k_n > -\gamma,\, \text{Re}k_n \leq k\} \sim k^{1+d_H} \quad (1)$$

where d_H is the *partial Hausdorff dimension of the repeller* [8, §4.4]. This relation generalizes the Weyl law for the density of states of a closed system to chaotic open systems. The notation $f(k) \sim g(k)$ means that for some constants $C > c > 0$, $cg(k) \leq f(k) \leq Cg(k)$ for k large. The constants depend on γ but not on k.

The repeller is defined as the set of points in phase space which do not escape to infinity at both positive or negative times. The dimension of the repeller is given by $2d_H + 2$, where we did not restrict ourselves to an energy surface. It may be that in finer analysis a different notion of dimension may have to be used, perhaps distinguishing between the upper and lower bounds. In mathematical works [3] the upper bounds are given in terms of the Minkowski dimension, which in symmetric examples coincides with Hausdorff dimension. For closed two dimensional systems, such as compact surfaces of constant negative curvature, we have real zeros only and $N(k) = \#\{k_n : k_n \leq k\} \sim k^2$,which is consistent with (1) as $d_H = 1$ then: everything is trapped.

Our motivation comes from rigorous work on quantum resonances and in particular from the work of Sjöstrand [3] on geometric upper bounds on their density. The optimal nature of that bound was recently shown by a numerical experiment [4] involving a computation of quantum resonances for semiclassical Schrödinger operators with chaotic classical dynamics.

Here we consider a different but related problem. Suppose that $Z(k)$ is the semi-classical Selberg-Ruelle zeta function, with k the wave number. In some situations the zeros of its meromorphic continuation approximate semi-classically the quantum resonances of the quantized system – see [5] for an early study, and [6] for a review of recent theoretical and experimental results. Because of the exact Selberg trace formula, this is rigorously known for surfaces of constant negative curvature [7]

We then claim (see §4) that for zeta functions associated to configurations of hard discs in the plane, eq.(1) holds. The result is demonstrated by numerical computations of resonances of n-disks on a plane obtained from the poles of a semi-classical Ruelle zeta function calculated using a cycle expansion.

2 Semiclassical zeta functions and resonances

The trace formulæ of Selberg, Gutzwiller, and Balian-Bloch (see for instance [9] for a survey and references) provide one of the most elegant and useful ways of expressing the classical quantum correspondence. To formulate it in terms of the semi-classical zeta function we consider a semi-classical quantization $\widehat{H}$ of a classical Hamiltonian H: for instance $H = p^2 + V(q)$ and $\widehat{H} = -\hbar^2\Delta + V(x)$.

The contribution of periodic orbits to the trace of the resolvent is is given by

$$\operatorname{tr} \frac{1}{E - \widehat{H}}\Big|_{\text{p.o.}} = (\log Z)'(E)(1 + \hbar a_1(E) + \hbar^2 a_2(E) + \cdots), \tag{2}$$

where $Z(E)$ is the semiclassical zeta function:

$$\exp\left(- \sum_{\gamma} \sum_{n=1}^{\infty} -\frac{1}{n} \frac{e^{i\nu_{\gamma,n} + in\frac{S_\gamma(E)}{\hbar}}}{|I - P_\gamma^n|^{\frac{1}{2}}} \right). \tag{3}$$

Here γ's are primitive periodic orbits, T_γ's their periods, $\nu_{\gamma,n}$, the Maslov indices at n iterations, S_γ's classical actions, and P_γ's the linear Poincaré maps. The formula (2) is somewhat formal but can be made rigorous – see [10] for a recent presentation in a generalized setting. We also note that, suitably modified, an exact formula for a class of constant curvature open chaotic systems is given in [7].

For open systems, the quantum resonances are defined as the poles of the meromorphic continuation of the resolvent (or Green's function) $(E - \widehat{H})^{-1}$. Neglecting higher higher order terms in (2) suggests that in the semi-classical limit the resonances should be approximated by the complex zeros of the analytic continuation of $Z(E)$. There exists enough evidence now that this is the case [5],[6].

Since a resonance corresponding to $E = E_0 - i\Gamma$ propagates as

$$\exp(-itE/h) = \exp(-itE_0/\hbar - t\Gamma/\hbar)$$

(hence E_0 is interpreted as the rest energy, and Γ as the rate of decay), the "visible" resonances should satisfy $\Gamma < C\hbar$: if $\Gamma \gg \hbar$ the state decays too fast to be seen. As a density of resonance states near a given energy level,

E_0, it is thus natural to consider

$$N_\hbar(E_0,\delta) = \\ \#\{E - i\Gamma \;:\; |E-E_0| < \delta\,,\; \Gamma < Ch\}\,.$$

From [3],[4] we expect that for hyperbolic classical flows

$$\begin{aligned} N_\hbar(E_0,\delta) \sim & h^{-d(E_0,\delta)}\,, \\ d(E_0,\delta) = & \frac{1}{2}\dim\{(q,p)\;: \\ & |H(q,p)-E_0| < \delta\,, \\ & \Phi^t(q,p) \not\to \infty\,,\; t\to\pm\infty\} \end{aligned} \tag{4}$$

The set appearing in the definition of $d(E_0,\delta)$ is the *trapped set* or the repeller of the classical flow. It is not clear at this point what notion of the dimension should be used. The upper bounds [3] are given in terms of the Minkowski dimension.

In view of the semi-classical connection between the zeros of $Z(E)$ and the resonances, it is natural to consider the analogue of (4) for those zeros.

3 Hard disc scattering

In the case of hard disc scattering in two dimensions (see [6] and references given there) the quantum Hamiltonian is given by $-\hbar^2\Delta_D$ where Δ_D is the Dirichlet Laplacian. It is then natural to introduce a new variable k, $k^2 = E/\hbar^2$. Semiclassical asymptotics correspond to the limit $k\to\infty$, and the semiclassical density of resonances considered above should be replaced by

$$N_\gamma(k) = \#\,\{k_n : \mathrm{Im}k_n > -\gamma,\, \mathrm{Re}k_n \le k\}\ .$$

The Hausdorff dimension of the repeller in (4) is now independent of the energy level and we will denote it by D_H.

The Zeta function can be considered as a function of k and it takes a somewhat simpler form: $Z(k) =$

$$\begin{aligned} &\exp\left(-\sum_\gamma\sum_{n=1}^\infty \frac{(-1)^{nm_\gamma}e^{inkT_\gamma}}{n}\frac{1}{|I-P_\gamma^n|^{\frac{1}{2}}}\right) \\ &= \prod_{j=1}^\infty\prod_\gamma(1-(-1)^{jm_\gamma}e^{ijT_\gamma}\Lambda_\gamma^{-j-\frac{1}{2}})\,, \end{aligned} \tag{5}$$

where $\Lambda_\gamma > 1$ is the larger of the two eigenvalues of P_γ, and m_γ is the number of reflections of γ. As was explained in §2 we want to find the density of zeros of $Z(k)$.

4 Computational results

Effective ways of evaluating the analytic continuations of semi-classical (and dynamical) zeta functions have been developed by several authors. The cycle expansion method [11] has proved itself to be particularly successful. We used it following the earlier computations performed for the purpose of comparisons with experimental data [12]. We also chose configurations for which the dimensions of the repellers were readily available [8]: three symmetrically spaced discs of radii $a = 1$, with centers $r = 6$ apart, and four discs with radii $a = 1$ placed at the vertices of the square of side $r = 6$. The Hausdorff dimension, D_H is given in terms of the partial dimension, d_H, $D_H = 2d_H + 2$ and in the examples above

$$d_H \simeq \begin{cases} 0.28952 & \text{three discs} \\ 0.42607 & \text{four discs} \end{cases}$$

The priciple (4) applied to the density $N_\gamma(k)$ defined in §3 suggests the law

$$\frac{\log N_\gamma(k)}{\log k} \simeq D_H/2 = 1+d_H\ , \quad k \to \infty\,. \tag{6}$$

The numerical results are shown in Fig.1 and Fig.2: they confirm the validity of (6) for the symmetric three and four disc configurations.

5 Conclusions

The computational results presented here produce convincing evidence for the connection between the density of resonant states and the fractal dimension of the repeller on which they concentrate. It is not clear yet what notion of dimension should be used and it is quite possible that different dimensions might occur in lower and upper bounds once less symmetric examples are considered. Since zeta functions were used in the computation, we have also provided evidence that the density of zeros of zeta functions is related to the dimension of the repeller. That this might occur has already been suggested [3].

It is worth noting that while fractality arises from R or its boundary ∂R in [2],[13], in the present work both R and ∂R are smooth and instead the classical phase space is fractal.

One might ask if the results depend on the choice of the width γ of the counting strip – see (1) and the explanation of the $\sim$ notation following it. The answer is that the width affects the prefactor but does not affect the $k-$dependence that is the focus of the present work. It is known from previous computations of both quantum resonances and the semi-classical approximations given by Zeta function [4],[5] that for γ small we may have $N_\gamma(k) \equiv 0$. Hence we expect (6) to hold for γ large enough but with the right hand side independent of γ.

The connections described here between quantum spectral properties and classical fractal properties of the associated repeller of open chaotic systems parallels a similar connection established earlier between the quantum spectral autocorrelation and the classical decay rate. The microwave spectra $T(k)$ of n-disk billiards was analyzed in terms of the autocorrelation $C_T(\kappa) = \langle T(k-(\kappa/2))T(k+(\kappa/2))\rangle_k$ where $\langle\rangle$ represents a suitable average[12]. Experimentally it has been shown that the autocorrelation can be well-described by $C_T(\kappa) = \sum_{\pm,n=1}^{\infty} b_n\gamma'_n/\gamma'^2_n + (\kappa \pm \gamma''_n)^2$,with $\gamma'_n \pm i\gamma''_n$ the classical Ruelle-Pollicott (RP) resonances in wave vector space. Semiclassical arguments supporting the experimental results were presented in [6]. The leading RP resonance γ'_0 ($\gamma''_0 = 0$) represents the classical decay rate and is related to the information dimension d_I of the repeller by $\gamma'_0 = \lambda(1-d_I)$, where λ is the average Lyapunov exponent. The higher RP resonances $\tilde{\gamma}_n\{n > 0\}$ represent fine structure properties of the fractal repeller comprised of the manifold of trapped orbits.

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