OVERDAMPED QNM FOR SCHWARZSCHILD BLACK HOLES

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Abstract. We prove that the number of quasinormal modes (QNM) for Schwarzschild and Schwarzschild–de Sitter black holes in a disc of radius r is bounded from below by cr^3, showing that the recent upper bound by Jézéquel [Je23] is sharp. The argument is an application of a spectral asymptotics result for non-self-adjoint operators which provides a finer description of QNM, explaining the emergence of a distorted lattice and generalizing the lattice structure in strips described in [SáZw97] (see Figure 1). Our presentation gives a general result about exponentially accurate Bohr–Sommerfeld quantization rules for one dimensional problems. The resulting description of QNM allows their accurate evaluation "deep in the complex" where numerical methods break down due to pseudospectral effects (see Figure 2).

1. Introduction and statement of the results

Quasinormal modes remain a topic of interest in physics and mathematics – see for instance the following recent references [Ca*18] and [GaWa21] respectively; one can also find there pointers to the vast literature of the subject. In this note we consider a purely mathematical question of lower global bounds on the number of quasinormal modes for Schwarzschild (S) or Schwarzschild–de Sitter (SdS) black holes. Our motivation comes from a recent paper by Jézéquel [Je23] who proved a global upper bounds for the number of quasinormal modes for SdS black holes: if we denote by $\mathrm{QNM}(m,\Lambda)$ the set of quasinormal modes for an SdS black hole of mass m with the cosmolotical constant Λ (see §2)

$$|\mathrm{QNM}(m,\Lambda)\cap D(0,r)| \leq C(m,\Lambda)r^3, \quad 0<\Lambda<3/m. \tag{1.1}$$

We recall from [SáZw97] that for $\Lambda>0$, $\mathrm{QNM}(m,\Lambda)$ is a discrete set, while for $\Lambda=0$ (S black hole) the situation is more complicated and not completely understood mathematically. However we have

Theorem 1. *For $0<t\ll 1$ and $0<\Lambda<1/9m^2$, let $A_t(r):=\{\lambda : 1\leq|\lambda|\leq r, \arg\lambda > -t\}$. Then, there exist $c(t,m,\Lambda)>0$, such that, as $r\to\infty$,*

$$|\mathrm{QNM}(m,\Lambda)\cap A_t(r)| = c(t,m,\Lambda)r^3 + o(r^3) \tag{1.2}$$

When $\Lambda=0$ the asymptotic equality in (1.2) should be replaced by $\geq$.

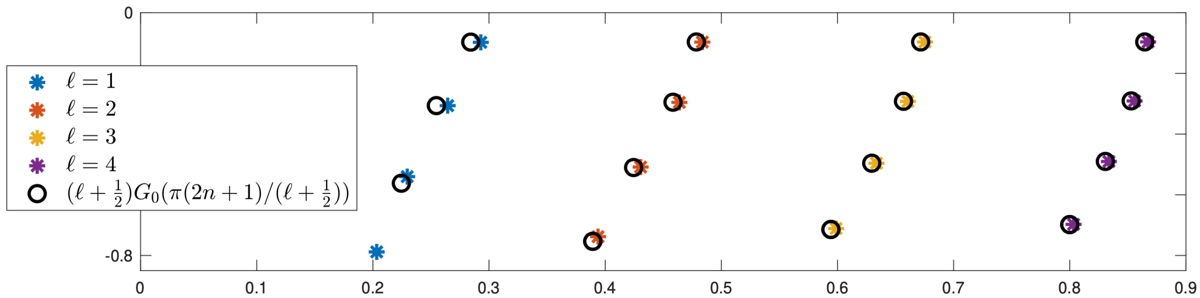

FIGURE 1. A comparison of numerically computed QNM in [Ja17, Figure 6] for the case of $m=1$, $\Lambda=0$ and the leading term in the semiclassical expression for QNM (1.3) given by (1.6). Even at large values of $h=(\ell+\frac{1}{2})^{-1}=\frac{2}{3},\frac{2}{5},\frac{2}{7},\frac{2}{9}$, one sees a reasonable agreement as well as the emergence of the distorted lattice deeper in the complex (as opposed to the lattice in strips described in Sá Barreto–Zworski [SáZw97].

In particular, this shows that Jézéquel's estimate is sharp. Theorem 1 is a consequence of a more precise result about the distribution of quasinormal modes obtained from a one dimensional scattering result in §5. It generalizes the result of [SáZw97], which was based on [Sj86] (applied in the special 1D case). The constant $c(t,m,\Lambda)$ can be computed using the function G_0 appearing in Theorem 2 – see (1.5).

Theorem 1 is a consequence of a finer description of quasinormal modes for the Regge–Wheeler potential (see §2). The main point is that for each angular momentum ℓ we obtain the description of quasinormal modes, $\mathrm{QNM}_\ell(m,\Lambda)$, with $\operatorname{Im}\lambda \geq -t\ell$, $0<t\ll 1$ and with $e^{-c|\lambda|}$ accuracy (though that level of finery is irrelevant for Theorem 1). In particular, this gives a mathematical description of the deformed lattice seen in numerical calculations. For smaller neighbourhoods of the real axis ($\operatorname{Im}\lambda > |\lambda|^{1-\delta}$ for any $\delta>0$), Iantchenko [Ia17] used the results of Kaidi–Kerdelhué [KaKe00] to describe quasinormal modes with $\mathcal{O}(\langle\lambda\rangle^{-\infty})$ accuracy.

Theorem 2. *For $0\leq\Lambda<1/9m^2$ and $\ell\gg 1$, the quasinormal modes (of multiplicity $2\ell+1$), with $|\lambda|>c_0\ell$ (for any positive c_0) and $\arg\lambda>-\theta$, $0<\theta\ll 1$ are given by*

$$\lambda_{\ell,n} = h^{-1}G(2\pi(n+\tfrac{1}{2})h,h), \quad h:=(\ell+\tfrac{1}{2})^{-1}, \quad G(x,h)\sim\sum_{j=0}^{\infty}G_j(x)h^j, \tag{1.3}$$

where the functions G_j are holomorphic near 0 and satisfy bounds $|G_j|\leq A^{j+1}j!$, $A>0$ (that is, G is an analytic symbol), G_0 is the inverse of a complex action (see Theorem A.2) and $G_1\equiv 0$.

Remark. Just as in the self-adjoint case for Schrödinger operators with h-independent potentials (see [CdV] for a clear presentation) we expect $G_{2j+1}\equiv 0$ but as our proofs work for more general operators we restrict ourselves to showing that $G_1 \equiv 0$. That holds for all operators with 0 subprincipal symbol.

The definition of G_0 is the inverse of the complex action (see Theorem A.2 gives

$$G_0(x) = \frac{(1-9\Lambda m^2)^{\frac{1}{2}}}{3\sqrt{3}m} \left(1 - \frac{ix}{2\pi} + \mathcal{O}(x^2)\right), \tag{1.4}$$

which describes the lattice of QNM from [SáZw97]. The constant in (1.2) is given by

$$c(t, m, \Lambda) = \pi^{-1}(1-9\Lambda m^2)^{-\frac{3}{2}} 3^{\frac{7}{2}} m^3 |\{x \in \mathbb{R}_+ : \arg G_0(x) > -t\}|, \tag{1.5}$$

where $|E|$ is the length of an interval E. For $\Lambda = 0$ and $m = 1$ additional terms in the expansion of G_0 are calculated by Mathematica as follows:

$$G_0(x) = \frac{1}{3\sqrt{3}} - \frac{ix}{6\sqrt{3}\pi} - \frac{5x^2}{432\sqrt{3}\pi^2} - \frac{235ix^3}{93312\sqrt{3}\pi^3} + \frac{17795x^4}{40310784\sqrt{3}\pi^4} + \cdots \tag{1.6}$$

Despite the mathematical assumption on the smallness of the effective h the asymptotic formula is effective starting with $\ell = 1$ – see Figure 1. The terms G_j are in principle computable, and using different methods and conventions similar expansions have been proposed in the physics literature as early as [IyWe87]. For an interesting recent physics perspective see [AAH22] and reference given there. Here we remark that as G in (1.3) is an analytic symbol, the usual truncation of the expansion at $n = [(eA)/h]$ approximates $\lambda_{n,\ell}$ up to errors of the size $e^{-c_0\ell}$ – see §3.

This result is an immediate consequence of a more general semiclassical Theorem 3 about spectra of non-self-adjoint operators satisfying suitable ellipticity conditions near infinity. It differs from the results of Hitrik [Hi04] by changing the class of operators and by providing analyticity of the expansion. Rather than to use exact WKB methods specialized to one (complex) dimensional holomorphic equations, we use general methods of analytic microlocal theory [HiSj15],[Sj82] and the essential one dimensional feature is the use of Vey's theorem [Ve77] (holomorphic and symplectic version of Morse lemma) in (phase space) dimension two. It then provides a geometric interpretation of G_0 as the inverse of a holomorphic action.

For regular energy levels, the recent thesis by Duraffour [Du24] provides Bohr–Sommerfeld quantization rules with analytic symbols in the same way as is done in Theorems 2 and 3 for fixed neighbourhoods of critical levels.

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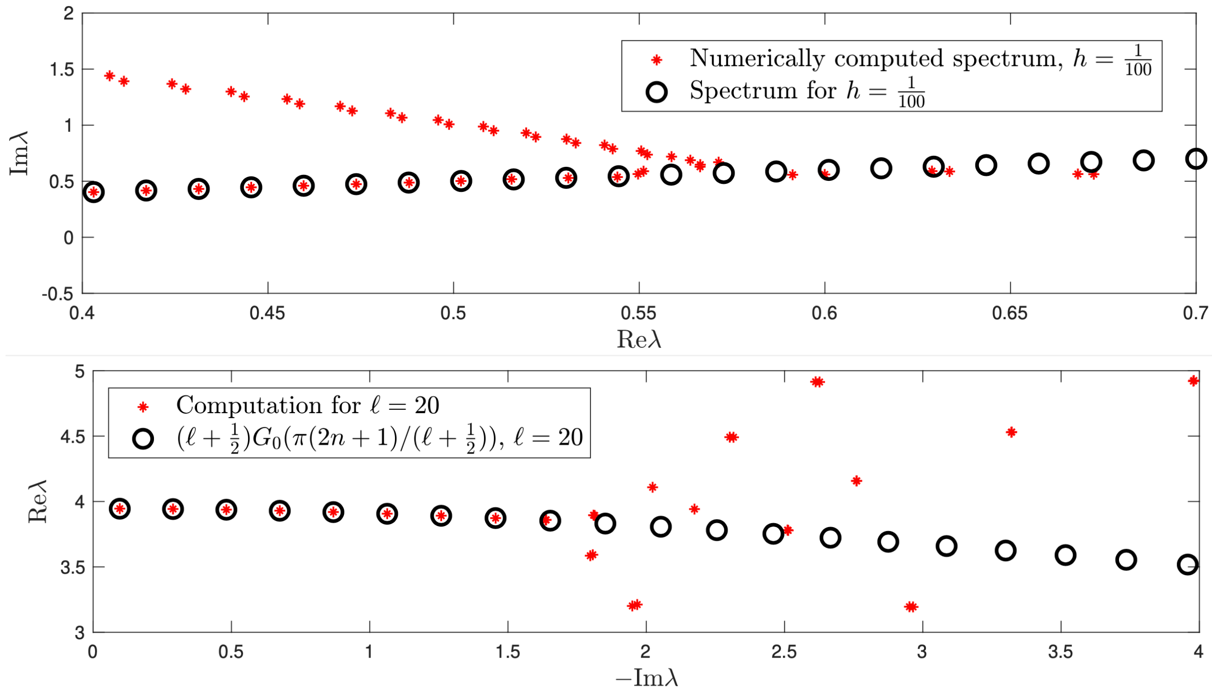

FIGURE 2. The top plot shows the spectrum of rotated harmonic oscillator $-h^2\partial_x^2 + ix^2$ used by Davies and Embree–Trefethen [TrEm05, I.5] to illustrate spectral instability for non-normal pseudodifferential operators. The eigenvalues (explicitly given by $e^{\pi i/4}h(2n+1)$ are compared to eigenvalues computed numerically using as the basis the first 151 Hermite functions (eigenfunctions of $-h^2\partial_x^2 + ix^2$). This illustrates the fragility of eigenvalues "deep in the complex". The bottom plot shows a calculation of QNM for $m = 1$, $\Lambda = 0$ and $\ell = 20$ using the Mathematica code [Ja17]. We again see divergence from the (mathematically) established (modulo the issues of the size of the region).

2. Regge–Wheeler potentials

In this section we review the definition of quasinormal modes for Schwarzschild and Schwarzschild–de Sitter black holes using the method of complex scaling.

2.1. **Complexified Regge–Wheeler potentials.** We recall the Regge–Wheeler potential defined for $0 \leq \Lambda < 1/9m^2$:

$$\begin{aligned} &V_\ell(x) := \alpha(r)^2 r^{-2}(\ell(\ell+1) + r\partial_r(\alpha(r)^2)), \\ &x'(r) = \alpha(r)^{-2}, \quad \alpha(r)^2 := 1 - \frac{2m}{r} - \tfrac{1}{3}\Lambda r^2. \end{aligned} \tag{2.1}$$

We recall also that

$$\alpha(r)^2 = \begin{cases} r^{-1}(r-2m), & \Lambda = 0, \\ \frac{1}{3}\Lambda^2 r^{-1}(r-r_0)(r-r_-)(r_+-r), & \Lambda > 0, \end{cases}$$

where $r_0 < 0 < r_- < r_+$. We have $x : (2m,\infty) \to (-\infty,\infty)$ $(\Lambda = 0)$ and $x : (r_-,r_+) \to (-\infty,\infty)$ $(\Lambda > 0)$ and the transformations are unique up to addittive constants. Explicitly, for $\Lambda = 0$,

$$x(r) = r + 2m\log(r-2m), \quad 2m < r < \infty, \quad r(x) = 2 + 2W_0(\tfrac{1}{2}e^{\frac{1}{2}x-1}), \tag{2.2}$$

where W_0 is the first branch of the Lambert W-function, $W_0(te^t) = t$, $t > 0$. For $\Lambda > 0$,

$$x(r) = a_0\log(r-r_0) + a_-\log(r-r_-) + a_+\log(r_+-r), \quad r_- < r < r_+,$$

where $\mp a_\pm, a_0 > 0$ and we chose the principal branch of the logarithm (real on the positive real axis).

The transformation $r \to x(r)$ is holomorphic and conformal for r in open neighbourhoods of $(2m,\infty) \subset \mathbb{C} \setminus \{2m\}$ and $(r_-,r_+) \subset \mathbb{C} \setminus \{r_-,r_+\}$ respectively and we have a unique inverse $r = r(x)$, where x is a neighbourhood of $\mathbb{R} \subset \mathbb{C}$. For $\Lambda = 0$ and $|r| \gg 2m$,

$$x(r) = r + \log r + \log(1-2m/r) = r + \log r + \frac{2m}{r} + \cdots, \quad |\arg r| \le \theta < \pi,$$

which produces a holomorphic inverse

$$r(x) = x - \log x + \sum_{k=1}^{\infty}\sum_{\ell=0}^{\infty} c_{k\ell}\frac{\log^\ell x}{x^k}, \quad |x| \gg 1, \quad |\arg x| \le \theta < \pi.$$

Let $\Lambda_{\underline{r}}$ denote the logarithmic plane branched at $\underline{r}$, that is the Riemann surface of $r = f(z) := \underline{r} + \exp z$, $f : \mathbb{C} \to \Lambda_{\underline{r}}$ bi-holomorphically. Suppose $\pi : \Lambda_{\underline{r}} \to \mathbb{C} \setminus 0$ is the natural projection and consider $D_\rho \subset \Lambda_{\underline{r}}$ defined by $|\pi(r)| < \rho$.

If we take $\underline{r} = 2m$ and $\Lambda = 0$, then for ρ sufficiently small $x : D_\rho \to \Omega_\rho \supset H_\rho := \{x : \operatorname{Re} x < c_0\log\rho\}$ is a bi-holomorphic map. We then have an inverse $r = r(x)$ defined in H_ρ. We note that $\alpha(r)^2 r^{-2}$ is holomorphic in D_ρ and goes to 0 as $|r| \to 0$. We conclude that

$$\alpha(r(x))^2 r(x)^{-2} \in \mathscr{O}(H_\rho), \quad \alpha(r(x))^2 r(x)^{-2} \to 0, \quad \operatorname{Re} x \to -\infty.$$

(In fact the decay is exponential though this is not relevant in this paper.) The same property is valid for $r^{-1}\partial_r(\alpha(r)^2)|_{r=r(x)}$.

The same arguments work for $\Lambda > 0$ where we consider the logarithmic planes $\Lambda_{r_\pm}$ with $r_\pm$ corresponding to $x = \pm\infty$. We summarize these results in

Lemma 2.1. *In the notation of* (2.1) *define the semiclasssical Regge–Wheeler potential as*

$$W(x,h) := (\ell + \tfrac{1}{2})^{-2}V_\ell(x) = W_0(x) + h^2W_1(x), \quad h = (\ell + \tfrac{1}{2})^{-1}. \tag{2.3}$$

Then for $0 \leq \Lambda < 1/9m^2$, *the potentials* $W_j(x)$ *have holomorphic extensions to* $\Omega_{\theta,\delta} := \{z : |\operatorname{Im} z| < \delta\} \cup \{z : |\arg(\pm z)| < \theta, |z| > 1/\delta_0\}$, *and*

$$W_j \in \mathscr{O}(\Omega_{\theta,\delta}), \quad W_j(z) \to 0, \ |z| \to \infty, \ z \in \Omega_{\theta,\delta}, \tag{2.4}$$

where $0 < \theta < \pi/2$ *and* $0 < \delta < \delta_0$ *(for some* δ_0*).*

Remark. We did not evaluate δ_0 as in the application we will use Lemma 2.1 only for small θ. But it might interesting to investigate large angle global complex scaling for $W(x,h)$.

2.2. **Complex scaling.** Quasinormal modes, or scattering resonances, for the operator $P = (hD_x)^2 + W(x,h)$ are defined by considering P as a holomorphic operator in $\Omega_{\theta_0,\delta}$, $0 < \theta_0 < \pi/2$ (see Lemma 2.1) and restricting it to a curve, Γ_θ, asymptotic to $e^{i\theta}\mathbb{R}$ at infinity. Resonances of P in $\arg\lambda > -\theta$ are then the eigenvalues of $P_\theta := P|_{\Gamma_\theta}$. We refer to [Sj96, §5] and [Sj02, §7.2] for the presentation of the method in a way applicable to our operator.

The choice of Γ_θ is tailored to the structure of the potential and we start by putting

$$V(x) = \alpha(r(x+x_0))^2 r(x+x_0)^{-2} - \alpha(r(x_0))^2 r(x_0)^{-2},$$

where $r(x_0)$ is the unique critical point of $\alpha(r)^2 r^{-2}$. We then have

$$V(x) = -c_0 x^2 + \mathscr{O}(x^3), \quad c_0 > 0, \quad xV'(x) < 0, \quad x \neq 0. \tag{2.5}$$

The complex scaling is performed as follows. We consider

$$\Gamma_\theta := \{2^{-1/2}(x + if(x)) : x \in \mathbb{R}\} \subset \mathbb{C} \tag{2.6}$$

where

$$\begin{gathered} f(x) = \theta x + (1-\theta)\chi(x), \quad \chi(0) = 0, \ \chi'(x) = \psi(x) = \psi_1(x/\delta), \\ \psi_1 \in C^\infty_{\rm c}((-2,2);[0,1]), \qquad \psi_1(x) = 1, \quad |x| \leq 1. \end{gathered} \tag{2.7}$$

In particular,

$$\Gamma_\theta \cap \{|z| < \delta\} = \exp(i\pi/4)\mathbb{R} \cap \{|z| < \delta\}. \tag{2.8}$$

We have the following key ellipticity statement:

Lemma 2.2. *Suppose that* Γ_θ *is given by* (2.6) *where* f *satsfies* (2.7), *and*

$$T^*\Gamma_\theta = \{(2^{-1/2}(x + if(x)), 2^{1/2}(1 + if'(x))^{-1}\xi) : x, \xi \in \mathbb{R}\}. \tag{2.9}$$

Then, we can choose for $0 < \delta, \theta \ll 1$ *in* (2.7) *so that* $\forall\, \varepsilon > 0 \ \exists\, \gamma, c_0 > 0$ *so that*

$$|\zeta^2 + V(z) - \omega| \geq c_0 \langle\zeta\rangle^2, \quad \text{if} \ \ |\omega| < \gamma, \quad |(z,\zeta)| > \varepsilon, \quad (z,\zeta) \in T^*\Gamma_\theta. \tag{2.10}$$

Proof. It is enough to show that

$$\zeta^2 + V(z) = 0, \quad (z,\zeta) \in T^*\Gamma_\theta \implies (z,\zeta) = (0,0). \tag{2.11}$$

For $x > 2\delta$ we use the fact that, with $c_1 = \int \psi_1(x)dx$,

$$\zeta^2 + V(z) = 2(1+i\theta)^{-1}\xi^2 + V(2^{-1/2}(x + i(\theta x + (1-\theta)c_1\delta))).$$

Hence,

$$\operatorname{Im}(\zeta^2 + V(z)) = (-2i\theta + \mathcal{O}(\theta^3))\xi^2 + 2^{-1/2}(\theta x + \delta(1-\theta)c_1)V'(2^{-1/2}x).$$

Since $V'(2^{-1/2}x) < 0$ we see that (2.11) holds. Similar argument works for $x < -2\delta$.

It remains to check (2.11) for $|x| < 2\delta$. We start with the quadratic part of the potential: $V_0(x) := -c_0x^2$ and define (with the parametrization in (2.9))

$$\begin{aligned} W(x) &:= 2c_0^{-1}(1+if'(x))^2V_0(z) = -((x+if(x))(1+if'(x)))^2 \\ &= (f(x)+xf'(x))^2 - (x - f(x)f'(x))^2 + 2i(x - f(x)f'(x))(f(x)+xf'(x)), \end{aligned}$$

Since $(x\chi(x))' = \delta(y\chi_1(y))'|_{y=x/\delta}$, $\chi_1(0) = 0$, $\chi_1'(y) = \psi_1(y)$ (see (2.7)) and $(y\chi_1(y))' = yg_1(y)$, $g_1(y) > 0$,

$$f(x) + xf'(x) = 2\theta x + (1-\theta)(x\chi(x))' = xg(x), \quad g(x) > 2\theta.$$

We then put

$$x - f(x)f'(x) = xh(x), \quad h \in C^\infty, \quad |h| \le M. \tag{2.12}$$

(The bound on h follows from writing f using χ_1 and ψ_1; M is independent of δ.) Then

$$W(x) = x^2(g(x)^2 - h(x)^2 + 2ig(x)h(x)), \quad |h(x)| \le M, \quad g(x) > 2\theta. \tag{2.13}$$

and, for $|x| < 2\delta$, (changing g and h to g/c_0 and h/c_0)

$$\begin{aligned} |c_0W(x) + 4\xi^2|^2 &= (c_0x^2(g(x)^2 - h(x)^2) + 4\xi^2)^2 + 4x^4g(x)^2h(x)^2 \\ &\ge c_2^2\theta^2(\theta^2x^2 + \xi^2)^2, \end{aligned} \tag{2.14}$$

with $c_2 > 0$ independent of θ and δ. (In fact, put $a = g(x)$ and $b = h(x)$, noting that $a > \theta/c_0$. If $b^2/a^2 \le \frac{1}{2}$ then (2.14) is immediate. Otherwise

$$\begin{aligned} (x^2(a^2-b^2) + 4\xi^2)^2 + 4x^4a^2b^2 &= (a^2+b^2)^2x^4 + 2(a^2-b^2)x^2(4\xi^2) + (4\xi)^4 \\ &\ge \gamma((a^2+b^2)^2x^4 + (4\xi)^4) \ge \gamma((\theta/c_0)^2x^4 + (4\xi)^4) \end{aligned}$$

if $(a^2-b^2)^2 \le (1-\gamma)^2(a^2+b^2)^2$. which gives (2.14). Since $\frac{1}{2} \le b^2/a^2 \le M^2c_0^2/\theta^2$ (see (2.13)) this holds with $\gamma \ge C\theta^2$, assuming that θ is small enough, independently of δ.)

Since $V_0(z) + \zeta^2 = \frac{1}{2}(1+if'(x))^{-2}(c_0W(x) + 4\xi^2)$, we obtained a quantitative version of (2.11) with V replaced by V_0, which then gives (2.11) for V (since $V(z) - V_0(z) = \mathcal{O}(z^3)$) for $|x| < 2\delta$, if δ is small enough. ∎

2.3. **Resonance free regions.** For $\lambda_{\ell,m}$, in Theorem 2 are given by the square roots of the eigenvalues of P_θ

$$P_\theta = P_{\Gamma_\theta}, \quad P := (hD_x)^2 + W(x,h), \quad h = (\ell + \tfrac{1}{2})^{-\frac{1}{2}}, \tag{2.15}$$

see Lemma 2.1 and §2.2. We now explain why to obtain the theorem we only need find eigenvalues of these eigenvalues in $\{|z - W_0(x_0)| < \delta\}$, $\delta > 0$ where x_0 is the critical point of W_0. Suppose that $0 < E$ satisfies $|W_0(x_0) - E| > \delta$. Then, as W_0 has a unique maximum at W_0, E is a *non-trapping* energy level for $p := \xi^2 + W_0(x)$ in the sense $p^{-1}(E) \subset T^*\mathbb{R}$ is connected and unbounded. Since W_0 is globally analytic it follows that

$$\exists\, \varepsilon_0, h_0 \quad \operatorname{Spec}(P_\theta) \cap \{z : |z - E| < \varepsilon_0\} = \emptyset, \quad 0 < h < h_0, \tag{2.16}$$

see [Sj02, §12.5] for a self-contained presentation and references. This implies that for θ_0 small enough. Rescaling by $h = 1/(\ell + \frac{1}{2})$ implies that for any (small) $c_0, \delta_0 > 0$ there exists t_0 such that

$$\operatorname{QNM}_\ell(m,\Lambda) \cap U(c_0,\delta_0,t_0) = \emptyset, \quad E_0 = W_0(x_0),$$
$$U(c_0,\delta_0,t_0) := \{\lambda : \operatorname{Re}\lambda \in [c_0\ell, (E_0 - \delta_0)\ell] \cup [(E_0+\delta_0)\ell, \ell/c_0],\ \operatorname{Im}\lambda_0 > -t_0 \operatorname{Re}\lambda\}.$$

To see that

$$\operatorname{QNM}_\ell(m,\Lambda) \cap \{\lambda : \operatorname{Re}\lambda \in [\ell/c_0, \infty),\ \operatorname{Im}\lambda_0 > -t_0 \operatorname{Re}\lambda\} = \emptyset, \tag{2.17}$$

we need a different choice of h in the passage from the Regge–Wheeler equation $D_x^2 + V_\ell(x)$ (see (2.1)) to $(hD_x)^2 + W(x,h)$. For that we write

$$P - z = h^2(D_x^2 + V_\ell(x) - \lambda^2) = (hD_x)^2 + h^2(\ell + \tfrac{1}{2})^2 W_0(x) + h^2 W_1(x) - z, \quad z := (\lambda h)^2.$$

If $h < (\ell + \frac{1}{2})^{-1}\sqrt{E}_0(1-\delta)$, $\delta > 0$, then the energy level $z = 1$ is non-trapping for P. Hence, as in (2.16), there exists h_0 and ε_0, such that if $h < \ell + \frac{1}{2})^{-1}E_0(1-\delta)$. $\ell \gg 1$, then P has no resonances z (elements of the spectrum of P_θ) for $|z - 1| < \varepsilon_0$. We now take $1/h = \operatorname{Re}\lambda > (\ell + \frac{1}{2})/\sqrt{(}E_0(1-\delta))$ and that give (2.17).

This shows that to obtain Theorem 2 it is sufficient to describe the eigenvalues of P_θ near $W_0(x_0)$, the maximal value of W_0.

When $\Lambda > 0$ we can replace c_0 by 0 and then Theorem 2 implies (1.2) in Theorem 1. This follows from [SáZw97, Proposition 4.4, (4.9)] and the fact that for $\Lambda > 0$ quasinormal modes form a discrete set in $\mathbb{C}$ (see [SáZw97, §2] and also the careful presentation in [BH08]).

Remark. The same conclusion should be valid for the case of $\Lambda = 0$ but as far as the authors are aware there is no mathematical argument for the discreteness of $\operatorname{QNM}(m,0)$ in $\{\lambda : \operatorname{Im}\lambda > -t_0|\operatorname{Re}\lambda|\}$ and for

$$\operatorname{QNM}_\ell(m,0) \cap \{|\operatorname{Re}\lambda| < c_0\ell, \operatorname{Im}\lambda > -t_0 \operatorname{Re}\lambda\} = \emptyset, \quad 0 < c_0, t_0 \ll 1.$$

The latter would be needed to obtain (1.2) for $\Lambda = 0$.

3. Analytic symbols

A (formal) classical analytic symbol in $\Omega \subset \mathbb{C}^d$ is a (formal) expression

$$\begin{gathered} a(z;h) := \sum_{k=0}^{\infty} h^k a_k(z), \quad a_k \in \mathscr{O}(\Omega), \\ \forall\, K \Subset \Omega \; \exists\, C = C(K) \;\; |a_k(z)| \leq C^{k+1} k^k, \;\; z \in K, \;\; k = 0, 1, 2, \ldots . \end{gathered} \tag{3.1}$$

Since all the symbols we use are classical we will drop that specification.

For open $\Omega_1 \Subset \Omega$ we have a *realization* of $a(z;h)$ given by the following holomorphic function:

$$a_{\Omega_1}(z;h) := \sum_{k=0}^{[(eC(\Omega_1)h)^{-1}]} a_k(z) h^k, \quad \Omega_1 \subset \Omega_2 \Subset \Omega \;\Rightarrow\; a_{\Omega_1} \equiv a_{\Omega_2} \text{ in } H_0(\Omega_1). \tag{3.2}$$

(See (5.4) for another way to *realize* a as a holomorphic function.) We refer to [Sj82, §1] or [HiSj15, §2.2] for a detailed account.

For $a = a(x, \xi; h)$ and $b = b(x, \xi; h)$, analytic symbols in $\Omega \subset \mathbb{C}^n_x \times \mathbb{C}^n_\xi$ we define

$$a \# b(x, \xi; h) := \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial_\xi^\alpha a(x, \xi; h) (hD_x)^\alpha b(x, \xi; h). \tag{3.3}$$

Then $a\#b$ is a analytic symbol in Ω. We now recall the fundamental result of Boutet de Monvel–Krée: if $a \neq 0$ on Ω and $\Omega_0 \Subset \Omega$ is an open set, then the formal symbol b defined by

$$a \# b = 1 \tag{3.4}$$

is a formal analytic symbol in Ω_0. The proof given by Sjöstrand in [Sj82] uses norms on formal pseudodifferential operators

$$\hat{a} = a(x, hD_x, h), \qquad a(x, \xi, h) = \sum_{\ell=0}^{\infty} a_\ell(x, \xi) h^\ell, \qquad \hat{a} \circ \hat{b} = \widehat{a \# b}.$$

To recall the definition of the norms we assume for simplicity that $\Omega = \{(x, \xi) : x_j, \xi_j \in \mathbb{C}, |x_j| < r_j, |\xi_j| < \rho_j\} \subset \mathbb{C}^{2n}$ is a polydisc. We then define a family of polydiscs by

$$\Omega(t) := \{(x, \xi) : x_j, \xi_j \in \mathbb{C}, |x_j| < r_j - t, |\xi_j| < \rho_j - t\}, \quad t < t_0 := \min_j \min(r_j, \rho_j).$$

Then $\Omega(t) \subset \Omega(s)$ if $t > s$ and by the Cauchy estimates

$$\sup_{\Omega(t)} |D^\alpha u| \leq (t - s)^{-|\alpha|} \alpha! \sup_{\Omega(s)} |u|, \quad s < t < t_0, \quad u \in \mathscr{O}(\Omega).$$

This allows the following definition:

$$\begin{aligned} \|\hat{a}\|_\rho &= \sum_{k=0}^{\infty} \frac{\rho^k}{k!} f_k(\hat{a}), \qquad f_k(\hat{a}) := \sup_{0\leq s<t<t_0} \frac{(t-s)^k \|A_k\|_{t,s}}{k^k}, \\ A_k(x,\xi;D_x) &:= \sum_{|\alpha|+\ell=k} \frac{1}{\alpha!} \partial_\xi^\alpha a_\ell(x,\xi) D_x^\alpha, \qquad \|B\|_{t,s} = \sup_{u\in\mathscr{O}(\Omega)} \frac{\sup_{\Omega(t)}|Bu|}{\sup_{\Omega(s)}|u|}. \end{aligned} \tag{3.5}$$

(Formally, $a(x,\xi+hD_x) = \sum_{k=0}^{\infty} h^k A_k$.)

The key property is given by

$$\|\hat{a}\circ\hat{b}\|_\rho \leq \|\hat{a}\|_\rho \|\hat{b}\|_\rho. \tag{3.6}$$

One immediate consequence is that the formal symbol b defined by $\hat{b} = \exp\hat{a}$ is an analytic symbol.

We also use the following technical result of Hitrik–Sjöstrand [HiSj08, Proposition 3.2]. (We stress that the two page proof can be read independently of the rest of the paper and depends only on the definitions reviewed above.)

Lemma 3.1. *For analytic sumbols a and b, there exists a constant $C(b)$ (independent of h and ρ) such that*

$$\|[\hat{a},\hat{b}]\|_\rho \leq C(b)\rho\|\hat{a}\|_\rho.$$

We need the following application of this lemma:

$$\|\operatorname{ad}_{\hat{a}}^p \hat{q}\|_\rho \leq \rho 2^p C(q) \|\hat{a}\|_\rho^p. \tag{3.7}$$

Proof of (3.7). We proceed by induction on p, with $p=1$ given by the lemma. Then

$$\|\operatorname{ad}_{\hat{a}}^{p+1}\hat{q}\|_\rho = \|\hat{a}\circ\operatorname{ad}_{\hat{a}}^p\hat{q} - \operatorname{ad}_{\hat{a}}^p\hat{q}\circ\hat{a}\| \leq 2\|\hat{a}\|_\rho \|\operatorname{ad}_{\hat{a}}^p\hat{q}\|_\rho \leq 2^{p+1}C(q)\rho\|a\|_\rho,$$

completing the proof. ∎

The key result we need is an analytic symbol version of an averaging result. We present it in a special case needed here:

Proposition 3.2. *Suppose that $q = q(z,\zeta,h)$ is an analytic symbol,*

$$q = \sum_{\ell=0}^{\infty} h^\ell q_\ell, \quad q_0(z,\zeta) = g(z\zeta), \quad q_j(z,\zeta) = g_j(z\zeta), \quad j\leq 1 \tag{3.8}$$

Then there exist analytic symbols $a(z,\zeta,h)$ and

$$G(w,h) = \sum_{\ell=0}^{\infty} h^\ell G_\ell(w), \quad G_0(w) = g(w),$$

such that, in the sense of formal asymptotic expansions,

$$e^{\hat{a}}\circ\hat{q}\circ e^{-\hat{a}} = \hat{b}, \quad b(z,\zeta,h) = G(z\zeta,h). \tag{3.9}$$

Remark. The assumption on q_1 in (3.8) is not a serious restriction. Using Lemma 3.3 below we can can find an analytic symbol a_0 such that

$$e^{\hat{a}_0} \circ \hat{q} \circ e^{-\hat{a}_0} = \hat{b}_0 + h^2 \hat{b}_1, \quad b_0(z, \zeta, h) = g(z\zeta) + hg_1(z\zeta), \tag{3.10}$$

and b_1 is an analytic symbol. The point of Proposition 3.2 is that this can be iterated to infinite order with analytic symbols.

The proof uses the following lemma. We note that We note that if $g(w) = w$ (3.12) is the same as

$$\{g(z\zeta), b(z, \zeta)\} = a(z, \zeta) + a_0(z\zeta), \tag{3.11}$$

where $\{f, g\} := \partial_\zeta f \partial_z g - \partial_z f \partial_\zeta g$ is the Poisson bracket with respect to the complex symplectic form

$$\sigma := d\zeta \wedge dz.$$

Lemma 3.3. *Let $\Omega(\delta) = D(0, \delta) \times D(0, \delta) \subset \mathbb{C}^{2n}$ and suppose that $r \in \mathscr{O}(\Omega)$, $\Omega(\delta) \Subset \Omega$. Then there exists $a \in \mathscr{O}(\Omega(\delta))$ and $r_0 \in \mathscr{O}(D(0, \delta^2))$ such that for a constant K depending only on g,*

$$\begin{gathered} i(z\partial_z - \zeta\partial_\zeta)a(z, \zeta) = -r(z, \zeta) + \langle r \rangle(z, \zeta), \quad \langle r \rangle(z, \zeta) = r_0(z\zeta) \\ \sup_{D(0,\delta^2)} |r_0| \leq K \sup_{\Omega(\delta)} |a|, \quad \sup_{\Omega(\delta)} |a| \leq K \sup_{\Omega(\delta)} |r - \langle r \rangle|. \end{gathered} \tag{3.12}$$

Proof. We note that that we can assume that $\delta = 1$ and that we can write

$$ir(z, \zeta) = \sum_{m,n \in \mathbb{N}} r_{mn} z^m \zeta^m, \qquad |r_{mn}| \leq \sup_{\Omega(1)} |a|,$$

which gives

$$\begin{gathered} a(z, \zeta) := \sum_{m \neq n} r_{mn}(m - n)^{-1} z^m \zeta^n, \quad r_0(w) := i \sum_{m \in \mathbb{N}} r_{mm} w^m, \\ |a(z, \zeta)| \leq C(1 - |z|)^{-1}(1 - |\zeta|)^{-1}, \quad |r_0(w)| \leq C(1 - |w^2|)^{-1}. \end{gathered}$$

This shows that solutions exists in $\Omega(1)$ and $D(0, 1)$ respectively.

Writing $z = re^{i\theta}$, $\zeta = \rho e^{i\theta}$, $0 < r, \rho < 1$ we consider the functions a and r_0 on tori

$$T(r, \rho) := \{(re^{i\theta}, \rho e^{i\varphi}) : 0 \leq \theta, \varphi < 2\pi\},$$

where they are given by

$$ir(z, \zeta)|_{T(r,\rho)} = R(\theta, \varphi), \quad r_0(z\zeta)|_{T_{r,\rho}} = R_0(\theta, \varphi), \quad (\partial_\theta - \partial_\varphi)A = iR - iR_0,$$

with Fourier series expansions

$$\begin{gathered} R(\theta, \varphi) = \sum r_{nm} r^n \rho^m e^{i(m\theta + n\varphi)}, \quad A(\theta, \varphi) = \sum \frac{r_{nm}}{n - m} r^n \rho^m e^{i(m\theta + n\varphi)}, \\ R_0(\theta, \varphi) = \sum r_{nn} r^n \rho^n e^{n(\theta + \varphi)}. \end{gathered}$$

It is now easy to check that A and R_0 have the following integral representations

$$A(\theta,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} tR(\theta+t,\varphi-t)dt, \quad R_0(\theta,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} R(\theta+t,\varphi+t)dt.$$

This gives uniform bounds

$$\sup_{T(r,\rho)} |a(z,\zeta)| \leq K \sup_{\Omega(1)} |r| \quad \sup_{T(r,\rho)} |r_0(z\zeta)| \leq K \sup_{\Omega(1)} |r|, \quad 0 < r < 1,\ 0 < \rho < 1.$$

By continuity the bounds remain valid for $0 \leq r, \rho < 1$ and (3.12) follows. $\square$

Proof of Proposition 3.2. We modify (with a simpler goal) the proof in [HiSj08, §3]. We first recall standard notation:

$$\mathrm{Ad}_{\hat a}\, \hat q := e^{\hat a} \circ \hat q \circ e^{-\hat a}, \quad \mathrm{ad}_{\hat b}\, \hat q := [\hat a, \hat q], \quad \mathrm{Ad}_{\hat a}\, \hat q = \sum_{k=0} \frac{1}{k!} \mathrm{ad}_{\hat a}^k\, \hat q,$$

where all the expressions are understood as compositions of formal pseudodifferential operators quantizing formal analytic symbols.

We start by reducing the problem to the case of $q_0(z\zeta) = z\zeta$. For that we find a holomorphic function $f = q_0^{-1}$, near 0. We then define an analytic symbol

$$q_{\mathrm{new}} = f(0) + f'(0)q + \tfrac{1}{2} f''(0) q \# q + \cdots,$$

where, as formal pseudodifferential operators, $f(\hat q) = \widehat{q_{\mathrm{new}}}$, and

$$q_{\mathrm{new}}(z,\zeta) = iz\zeta + hq_{\mathrm{new},1}(z,\zeta) + h^2 q_{\mathrm{new},2}(z,\zeta) + \cdots, \quad q_{\mathrm{new},1}(z,\zeta) = g_{\mathrm{new},1}(z\zeta).$$

In the sense of asymptotic expansions,

$$f(Ad_{\hat a}\hat q) = \mathrm{Ad}_{\hat a}\, f(\hat q).$$

Hence in proving (3.9) we can replace q by q_{new}, that is assume that

$$q_0(z\zeta) = z\zeta. \tag{3.13}$$

We now have the following consequence of (3.7) in which we consider q as fixed and have all the constants depending on q. Suppose that $\|\hat b_1\|_\rho \leq \|\hat b_0\|_\rho$. Then

$$\begin{gathered} \mathrm{Ad}_{\hat b_0 + \hat b_1}\, \hat q = \mathrm{Ad}_{\hat b_0}\, \hat q + \mathrm{ad}_{\hat b_1}\, \hat q + R(\hat b_0, \hat b_1), \\ \|R(\hat b_0,\hat b_1)\|_\rho \leq C\rho \|\hat b_0\|_\rho \max(\|\hat b_0\|_\rho, \|\hat b_1\|_\rho) e^{2\max(\|b_0\|_\rho, \|b_1\|_\rho)}. \end{gathered} \tag{3.14}$$

To see this we put $J(k) = \{0,1\}^k \setminus (0,\cdots,0)$ and write

$$R(\hat b_0, \hat b_1) = \sum_{k=2} \frac{1}{k!} \sum_{j \in J(k)} \mathrm{ad}_{\hat b_{j_1}} \cdots \mathrm{ad}_{\hat b_{j_k}}\, \hat q.$$

An application of Lemma 3.1 as in the proof of (3.7) and the fact that $\|\hat b_1\|_\rho \leq \|\hat b_1\|$ show that

$$\|\, \mathrm{ad}_{\hat b_{j_1}} \cdots \mathrm{ad}_{\hat b_{j_k}}\, \hat q\|_\rho \leq C\rho k \max(\|b_0\|_\rho, \|b_1\|_\rho)^{k-1} \|b_1\|_\rho,$$

and hence

$$\begin{aligned}\|R(\hat b_0,\hat b_1)\|_\rho &\le C\rho\|b_1\|_\rho\sum_{k=2}^\infty \frac{1}{(k-1)!}(2\max(\|b_0\|_\rho,\|b_1\|_\rho))^{k-1}\\ &= K\rho\|b_1\|_\rho(e^{2\max(\|\hat b_0\|_\rho,\|\hat b_1\|_\rho)}-1),\end{aligned}$$

proving the estimate in (3.14). (We will use K for a fixed large constant depending only on q.)

We now use (3.13) and Lemma 3.3 to obtain

$$\hat a := \mathcal{L}(\hat r), \quad i\{a,q_0\} = -r + \langle r\rangle, \quad r_0(z,\zeta) = r_0(z\zeta).$$

Since q_0 is quadratic, we have an exact formula

$$\mathrm{ad}_{\mathcal{L}(\hat r)}\,\hat q_0 = -h\hat r + h\langle \hat r\rangle, \quad \|\mathcal{L}(\hat r)\|_\rho \le K\|\hat r\|_\rho. \tag{3.15}$$

Here, and below, we simplify the notation writing $\langle\hat r\rangle$ for $\widehat{\langle r\rangle}$.

In the notation of (3.8) we put

$$hr(z,\zeta,h) = \sum_{\ell=2} h^\ell q_\ell(z,\zeta), \quad \|\hat r\|_\rho \le Kh$$

and define

$$\hat a_0 := \mathcal{L}(\hat r), \quad \|\hat a_0\|_\rho \le K\|\hat r\|_\rho \le K^2 h.$$

We then use (3.14) with $b_0 = 0$, $b_1 = \hat a_0$ to obtain

$$\begin{aligned}\mathrm{Ad}_{\hat a_0}\,\hat q &= \hat q + \mathrm{ad}_{\hat a_0}\,\hat q + R(0,\hat a_0) = \hat q_0 + h\langle\hat r\rangle + R(0,\hat a_0)\\ &= \hat q_0 + h\langle \hat r\rangle + h\hat r_1,\end{aligned}$$

where

$$\|\hat r_1\|_\rho \le K\rho h^{-1}\|\hat a_0\|_\rho^2 e^{2\|\hat a_0\|_\rho} + h\|\hat r\|_\rho \le B\rho h.$$

Here B is a large constant which will be chosen later to close the argument.

We now assume that we found a_ℓ, $\ell \le j$ such that (with $r_0 := r$)

$$\begin{gathered}\mathrm{Ad}_{\hat a_0+\cdots \hat a_j}\,\hat q = \hat q_0 + h\sum_{\ell=0}^j\langle\hat r_\ell\rangle + h\hat r_{j+1},\\ \|\hat a_\ell\|_\rho \le K^2B^\ell\rho^\ell h, \quad \|\hat r_{\ell+1}\|_\rho \le KB^{\ell+1}\rho^{\ell+1}h, \quad 0\le\ell\le j.\end{gathered} \tag{3.16}$$

We now define

$$\hat a_{j+1} = \mathcal{L}(\hat r_{j+1}), \quad \|\hat a_{j+1}\|_\rho \le K^2B^{j+1}\rho^{j+1}h.$$

Then

$$\begin{aligned}\mathrm{Ad}_{\hat a_0+\cdots+\hat a_{j+1}}\,\hat q &= \mathrm{Ad}_{\hat a_0+\cdots \hat a_j}\,\hat q + \mathrm{ad}_{\hat a_1}\,\hat q + R(\hat a_0+\cdots \hat a_j, \hat a_{j+1})\\ &= \hat q_0 + h\sum_{\ell=0}^{j+1}\langle \hat r_\ell\rangle + h\hat r_{j+2},\end{aligned}$$

where

$$\hat{r}_{j+2} := h^{-1}R(\hat{a}_0 + \cdots \hat{a}_j, \hat{a}_{j+1}).$$

Using (3.14) and (3.15) we obtain

$$\begin{aligned}\|\hat{r}_{j+2}\|_\rho &\leq \rho h^{-1}\|\hat{a}_{j+1}\|_\rho \left(\sum_{\ell=0}^{j}\|\hat{a}_\ell\|_\rho\right) e^{2\max(\|a_{j+1}\|_\rho \sum_{\ell=0}^{j}\|\hat{a}_\ell\|_\rho)} + Kh\|\hat{r}_{j+1}\|_\rho \\ &\leq K^4\rho h(1-B\rho)^{-1}B^{j+1}\rho^{j+1}e^{2K^2h(1-B\rho)^{-1}} + hK^2B^{j+1}h\rho^{j+1} \\ &\leq KB^{j+2}\rho^{j+2}h,\end{aligned}$$

if we choose B large enough depending on K and h_0, $0 < h < h_0$.

Hence, by induction, we constructed, analytic symbols a_ℓ, $\ell \in \mathbb{N}$, satsifying (3.16). It follows that if $\rho < 1/B$ then $a := \sum_\ell a_\ell$ is an analytic symbol for which (3.9) holds. $\square$

We now need the following fact which we first establish on a formal level. For more on functional calculus for quadratic symbols see Dereziński [De93] and Hörmander [Hö95].

Lemma 3.4. *Suppose $F(z,\zeta,h)$ is a analytic symbol of the form*

$$F(z,\zeta;h) = f(z\zeta,h), \quad f(w,h) = \sum_{k=0}^{\infty} h^k f_k(w), \quad \sup_{D(0,\delta)} |f_k| \leq A^{k+1}k^k. \tag{3.17}$$

Then there exists a analytic symbol, $g(w,h) = \sum_{k=0}^{\infty} h^k g_k(w)$ such that, as formal pseudodifferential operators,

$$F^{\mathrm{w}}(z,hD_z;h) = g(zhD_z - ih/2, h). \tag{3.18}$$

Remark. Functional calculus of formal analytic pseudodifferential operators in reviewed in a self-contained way in [HelSj89, §a.2]. In our case,

$$g(zhDz - ih/2, h) := \frac{1}{2\pi i}\int_{\partial D(0,\varepsilon)} (\lambda - zhD_z)^{-1} g(\lambda + ih/2, h)d\lambda,$$

where $0 < \varepsilon \ll 1$ (independent of h) and, using (3.6), $(\lambda - zhD_z + ih/2)^{-1} = R(\lambda, z, hD_z, h)$, $\lambda \in \partial D(0,\varepsilon)$ where $R(\lambda, z, \zeta, h)$ is an analytic symbol for $(z,\zeta) \in \mathrm{neigh}_{\mathbb{C}^2}(0)$. If we write $\hat{p} = zhD_z$, $p := z\zeta$ then for a suffieciently small neighbourhood $\Omega = \mathrm{neigh}_{\mathbb{C}^2}(0)$ (in the definition (3.5)), $\|\hat{p}\|_\rho \leq \varepsilon/2$. This means that $(\lambda - \hat{p})^{-1} = \lambda^{-1}\sum_{k=0}^{\infty}(\hat{p}/\lambda)^k$ converges in the $\|\bullet\|_\rho$ norm and, as a formal pseudodifferential operator $R(\lambda, z, hD_z, h) = \lambda^{-1}\sum_{k=0}^{\infty}\lambda^{-k}(hzD_z)^k$. In particular, if $g(\lambda + ih/2, h) = \sum_{\ell=0}\sum_{p=0}^{\infty} g_{p\ell}h^\ell\lambda^p$, then

$$g(hD_z - ih/2, h) = \sum_{\ell=0}\sum_{p=0}^{\infty} g_{p\ell}h^\ell(hzD_z)^\ell,$$

with convergence of the corresponding analytic symbols.

Proof of Lemma 3.4. We will prove (3.18) for the usual quantization and for a modified g:

$$F(z, hD_z; h) = g(z, hD_x, h). \tag{3.19}$$

A modification for the Weyl quantization follows from [Zw12, Theorem 4.13] and the fact that $\exp(ithD_zD_\zeta)f(z\zeta, h) = f_t(z\zeta, h)$ and f_t is an analytic symbol.

Let us first assume that f is independent of h and that $f(0) = 0$ (the constant term contribution is obvious): $f(iw) = \sum_{n=1}^\infty f_n w^n$, where $|f_n| \leq B^{n+1}$, so that formally, (3.19) reads

$$\sum_{n=1}^\infty f_n z^n (h\partial_z)^n = \sum_{j=0}^\infty \sum_{n=1}^\infty g_{nj} h^j (zh\partial_z)^n, \quad g_j(iw) = \sum_{n=1}^\infty g_{nj} w^n.$$

An induction argument based on $(z\partial_z - n)(z^n\partial_z^n) = z^{n+1}\partial_z^{n+1}$ shows that

$$z^n\partial_z^n = p_n(z\partial_z), \quad p_n(t) := \prod_{j=0}^{n-1}(t-j) = \sum_{k=0}^{n-1} a_{k,n} t^{n-k}, \quad |a_{k,n}| \leq n^{k+1}.$$

This means that

$$\begin{aligned} g(iw, h) &= \sum_{n=1}^\infty f_n \sum_{k=0}^{n-1} h^k a_{k,n} w^{n-k} = \sum_{k=0}^\infty h^k \sum_{n=k+1}^\infty f_n a_{k,n} w^{n-k} \\ &= \sum_{k=0}^\infty h^k g_k(iw), \quad g_k(iw) := \sum_{p=1}^\infty f_{k+p} a_{k,k+p} w^p, \end{aligned}$$

and

$$\begin{aligned} |g_k(iw)| &\leq \sum_{p=1}^\infty B^{k+p+1}(k+p)^{k+1}|w|^p \leq B^{k+1}k^k \sum_{p=1} (Be|w|)^p \\ &\leq 2BeB^{k+1}k^{k+1}, \quad |w| \leq (2eB)^{-1}. \end{aligned}$$

Since (A.5) implies that $f_k(iw) = \sum_{p=0}^\infty f_{kp} w^p$, $f_{kp} \leq \delta^{-p} A^{k+1} k^k$, the general case follows from the h independent case. ☐

For completeness we also include the following, undoubtedly well known, Proposition. Its relevance comes from the fact that in the WKB literature (see for instance [AAH22]) one typically computes the quantum action $S(E, h)$ while the eigenvalues (or resonances) are given in terms of its inverse $G(z, h)$. In this paper we construct G directly but of course the connection with actions is there – see Theorem A.2 for an indication of this.

Proposition 3.5. *Suppose that $S(x, h)$, $x \in \mathrm{neigh}_{\mathbb{C}}(0)$ is a analytic symbol such that $S_0(0) = 0$, $S_0'(0) \neq 0$. Then there exists a analytic symbol $G(x, h)$, such that*

$$S(G(x, h), h) \equiv x, \quad x \in \mathrm{neigh}_{\mathbb{C}}(0). \tag{3.20}$$

The equivalence is in the sense of formal analytic symbols.

Proof. We define a simpler version of the semi-norms on analytic symbols:

$$\|A\|_\rho := \sum_{k=0}^\infty \frac{\rho^k}{k!}\|A_k\|, \quad A(x) = \sum_{k=0}^\infty A_k(x)h^k, \quad \|f\| := \sup_{x\in\Omega}|f(x)|, \tag{3.21}$$

where Ω is a fixed sufficiently small neighbourhood of 0. This norm is finite for *some* $\rho > 0$ if and only if A is a analytic symbol. We recall from (3.6) (or rather from a much simpler special case) that

$$\|H^\ell\|_\rho \leq \|H\|_\rho^\ell, \quad H(x,h)^\ell = \sum_{k=0}^\infty H_{\ell k}(x)h^k. \tag{3.22}$$

If $S(x,h) = \sum_{j=0}^\infty S_j(x)h^j$, then we can find $G_0(x)$ such that $S_0(G_0(x)) = x$ near 0. Replacing $S(x,h)$ by $S(G_0(x),h)$ we can assume that $S_0(x) = x$, write $S(x,h) = x - hF(x,h)$, and to postulate the form of $G(x,h)$ to be

$$x + \sum_{\ell=1}^\infty h^\ell G_\ell(x) =: x + hH(x,h) = x + h\left(\sum_{\ell=0}^\infty h^\ell H_\ell(x)\right).$$

Hence, (3.20) becomes

$$H(x,h) = F(x + hH(x,h)) = \sum_{\ell=0}^\infty \frac{h^\ell}{\ell!}F^{(\ell)}(x,h)H(x,h)^\ell, \tag{3.23}$$

and

$$\frac{1}{\ell!}F^{(\ell)}(x,h) = \sum_{k=0}^\infty F_{m\ell}(x)h^m, \quad F_{m\ell}(x) := \frac{1}{\ell!}F_m^{(\ell)}(x), \quad \|F_{m\ell}\| \leq A^{m+1}m! \tag{3.24}$$

The terms in the expansion of $H(x,h)$ are obtained from an iteration procedure based on (3.23) (note that $H_{0p}(x) = \delta_{0p}$):

$$H_k(x) = \sum_{p+m+\ell=k} F_{m\ell}(x)H_{\ell p}(x) = F_{k0}(x) + \sum_{\ell=1}^k\sum_{m=0}^{k-\ell}\sum_{p=0}^{k-\ell-m} F_{m\ell}(x)H_{\ell p}(x). \tag{3.25}$$

We now define

$$M_n := \sum_{k=0}^n \frac{\rho^k}{k!}\|H_k\|, \quad \|H\|_\rho = \sup_n M_n.$$

Since $H_{\ell k}$, $k < n$, depend only on H_p with $p < n$, we see that (3.22) implies

$$\sum_{k=0}^{n-1} \frac{\rho^k}{k!}\|H_{\ell k}\| \leq M_{n-1}^\ell, \tag{3.26}$$

We now want to estimate M_n in terms of M_{n-1}: for that we use (3.24) and (3.26) in (3.25) to obtain, with $B = A(1-\rho_0 A)^{-1}$, $0 < \rho < \rho_0 < 1/A$,

$$M_n \leq B + \sum_{\ell=1}^{n} \sum_{m+p=n-\ell} \frac{\rho^{m+\ell+p} \|F_{m\ell}\| \|H_{\ell p}\|}{(m+p+\ell)!} \leq B + \sum_{\ell=1}^{n} \sum_{p=0}^{n-1} \sum_{m=0}^{n-1} A(A\rho)^m \frac{\|H_{\ell p}\| \rho^p}{p!} \frac{\rho^\ell}{\ell!}$$
$$\leq B + B \sum_{\ell=1}^{n} \frac{M_{n-1}^\ell \rho^\ell}{\ell!} = B \exp\left(\rho M_{n-1}\right).$$

Putting $a_n := M_n/B$ we have the following iterative inequality $a_n \leq \exp\left(\rho B a_{n-1}\right)$, $a_0 \leq 1$. It has been known since Euler that a_n is bounded if $\rho B \leq 1/e$. For ρ small enough we then have $\|H\|_\rho < \infty$, proving that H, and hence G, are analytic symbols. $\square$

4. Review of pseudodifferential and Fourier integral operators

Here we recall, with proofs or precise references, the needed results from microlocal analytic theory.

4.1. **Global pseudodifferential operators.** We describe the action of pseudodifferential operators defined by globally analytic symbols on weighted spaces of holomorphic functions following [HiSj15][Chapter 1] and [Sj02][§12] (see also [Zw12][Chapter 13] for a gentle introduction).

We start with Φ_0, a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$, and the Hilbert space of weighted holomorphic functions:

$$H_{\Phi_0} = H_{\Phi_0}(\mathbb{C}^n) := \{u \in \mathscr{O}(\mathbb{C}^n) : \|u\|_{\Phi_0}^2 := \int_{\mathbb{C}^n} |u(x)|^2 e^{-2\Phi_0(x)/h} dL(x) < \infty\}, \quad (4.1)$$

where $dL(x)$ is the standard Lebesgue measure on $\mathbb{C}^n = \mathbb{R}^{2n}$. To this space we we associate a geometric object: a real linear subspace of $\mathbb{C}^{2n}$:

$$\Lambda_{\Phi_0} = \left\{ \left(x, \tfrac{2}{i}\partial_x \Phi_0(x)\right), \, x \in \mathbb{C}^n \right\}. \quad (4.2)$$

A function $1 \leq m \in C^\infty(\Lambda_{\Phi_0})$ is called an order function on Λ_{Φ_0} if for some $C_0 > 0$, $N_0 > 0$, we have

$$m(X) \leq C_0 \langle X - Y \rangle^{N_0} m(Y), \quad X, Y \in \Lambda_{\Phi_0}. \quad (4.3)$$

Since $\pi : \Lambda_{\Phi_0} \to \mathbb{C}^n$, $\pi(x,\xi) : x$ identifies Λ_{Φ_0} with $\mathbb{C}^n$ we can consider m as a function on $\mathbb{C}^n$. To weights m we associate a more general class of weighted spaces generalizing Sobolev spaces:

$$H_{\Phi_0,m} = H_{\Phi_0,m}(\mathbb{C}^n) := \mathscr{O}(\mathbb{C}^n) \cap L^2_{\Phi_0,m}, \qquad L^2_{\Phi_0,m} := \{u : \|u\|_{\Phi_0,m} < \infty\},$$
$$\|u\|_{\Phi_0,m}^2 := \int_{\mathbb{C}^n} m(x)^2 |u(x)|^2 e^{-2\Phi_0(z)/h} dL(x) \quad (4.4)$$

Let $P = P(x, \xi; h)$ be a holomorphic function in $\Lambda_{\Phi_0} + B_{\mathbb{C}^{2n}}(0, \delta)$, $\delta > 0$, and assume that

$$|P(x, \xi; h)| \leq Cm(x), \quad (x, \xi) \in \Lambda_{\Phi_0} + B_{\mathbb{C}^{2n}}(0, \delta), \tag{4.5}$$

and that there is a complete asymptotic expansion

$$P(x, \xi; h) \sim \sum_{k=0}^{\infty} h^k p_k(x, \xi), \quad p := p_0. \tag{4.6}$$

in the space of holomorphic functions satisfying (4.5). Eventually, we strengthen this to analytic symbol expansion in the sense of §3. We also demand that P is analytic near infinity:

$$|p(x, \xi)| \geq m(x)/C, \quad (x, \xi) \in \Lambda_{\Phi_0} + B_{\mathbb{C}^{2n}}(0, \delta), \quad |(x, \xi)| \geq R. \tag{4.7}$$

The Weyl quantization of P, $P^{\mathrm{w}}(x, hD_x; h)$is an unbounded operator on H_{Φ_0} defined by

$$P^{\mathrm{w}}(x, hD_x; h)u(x) = \frac{1}{(2\pi h)^n} \iint_{\Gamma_{\Phi_0}(x)} e^{\frac{i}{h}(x-y)\cdot\theta} P\left(\frac{x+y}{2}, \theta; h\right) u(y)\, dy\, d\theta, \tag{4.8}$$

where the contour of integration $\Gamma_{\Phi_0}(x) \subset \mathbb{C}^{2n}_{y,\theta}$ is given by

$$y \mapsto \theta = \frac{2}{i}\frac{\partial \Phi_0}{\partial x}\left(\frac{x+y}{2}\right) + \frac{i\delta}{2}\frac{\overline{(x-y)}}{\langle x-y\rangle}. \tag{4.9}$$

(We note that $((x+y)/2, \theta) \in B_{\mathbb{C}^{2n}}(0, \delta)$. For a discussion of contours of integration in this context see [Zw12][§13.2.1].) If $\widetilde{m}$ is another order function then an application of Schur's inequality [DyZw19][(A.5.3)] to (4.8, (4.9), gives

$$P^{\mathrm{w}}(x, hD_x; h) = \mathcal{O}(1) : H_{\Phi_0, \widetilde{m}} \to H_{\Phi_0, \widetilde{m}/m}, \tag{4.10}$$

see [HiSj15], [Sj02, Section 12.2]. In particular, for the domain of P^{w} as an unbounded operator on H_{Φ_0} we can take $H_{\Phi_0, m}$.

The holomorphy of P in a tubular neighbourhood of Λ_{Φ_0} allows us to consider other weights as well:

Proposition 4.1. *Suppose that $\Phi = \Phi_0 + f \in C^\infty(\mathbb{C}^n)$ satisfies*

$$f \in L^\infty(\mathbb{C}^n), \quad \|\nabla^k f\|_{L^\infty(\mathbb{C}^n)} \leq \varepsilon, \quad k = 1, 2, \tag{4.11}$$

where ε is sufficiently small depending on δ in (4.5). Then

$$P^{\mathrm{w}}(x, hD_x; h) = \mathcal{O}(1) : H_{\Phi, \widetilde{m}} \to H_{\Phi, \widetilde{m}/m}, \tag{4.12}$$

where the exponentially weighted spaces are defined as in (4.4), replacing Φ_0 by Φ.

Proof. Define a new contour

$$\Gamma_\Phi(x) : y \mapsto \theta = \frac{2}{i}\frac{\partial \Phi}{\partial x}\left(\frac{x+y}{2}\right) + \frac{i\delta}{2}\frac{\overline{(x-y)}}{\langle x-y\rangle}, \tag{4.13}$$

and performing a contour deformation in (4.8) we obtain a different formula for the action of $P^{\mathrm{w}}(x, hD_x; h)$ adapted to the weight Φ,

$$P^{\mathrm{w}}(x, hD_x; h)u(x) = \frac{1}{(2\pi h)^n}\iint_{\Gamma_\Phi(x)} e^{\frac{i}{h}(x-y)\cdot\theta} P\left(\tfrac{x+y}{2}, \theta; h\right) u(y)\, dy\, d\theta. \tag{4.14}$$

Along the contour $\Gamma_\Phi(x)$, we have

$$\begin{aligned} &-\Phi(x) + \operatorname{Re}\,(i(x-y)\cdot\theta) + \Phi(y) \\ &\quad = -\Phi(x) + \operatorname{Re}\left(2\partial_x\Phi(\tfrac{x+y}{2})\cdot(x-y)\right) + \Phi(y) - (\delta/2)|x-y|^2/\langle x-y\rangle \\ &\quad = -f(x) + \langle \nabla f\left(\tfrac{x+y}{2}\right), x-y\rangle_{\mathbb{R}^2} + f(y) - (\delta/2)|x-y|^2/\langle x-y\rangle. \end{aligned} \tag{4.15}$$

Here

$$-f(x) + \langle \nabla f\left(\tfrac{x+y}{2}\right), x-y\rangle_{\mathbb{R}^2} + f(y) \le 2\|\nabla f\|_{L^\infty(\mathbb{C}^n)}\,|x-y|\,, \tag{4.16}$$

and an application of Taylor's formula gives that

$$\begin{aligned} -f(x) + \langle \nabla f\left(\tfrac{x+y}{2}\right), x-y\rangle_{\mathbb{R}^2} + f(y) &= \int_0^1 (1-t)f''\left(\tfrac{x+y}{2} - t\left(\tfrac{x-y}{2}\right)\right)\tfrac{(x-y)}{2}\cdot\tfrac{(x-y)}{2}\,dt \\ -\int_0^1 (1-t)f''\left(\tfrac{x+y}{2} + t\left(\tfrac{x-y}{2}\right)\right)\tfrac{(x-y)}{2}\cdot\tfrac{(x-y)}{2}\,dt &\le \tfrac{1}{4}\|\nabla^2 f\|_{L^\infty(\mathbb{C}^n)}\,|x-y|^2\,. \end{aligned} \tag{4.17}$$

(The Hessian and the scalar product are taken in the sense of $\mathbb{R}^{2n}$.) It follows, in view of (4.5), (4.11), (4.15), (4.16), and (4.17) that the absolute value of the effective kernel of the operator in (4.12), (4.14) does not exceed a multiple of

$$\begin{aligned} &h^{-n} m\left(\frac{x+y}{2}\right)\frac{\widetilde{m}(x)\widetilde{m}(y)}{m(x)}\exp\left(\frac{1}{h}\left(\frac{|x-y|}{\min(1,|x-y|)} - \frac{\delta}{2}\frac{|x-y|^2}{\langle x-y\rangle}\right)\right) \\ &\qquad\qquad \le Ch^{-n}\exp\left(-\frac{\delta}{4h}\frac{|x-y|^2}{\langle x-y\rangle}\right)\langle x-y\rangle^N, \end{aligned}$$

for some $N \ge 0$, provided that ε in (4.11) is sufficiently large. The conclusion (4.12) follows therefore from an application of Schur's inequality [DyZw19][(A.5.3)]. □

The next proposition gives a way of approximating the action of P by multiplication. The method goes back to the Cordoba–Fefferman proof of the sharp Gårding inequality and the version here comes from [Sj02, Section 12.4], see also [HiSj15, Proposition 1.4.4]. For the reader's convenience we present the proof.

Proposition 4.2. *Suppose that $\psi \in C^1(\mathbb{C}^n)$, $\psi, \nabla\psi \in L^\infty(\mathbb{C}^n)$, and set $\xi(x) = (2/i)\partial_x\Phi(x)$. Then for order functions m_j, $j = 1, 2$ satisfying $m_1 m_2 \geq m$,*

$$\begin{aligned}(\psi P^{\mathrm{w}}(x, hD_x; h)u_1, u_2)_{L^2_\Phi} = \int_{\mathbb{C}^n} \psi(x)\, p\,(x, \xi(x))\, u_1(x)\overline{u_2(x)}e^{-\frac{2}{h}\Phi(x)}\, dL(x) \\ + \mathcal{O}(h)\|u_1\|_{H_{\Phi,m_1}}\|u_2\|_{H_{\Phi,m_2}}, \quad u_j \in H_{\Phi,m_j}.\end{aligned} \tag{4.18}$$

Furthermore, for $u \in H_{\Phi,m}$,

$$\begin{aligned}&(\psi\, P^{\mathrm{w}}(x, hD_x; h)u, P^{\mathrm{w}}(x, hD_x; h)u)_{L^2_\Phi} = \\ &\int_{\mathbb{C}^n} \psi(x)\ |p\,(x, \xi(x))|^2\, |u(x)|^2\, e^{-2\Phi(x)/h}\, dL(x) + \mathcal{O}(h)\|u\|^2_{H_{\Phi,m}}.\end{aligned} \tag{4.19}$$

Proof. Taylor's formula gives (recall that $\xi(x) = (2/i)\partial_x\Phi(x)$):

$$\begin{aligned}p\left(\tfrac{x+y}{2}, \theta\right) = p(x, \xi(x)) + \tfrac{1}{2}\partial_x p(x, \xi(x)) \cdot (y - x) \\ + \partial_\xi p(x, \xi(x)) \cdot (\theta - \xi(x)) + r(x, y, \theta),\end{aligned} \tag{4.20}$$

where

$$r(x, y, \theta) = \int_0^1 (1 - t)\partial_t^2 \left[p\left(x + t\tfrac{y-x}{2}, \xi(x) + t(\theta - \xi(x))\right)\right] dt.$$

Using (4.5) together with the Cauchy inequalities, as well as the fact that $\theta - \xi(x) = \mathcal{O}(|x - y|)$ along the contour (4.13), we get

$$|r(x, y, \theta)| \leq Cm(x)\langle y - x\rangle^{N_0}\, |y - x|^2\,, \quad (y, \theta) \in \Gamma_\Phi(x). \tag{4.21}$$

Let

$$\begin{aligned}Ru(x) &= \frac{1}{(2\pi h)^n}\iint_{\Gamma_\Phi(x)} e^{\frac{i}{h}(x-y)\cdot\theta} r\,(x, y, \theta)\, u(y)\, dy\, d\theta \\ &=: \int k(x, y; h)u(y)dL(y).\end{aligned} \tag{4.22}$$

The bound of R as an operator $H_{\Phi,\widetilde{m}} \to L^2_{\Phi,\widetilde{m}/m}$ is given by the bound of the operator with the the reduced kernel (see [Zw12][Discussion after (13.4.3)]):

$$e^{-\Phi(x)/h}\frac{\widetilde{m}(x)}{m(x)\widetilde{m}(y)}k(x, y; h)e^{\Phi(y)/h}.$$

Estimates (4.15), (4.16), (4.17), and (4.21) show that it is bounded by a multiple of

$$h^{-n}\exp\left(-\frac{\delta}{4h}\frac{|x - y|^2}{\langle x - y\rangle}\right)\langle y - x\rangle^N\, |y - x|^2\,. \tag{4.23}$$

An application of Schur's inequality [DyZw19][(A.5.3)] shows that the right hand side in (4.23) is the integral kernel of a convolution operator on $L^2(\mathbb{C}^n)$ of norm $\mathcal{O}(h)$. It follows that

$$R = \mathcal{O}(h) : H_{\Phi,\widetilde{m}} \to L^2_{\Phi,\widetilde{m}/m} \tag{4.24}$$

Inserting (4.14) in (4.14) (and using the boundedness of the lower order terms) gives

$$P^{\mathrm{w}}(x,hD)u = p(x,\xi(x))u(x) + \frac{1}{(2\pi h)^n}\iint_{\Gamma_\Phi(x)} e^{\frac{i}{h}(x-y)\cdot\theta}\partial_\xi p(x,\xi(x))\cdot(\theta-\xi(x))u(y)dy\,d\theta$$
$$+ R_1u(x) + R_2u(x),$$

where $R_1 = \mathcal{O}(h) : H_{\Phi,\widetilde{m}} \to L^2_{\Phi,\widetilde{m}/m}$ (coming from R in (4.22) and lower order terms) and

$$\begin{aligned} R_2u(x) &= \frac{1}{4\pi h}\iint_{\Gamma_\Phi(x)} e^{\frac{i}{h}(x-y)\cdot\theta}\partial_x p(x,\xi(x))\cdot(y-x)u(y)dy\,d\theta \\ &= \frac{1}{4\pi h}\iint_{\Gamma_\Phi(x)} \partial_x p(x,\xi(x))\cdot ih\partial_\theta(e^{\frac{i}{h}(x-y)\cdot\theta})u(y)dy = 0. \end{aligned}$$

Noting that $\theta e^{\frac{i}{h}(x-y)\cdot\theta} = -hD_y e^{\frac{i}{h}(x-y)\cdot\theta}$ we obtain

$$P^{\mathrm{w}}(x,hD_x;h)u(x) = p(x,\xi(x))u(x) + \sum_{j=1}^{m}\partial_{\xi_j}p(x,\xi(x))(hD_{x_j}-\xi_j(x))u(x) + R_1u(x).$$

To obtain (4.18) we note that

$$(hD_{x_j}+\xi_j(x))e^{-2\Phi(x)/h} = (hD_{x_j}+2D_{x_j}\Phi(x))e^{-2\Phi(x)/h} = 0,$$

so that for $u_j \in H_{\Phi,m_j}$,

$$\begin{aligned} &\int_{\mathbb{C}^n}\psi(x)\partial_{\xi_j}p(x,\xi(x))(hD_{x_j}-\xi_j(x))u_1(x)\overline{u_2(x)}e^{-2\Phi(x)/h}dL(x) \\ &= -\int_{\mathbb{C}^n}u_1(x)(hD_{x_j}+\xi_j(x))[\psi(x)p(x,\xi(x))\overline{u_2(x)}e^{-2\Phi(x)/h}]dm(x) \\ &= -h\int_{\mathbb{C}^n}D_{x_j}(\psi(x)p(x,\xi(x))u_1(x)\overline{u_2(x)}e^{-2\Phi(x)/h}]dm(x) = \mathcal{O}(h)\|u_1\|_{H_{\Phi,m_1}}\|u_2\|_{H_{\Phi,m_2}}. \end{aligned}$$

The same argument gives (4.19). □

We conclude by applying (4.19) to obtain an elliptic estimate near infinity. It allows us to localize the spectral analysis to a neighbourhood of the critical point:

Proposition 4.3. *Suppose $P(x,\xi;h)$ satisfies (4.5), (4.6), (4.7), and let $\Phi = \Phi_0 + f$ with f satisfying (4.11). Then there exist $h_0 > 0$, $\eta > 0$, such that for $0 < h < h_0$ and $u \in H_{\Phi,m}$,*

$$\|u\|_{L^2_{\Phi,m}(\mathbb{C}^n\setminus B(0,2R))} \leq C\|P^{\mathrm{w}}(x,hD_x;h)u\|_{H_\Phi(\mathbb{C}^n\setminus B(0,R/2))} + e^{-\eta/h}\|u\|_{H_{\Phi,m}}. \tag{4.25}$$

Proof. Let $\chi \in C_0^\infty(\mathbb{C}^n;[0,1])$ satisfy $\chi(x) = 1$, $|x| \leq R$, and $\operatorname{supp}\chi \subset B(0,2R)$. We shall apply (4.19) with $\psi \in C^\infty(\mathbb{C};[0,1])$ such that

$$\psi(x) = 1,\ |x| \geq R, \quad \psi(x) = 0,\ |x| < \frac{R}{2},$$

and with the weight Φ replaced by

$$\widetilde{\Phi} = \Phi - \eta(1-\chi), \quad 0 < \eta \ll 1, \tag{4.26}$$

which still satisfies (4.11), for $\eta > 0$ small enough. For $u \in H_{\Phi,m}(\mathbb{C}^n)$, and $\tilde{\xi}(x) = (2/i)\partial_x \widetilde{\Phi}(x)$,

$$\int_{|x|\geq R} |p(x,\tilde{\xi}(x))|^2 |u(x)|^2 e^{-2\widetilde{\Phi}(x)/h}\, dL(x) \leq \|P^{\mathrm{w}}(x,hD_x;h)u\|^2_{H_{\widetilde{\Phi}}(|x|>R/2)} + \mathcal{O}(h)\|u\|^2_{H_{\widetilde{\Phi},m}}.$$

Combining this with the assumption (4.7) which gives

$$|p(x,\tilde{\xi}(x))| \geq m(x)/C, \quad |x| \geq R,$$

we get

$$\int_{|x|\geq R} |u(x)|^2\, m^2(x) e^{-2\widetilde{\Phi}(x)/h}\, dL(x) \leq C\|P^{\mathrm{w}}(x,hD_x;h)u\|^2_{H_{\widetilde{\Phi}}(|x|>R/2)} + \mathcal{O}(h)\|u\|^2_{H_{\widetilde{\Phi},m}},$$

and therefore, taking $h > 0$ small enough,

$$\int_{|x|\geq R} |u(x)|^2\, m^2(x) e^{-2\widetilde{\Phi}(x)/h}\, dL(x) \leq e^{2\eta/h}\|P^{\mathrm{w}}(x,hD_x;h)u\|^2_{H_{\Phi}(|x|>R/2)} + \mathcal{O}(h)\|u\|^2_{H_{\Phi,m}}.$$

Here we have also used that $\widetilde{\Phi}(x) = \Phi(x)$ for $|x| \leq R$ and that $\widetilde{\Phi} \geq \Phi - \eta$ on $\mathbb{C}^n$. In the region $|x| \geq 2R$, we have $\widetilde{\Phi} = \Phi - \eta$, and that gives (4.25). $\square$

Remarks. 1. In what follows, in order to enter the framework of analytic h–pseudodifferential operators, we shall strengthen condition (4.6) by assuming that

$$|p_k(x,\xi)| \leq C^{k+1}k^k\, m(x), \quad (x,\xi) \in \Lambda_{\Phi_0} + B_{\mathbb{C}^{2n}}(0,\delta), \quad k = 0,1,2,\ldots \tag{4.27}$$

2. We observe that rather than assuming (4.11), it would have been sufficient to consider uniformly strictly plurisubharmonic weights $\Phi \in C^2(\mathbb{C}^n)$ such that $\Phi - \Phi_0 \in L^\infty(\mathbb{C}^n)$, $||\,\nabla(\Phi - \Phi_0)\,||_{L^\infty(\mathbb{C}^n)}$ is small enough, and $\nabla^2\Phi \in L^\infty(\mathbb{C}^n)$. Indeed, in the proof of Proposition 4.1, instead of introducing the contour given in (4.13), we deform to the contour

$$\theta = \frac{2}{i}\int_0^1 \frac{\partial \Phi}{\partial x}(tx + (1-t)y)\, dt + \frac{i\delta}{2}\frac{\overline{(x-y)}}{\langle x-y\rangle}.$$

Proposition 4.2 and Proposition 4.3 are still valid under these weaker assumptions on the weight.

4.2. **Local Fourier integral operators.** The local theory is more subtle than the global theory reviewed in detail in §4.2. One key point is that actions of operators are defined only up to exponentially small errors relative to the weights. We cannot present all the details but precise references are provided.

To prove Theorem 2 we will consider microlocal equivalence, with exponentially small errors, of analytic pseudodifferential operators acting on Hilbert spaces of the form

$$H_\Psi(V) = \mathscr{O}(V) \cap L^2(V, e^{-2\Psi/h}\, dL(x)), \tag{4.28}$$

where $V \Subset \mathbb{C}^n$ is a small open neighbourhood of $0 \in \mathbb{C}^n$ and $\Psi = \Phi_0, \Phi$, with Φ_0 quadratic strictly plurisubharmonic and Φ real analytic in V satisfying $\Phi(x) = \Phi_0(x) + \mathcal{O}(x^3)$. In fact, in our applications the quadratic weight Φ_0 will be strictly convex, and in the following discussion we shall make this stronger assumption,

$$\Psi(x) \ \text{ is strictly convex and } \ \Psi(x) = \mathcal{O}(x^2), \quad x \to 0. \tag{4.29}$$

We start by recalling pseudodifferential operators with classical analytic symbols acting on $H_\Psi(V)$ in (4.28):

$$Pu(x;h) = \frac{1}{(2\pi h)^n} \iint_{\Gamma(x)} e^{\frac{i}{h}\varphi(x,y,\theta)} P(x,\theta;h) u(y)\, dy\, d\theta, \quad u \in H_\Psi(V), \tag{4.30}$$

where

$$\varphi(x,y,\theta) = (x-y)\cdot\theta, \tag{4.31}$$

$P(x,\theta;h) = p(x,\theta) + \mathcal{O}(h)$ is a classical analytic symbol, and $\Gamma(x) \subset \mathbb{C}^{2n}_{y,\theta}$ is a good contour for $(y,\theta) \mapsto -\mathrm{Im}\,((x-y)\cdot\theta) + \Psi(y)$, see [HiSj15, §2.4.c] for more on that terminology. Since we assumed (4.29), it can be taken independent of x:

$$\Gamma : y \mapsto \theta = \tfrac{2}{i}\partial_y \Psi(y), \quad y \in \mathrm{neigh}_{\mathbb{C}^n}(0). \tag{4.32}$$

In fact,

$$\begin{aligned} -\,\mathrm{Im}\,((x-y)\cdot\theta|_\Gamma) + \Psi(y) &= -\,\mathrm{Im}(\tfrac{2}{i}\partial_y\Psi(y)\cdot(x-y)) + \Psi(y) \\ &= \Psi(y) + \partial_{\mathrm{Re}\, y}\Psi(y)\cdot \mathrm{Re}(x-y) + \partial_{\mathrm{Im}\, y}\Psi(y)\cdot\mathrm{Im}(x-y) \\ &\leq \Psi(x) - |x-y|^2/C, \end{aligned}$$

by the strict convexity of Ψ.

We stress that if V is sufficiently small, then $Pu \in H_\Psi(V)$ but the definition (4.30) depends on the realization of the analytic symbol $P(x,\theta;h)$ – see (3.2), [Sj82, Exemple 1.1] (or [HiSj15, §2.2]), and the errors from the contour for $|x-y| > \varepsilon_0$ produce contributions in $H_{\Psi-\varepsilon_1}(V)$, $\varepsilon_1 > 0$. Ultimately, this produces an overall ambiguity which is however exponentially small, which is consistent with our goal. We also note that $P : H_\Psi(V) \to H_\Psi(V)$ is bounded uniformly in h.

The microlocal equivalence of analytic pseudodifferential operators is obtained using local *analytic Fourier integral operators* which quantize locally defined holomorphic symplectomorphisms:

$$\kappa : \mathrm{neigh}_{\mathbb{C}^{2n}}(0,0) \to \mathrm{neigh}_{\mathbb{C}^{2n}}(0,0), \quad \kappa(0,0) = (0,0). \tag{4.33}$$

We assume that, in the notation of (4.2),

$$\kappa(\mathrm{neigh}_{\Lambda_{\Phi_0}}(0,0)) = \mathrm{neigh}_{\Lambda_\Phi}(0,0). \tag{4.34}$$

These assumptions on κ provide a particularly nice generating function: we record that in the following lemma from [MeSj03, Section 2] (the proof there is geometric and is independent of the rest of the paper):

Lemma 4.4. *Suppose that κ in (4.33) satisfies $\kappa^*(d\xi \wedge dx) = d\xi \wedge dx$ and (4.34). Then (near 0)*

$$\{(y,\eta), \kappa(y,\eta))\} = \{(y, -\varphi'_y(x,y,\theta), x, \varphi'_x(x,y,\theta)) \,:\, \varphi'_\theta(x,y,\theta) = 0\} \tag{4.35}$$

where

$$\varphi(x,y,\theta) = \tfrac{2}{i}\left(F(x,\theta) - \Psi_0(y,\theta)\right), \quad \Psi_0 \in \mathscr{O}(\mathbb{C}^{2n}), \quad \Psi_0(x,\bar{x}) = \Phi_0(x), \tag{4.36}$$

and $F \in \mathscr{O}(\mathrm{neigh}_{\mathbb{C}^{2n}}(0))$, satisfies $\det F''_{x\theta} \neq 0$, and

$$-C\,|x - \tilde{\kappa}(y)|^2 \leq 2\mathrm{Re}\,F(x,\bar{y}) - \Phi(x) - \Phi_0(y) \leq -\,|x - \tilde{\kappa}(y)|^2/C, \tag{4.37}$$

where $\tilde{\kappa}(y) := \pi(\kappa(y,(2/i)\partial_y\Phi_0(y)))$ and $(x,y) \in \mathrm{neigh}_{\mathbb{C}^{2n}}(0)$.

Remark. When $\Phi = \Phi_0$ and κ and $\tilde{\kappa}$ are the identity transformations on $\mathbb{C}^{2n}$ and $\mathbb{C}^n$ respectively, then $F(x,\theta) = \Psi_0(x,\theta)$ and (4.37) is a standard consequence of the strict plurisubharmonicity of Φ_0. Of course in that case the obvious choice of φ for which (4.35) holds is given (4.31).

We recall the following fundamental property of the generating function φ (valid for more general generating functions as well):

$$\begin{gathered} \partial_{y,\theta}(-\operatorname{Im}\varphi(0,y,\theta) + \Phi_0(y))|_{(y,\theta)=(0,0)} = 0, \\ \mathrm{sgn}\left(\partial^2_{y,\theta}(-\operatorname{Im}\varphi(0,y,\theta) + \Phi_0(y))|_{(y,\theta)=(0,0)}\right) = 0. \end{gathered} \tag{4.38}$$

This implies that for x near 0, we have the unique critical point,

$$\partial_{y,\theta}(-\operatorname{Im}\varphi(x,y,\theta) + \Phi_0(y))|_{(y,\theta)=(y_{\mathrm{c}}(x),\theta_{\mathrm{c}}(x))} = 0,$$

and

$$-\operatorname{Im}\varphi(x, y_{\mathrm{c}}(x), \theta_{\mathrm{c}}(x)) + \Phi_0(y_{\mathrm{c}}(x)) = \Phi(x).$$

It also shows the existence of a good contour $\Gamma(x)$ passing through the critical point $(y_c(x), \theta_c(x))$, along which we have

$$-\operatorname{Im}\varphi(x,y,\theta) + \Phi_0(y) - \Phi(x) \leq -|y - y_c(x)|^2/C - |\theta - \theta_c(x)|^2/C,$$

$x \in \mathrm{neigh}_{\mathbb{C}^n}(0)$. In the case of φ given in Lemma 4.4, the unique critical point is given by $x = \tilde{\kappa}(y)$, $\theta = \bar{y}$, and we can simply take $\Gamma(x)$ to be x–independent given by $\Gamma_V : y \mapsto \theta = \bar{y}$, $y \in V$.

In the notation of (4.37), $\tilde{\kappa} : V \to U$ is a diffeomorphism for some $V, U \subset \mathbb{C}^n$, small open neighborhoods of 0. Local Fourier integral operators taking $H_{\Phi_0}(V)$ to $H_\Phi(U)$

are defined as follows. Let $a = a(x, y, \theta; h) = a_0(x, y, \theta) + \mathcal{O}(h)$ be a classical analytic symbol in $\mathrm{neigh}_{\mathbb{C}^{3n}}(0)$. For $u \in H_{\Phi_0}(V)$ we put

$$\begin{aligned} Au(x) &= \frac{1}{(2\pi h)^n} \iint\limits_{\Gamma_V} e^{\frac{i}{h}\varphi(x,y,\theta)} a(x, y, \theta; h) u(y)\, dy\, d\theta \\ &= \frac{i^n}{(\pi h)^n} \int_V e^{\frac{2}{h}F(x,\overline{y})} a(x, y, \overline{y}; h) u(y)\, e^{-2\Phi_0(y)/h}\, dL(y). \end{aligned} \tag{4.39}$$

As in (4.30), $Au \in H_\Phi(U)$ is well defined modulo errors in $H_{\Phi-\varepsilon}(U)$, $\varepsilon > 0$.

The first formula for Au in (4.39) makes sense for more general phase functions but we do not stress this point here. The advantage of the second representation based on Lemma 4.4 is the simplicity of the contour (integration over $V \subset \mathbb{C}^n$) in the definition.

The properties of A are summarized in the following proposition. For the proofs, see [Sj82, Theorem 4.5, Proposition 12.9], [Leb85, Section III.7].

Proposition 4.5. *Suppose that $V \Subset \mathbb{C}^n$ is a small open neighbourhood of 0, which is mapped diffeomorphically onto $U \Subset \mathbb{C}^n$ by $\tilde{\kappa}$, see (4.37). Then*

$$A = \mathcal{O}(1) : H_{\Phi_0}(V) \to H_\Phi(U). \tag{4.40}$$

Moreover, if a in (4.39) satisfies $a_0(0) \neq 0$, then there exists a classical analytic symbol b with $b_0(0) \neq 0$ and $\Gamma(y)$, a good contour for $(x, \theta) \mapsto \operatorname{Im} \varphi(x, y, \theta) + \Phi(x)$, such that

$$(Bv)(y) := \frac{1}{(2\pi h)^n} \iint\limits_{\Gamma(y)} e^{-\frac{i}{h}\varphi(x,y,\theta)} b(x, y, \theta; h) v(x)\, dx\, d\theta, \tag{4.41}$$

is a microlocal inverse of A in the sense that

$$B = \mathcal{O}(1) : H_\Phi(U) \to H_{\Phi_0}(V), \tag{4.42}$$

and for each $W_1 \Subset V$, $W_2 \Subset U$, there exists $\eta > 0$ such that

$$\begin{aligned} I - AB &= \mathcal{O}(1) e^{-\eta/h} : H_\Phi(U) \to H_\Phi(W_2), \\ I - BA &= \mathcal{O}(1) e^{-\eta/h} : H_{\Phi_0}(V) \to H_{\Phi_0}(W_1). \end{aligned} \tag{4.43}$$

The composition of Fourier integral operators in the complex domain is presented in [Sj82, Chapter 4], [HiSj15, Section 2.5], In particular, in the notation above,

$$P \circ A : H_{\Phi_0}(V) \to H_\Phi(U) \tag{4.44}$$

is a Fourier integral operator of the form (4.39), with an analytic symbol given by

$$\frac{1}{(2\pi h)^n} e^{-i\varphi(x,y,\theta)/h} P\left(e^{i\varphi(\cdot,y,\theta)/h} a(\cdot, y, \theta; h)\right)(x).$$

Assuming that A is elliptic and letting B be a microlocal inverse of A from Proposition 4.5, we also obtain that

$$Q := B \circ P \circ A : H_{\Phi_0}(V) \to H_{\Phi_0}(V) \tag{4.45}$$

is an analytic h–pseudodifferential operator, with the principal symbol given by

$$q = p \circ \kappa. \tag{4.46}$$

This is the form of Egorov's theorem we will use.

5. A general semiclassical result

We present a general semiclassical result in one dimension generalizing the result in [Hi04] by allowing more general behaviour at infinity and, perhaps more importantly, describing eigenvalues with exponential accuracy in h, uniformly in a neighbourhood of a non-degerate minimum of the classical observable. Although not surprising, we do not know a detailed reference for this even in the self-adjoint case.

Hence, we assume that $n = 1$, and that $H^{\mathrm{w}}(x, hD, h)$ is a semiclassical pseudodifferential operator satisfying the following assumptions for some order function $m \geq 1$ (see [Zw12, §4.4] or (4.3)):

$$\begin{gathered} H \in S(\mathbb{R}^{2n}, m), \quad H(\rho) \sim H_0(\rho) + hH_1(\rho) + \cdots, \quad \rho = (x, \xi), \quad H_0(0) = 0, \\ dH_0(0) = 0,\ d^2H_0(0) \text{ is elliptic, and } \{\langle d^2H_0(0)v, v\rangle : v \in \mathbb{R}^2\} \neq \mathbb{C}, \\ H(\rho, h) \text{ extends to } \{|\operatorname{Im}\rho| < c_0\} \subset \mathbb{C}^2,\ c_0 > 0, \text{ as an analytic symbol,} \\ \forall\, \varepsilon > 0\, \exists\, \delta > 0\ |\rho| > \varepsilon \implies |H_0(\rho)| \geq \delta m(\rho),\ \rho \in \mathbb{R}^2. \end{gathered} \tag{5.1}$$

Ellipticity of $d^2H_0(0)$ means that $\langle d^2H_0(0)v, v\rangle \neq 0$ for all $v \in \mathbb{R}^2 \setminus \{0\}$. This and the last condition on $d^2H_0(0)$ above imply that there exists $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\langle \lambda d^2H_0 v, v\rangle > 0$, $v \in \mathbb{R}^2 \setminus \{0\}$, see Lemma A.1.

The condition that the range of $\mathbb{R}^2 \ni v \mapsto \langle d^2H_0(0)v, v\rangle$ is not equal to $\mathbb{C}$, implies that there exists $\theta_0 \in [0, 2\pi)$ such that $\langle d^2H_0(0)v, v\rangle + e^{i\theta_0} \neq 0$, $v \in \mathbb{R}^2$. That, in turn implies that if we take $\chi \in C_c^\infty(\mathbb{R}^2, [0,1])$, $\chi|_{B_{\mathbb{R}^2}(0,\varepsilon)} = 1$, or $\chi \equiv 1$, then $|H_0 + (\delta/2)e^{i\theta_0}\chi(\rho)| > \delta m(\rho)$, (recall that $m \geq 1$) if ε is small enough and δ is as in (5.1). Hence, $H^{\mathrm{w}} + \chi^{\mathrm{w}} - z$ and $H^{\mathrm{w}} + (\delta/2)e^{i\theta_0} - z$ are invertible for small h and $|z|$ (see [Zw12, Theorem 4.29]). Then

$$\operatorname{neigh}_{\mathbb{C}}(0) \ni z \mapsto (H^{\mathrm{w}} - z)^{-1} = (H^{\mathrm{w}} + \chi^{\mathrm{w}} - z)^{-1}(I - \chi^{\mathrm{w}}(H^{\mathrm{w}} + \chi^{\mathrm{w}} - z)^{-1})^{-1}$$

is meromorphic in z: $T(z) := \chi^{\mathrm{w}}(H^{\mathrm{w}} + \chi^{\mathrm{w}} - z)^{-1}$ is compact (see [Zw12, Theorem 4.28]) and invertibility of $H^{\mathrm{w}} + (\delta/2)e^{i\theta_0}$ shows that $I - T(\frac{1}{2}\delta e^{i\theta_0})$ is invertible. That means that analytic Fredholm theory applies [Zw12, §D.4] and indeed $(H^{\mathrm{w}} - z)^{-1}$ is meromorphic near 0. The spectrum of H^{w} is the set of poles with the usual formula for multiplicity. We can consider H^{w} as a closed unbounded operator on L^2, with the domain given by $H_h(m)$ (see [Zw12, §8.2]).

A typical example is given by $H(x, \xi) = H_0(x, \xi) + h^2H_2(x, \xi)$, where $H_0(x, \xi) = \xi^2 + W_0(x)$, $H_2 = W_1(x)$, where W_j are holomorphic in a strip around $\mathbb{R} \subset \mathbb{C}$, such that $W_0^{-1}(0) = \{0\}$, $W_0'(0) = 0$, $W_0''(0) \notin (-\infty, 0]$ (recall that W_j may be complex valued) – see §2.1. The ellipticity assumption here means that $|H_0(x,\xi)| \geq \delta(1+|\xi|^2)$ for $|x| > \varepsilon$.

Our general result is given in

Theorem 3. *Suppose that H satisfies (5.1). Then there exist constants $r_0, h_0, c_0 > 0$ such that for $0 < h < h_0$,*

$$\begin{aligned} \mathrm{Spec}(H^{\mathrm{w}}(x, hD, h)) \cap D_{\mathbb{C}}(0, r_0) &= \{\lambda_n(h) : n = 0, 1, \cdots\} \cap D_{\mathbb{C}}(0, r_0), \\ \lambda_n(h) &= G(2\pi(n+\tfrac{1}{2})h, h) + \mathcal{O}(e^{-c_0/h}), \end{aligned} \tag{5.2}$$

where $G(z,h)$ has an asymptotic expansion, $G_0(z) + hG_1(z)\cdots$, $G_j \in \mathscr{O}(\mathrm{neigh}_{\mathbb{C}}(0))$, $|G_j| \leq A^{j+1}j!$, in the sense that there exists $B > 0$ such that for all $N \in \mathbb{N}$, we have

$$|G(z,h) - \sum_{j=0}^{N-1} h^j G_j(z)| \leq B^{N+1} N! h^N. \tag{5.3}$$

The leading term G_0 is the inverse of a complex action defined in Theorem A.2. When $H_1 \equiv 0$ (see (5.1)) then $G_1 \equiv 0$.

In view of (5.2) and (5.3) the theorem gives an exponentially accurate description of eigenvalues of H^{w} near 0. To obtain $G(z,h)$ satisfying (5.3) from G_j's one can proceed in different ways. One was used in (3.2), see [HiSj15, §2.2], and here we present a variant in the spirit of exact WKB:

$$G(z,h) = h^{-1}\int_0^{\varepsilon} g(z,t)e^{-t/h}dt, \quad g(z,h) := \sum_{j=0}^{\infty} \frac{G_j(z)}{j!}h^j. \tag{5.4}$$

where $\varepsilon < 1/A$ is smaller that the radius of covergence of $h \mapsto g(z,h)$. The choice of ε produces exponentially small ambiguity, just as we saw in (3.2).

The first step in the proof of Theorem 3 consists of passing to the FBI transform side by means of a suitable metaplectic FBI-Bargmann transform. When doing so, we write, in the notation of (5.1),

$$H_0(y,\eta) = q(y,\eta) + \mathcal{O}((y,\eta)^3), \tag{5.5}$$

where q is the quadratic form corresponding to $d^2H_0(0)$. Lemma A.1 shows that $\mathrm{Re}\,(\lambda q)$ is positive definite on $\mathbb{R}^2$, for some $\lambda \in \mathbb{C}$. It follows from [Hi04, Proposition 2.4], [HiSjVi13, Proposition 2.1] that there exists a metaplectic unitary FBI transform, i.e. an FBI transform with a quadratic phase and a constant amplitude,

$$T : L^2(\mathbb{R}) \to H_{\Phi_1}(\mathbb{C}), \tag{5.6}$$

such that the conjugated operator $T \circ H^{\mathrm{w}}(y, hD_y; h) \circ T^{-1}$ acting on $H_{\Phi_1}(\mathbb{C})$, is of the form

$$T \circ H^{\mathrm{w}}(y, hD_y; h) \circ T^{-1} = P^{\mathrm{w}}(x, hD_x; h), \;\; P(x,\xi;h) \sim p_0(x,\xi) + hp_1(x,\xi) + \ldots, \tag{5.7}$$

where

$$p_0(x,\xi) = H_0(\kappa_T^{-1}(x,\xi)) = \mu x\xi + \mathcal{O}((x,\xi)^3), \quad \mathrm{Im}\,(\lambda\mu) > 0. \tag{5.8}$$

Here κ_T is the complex linear canonical transformation associated to T and the quadratic weight Φ_1 in (5.6) is strictly convex. See also [HiSj15, Theorem 1.4.2], [Zw12, Theorem 13.9]. The structure and properties of the conjugated operator $P^{\mathrm{w}}(x,hD;h)$ are reviewed in §4.1, and in particular, $P(x,\xi;h)$ is a holomorphic function in a tubular neighbourhood of Λ_{Φ_1} in $\mathbb{C}^2$, satisfying (4.5), (4.6), and (4.27).

Following [KaKe00], [Hi04], we shall now modify the weight function Φ_1 in a bounded region of $\mathbb{C}$, so that the modified weight agrees with the standard radial weight

$$\Phi_0(x) = \tfrac{1}{2}|x|^2 \tag{5.9}$$

in a small but fixed neighbourhood of the origin, while keeping the ellipticity of p_0 away from 0 (see the last condition in (5.1)). This reduction is very convenient in what follows, as monomials (the eigenfunctions of xhD_x) are then orthogonal in $L^2_{\Phi_0}(\mathbb{C})$.

Proposition 5.1. *For every sufficiently small open neighborhood $U \subset \mathbb{C}$ of 0 there exists a strictly convex weight $\widetilde{\Phi} \in C^\infty(\mathbb{C})$ satisfying*

$$\widetilde{\Phi}|_{\mathbb{C}\setminus U} = \Phi_1|_{\mathbb{C}\setminus U}, \quad \widetilde{\Phi}|_{U_0} = \Phi_0|_{U_0}, \quad U_0 = \mathrm{neigh}_{\mathbb{C}}(0) \Subset U, \quad \nabla^2\widetilde{\Phi} \in L^\infty(\mathbb{C}),$$

and such that

$$\forall\,\theta > 0\ \exists\,\delta > 0 \quad (x,\xi) \in \Lambda_{\widetilde{\Phi}},\ \ |(x,\xi)| \geq \theta \Longrightarrow |p_0(x,\xi)| \geq \delta m(x). \tag{5.10}$$

Proof. Following an argument of [KaKe00], we let $\psi \in C^\infty(\mathbb{R};[0,1])$ be such that $\psi(t) = 1$, $t \leq 1$, $\psi(t) = 0$, $t \geq 2$, and notice that the $C_0^\infty(\mathbb{C})$–function $\chi_\varepsilon(x) = \psi(\varepsilon \log|x|)$ satisfies for $\varepsilon \in (0,1]$,

$$\nabla\chi_\varepsilon(x) = \mathcal{O}\left(\varepsilon|x|^{-1}\right), \quad \nabla^2\chi_\varepsilon(x) = \mathcal{O}\left(\varepsilon|x|^{-2}\right). \tag{5.11}$$

It follows from (5.11) that the $C^\infty(\mathbb{C})$–function

$$\Phi_{\varepsilon,\eta}(x) = \chi_\varepsilon\left(e^{2/\varepsilon}x/\eta\right)\Phi_0(x) + \left(1-\chi_\varepsilon\left(e^{2/\varepsilon}x/\eta\right)\right)\Phi_1(x) \tag{5.12}$$

is strictly convex for $\varepsilon > 0$ sufficiently small, uniformly in $\eta > 0$, and we have

$$\Phi_{\varepsilon,\eta}(x) = \Phi_0(x),\ |x| < \eta\, e^{-1/\varepsilon}, \quad \Phi_{\varepsilon,\eta}(x) = \Phi_1(x),\ |x| > \eta, \tag{5.13}$$

Furthermore,

$$||\,\nabla(\Phi_{\varepsilon,\eta} - \Phi_1)\,||_{L^\infty(\mathbb{C})} \leq \mathcal{O}(\eta), \quad \nabla^2\Phi_{\varepsilon,\eta} \in L^\infty(\mathbb{C}),$$

uniformly in $\varepsilon \in (0,1]$, $\eta > 0$. Using (4.2) we define $\Lambda_{\Phi_{\varepsilon,\eta}}$ which coincides with Λ_{Φ_0} in a neighbourhood of the origin and agrees with Λ_{Φ_1} away from another fixed neighbourhood of 0. In view of (5.11), (5.12) we have

$$\begin{aligned} \xi(x) &:= \tfrac{2}{i}\partial_x\Phi_{\varepsilon,\eta}(x) \\ &= \chi_\varepsilon\left(e^{2/\varepsilon}x/\eta\right)\tfrac{2}{i}\partial_x\Phi_0(x) + \left(1-\chi_\varepsilon\left(e^{2/\varepsilon}x/\eta\right)\right)\tfrac{2}{i}\partial_x\Phi_1(x) + \mathcal{O}(\varepsilon\,|x|), \end{aligned} \tag{5.14}$$

uniformly in $\eta > 0$, and therefore, for $\varepsilon > 0$ small enough,

$$\begin{aligned}\mathrm{Re}\,(ix\xi(x)) &= \chi_\varepsilon\,\mathrm{Re}\,(2x\,\partial_x\Phi_0(x)) + (1-\chi_\varepsilon)\,\mathrm{Re}\,(2x\,\partial_x\Phi_1(x)) + \mathcal{O}(\varepsilon\,|x|^2) \\ &= \chi_\varepsilon\langle x, \nabla\Phi_0(x)\rangle_{\mathbb{R}^2} + (1-\chi_\varepsilon)\langle x, \nabla\Phi_1(x)\rangle_{\mathbb{R}^2} + \mathcal{O}(\varepsilon\,|x|^2) \asymp |x|^2\,,\end{aligned} \tag{5.15}$$

uniformly in $\eta > 0$. Taking $\eta > 0$ sufficiently small but fixed, we obtain, using (5.8), (5.15),

$$|p_0(x, \xi(x))| \asymp |x|^2\,,\ x \in \mathrm{neigh}_{\mathbb{C}}(0). \tag{5.16}$$

Recalling also (5.1), (5.13), we obtain the following ellipticity property along $\Lambda_{\Phi_{\varepsilon,\eta}}$: $\forall\theta > 0\ \exists\delta > 0$ such that

$$x \in \mathbb{C},\ |x| \geq \theta \Longrightarrow |p_0(x, \xi(x))| \geq \delta m(x). \tag{5.17}$$

It follows that we can take $\widetilde{\Phi} = \Phi_{\varepsilon,\eta}$, for $\varepsilon > 0$, $\eta > 0$ small enough fixed, and this completes the proof. $\square$

Proposition 5.1 and the results of §4.1 show that

$$P^{\mathrm{w}}(x, hD_x; h) = \mathcal{O}(1) : H_{\widetilde{\Phi},m}(\mathbb{C}) \to H_{\widetilde{\Phi}}(\mathbb{C}). \tag{5.18}$$

The spectral analysis of P^{w} in the proof of Theorem 3 is carried out in a weighted space obtained from $H_{\widetilde{\Phi}}(\mathbb{C})$ by an additional modification of the weight in a small but fixed neighbourhood of the origin, implementing the holomorphic canonical transformation $\kappa : \mathrm{neigh}_{\mathbb{C}^2}(0) \to \mathrm{neigh}_{\mathbb{C}^2}(0)$ given in Theorem A.1, with $p(x, \xi) = p_0(x, \xi)$ in (5.8). We have $\kappa(\rho) = \rho + \mathcal{O}(\rho^2)$, and it follows that the I-Lagrangian R-symplectic manifold $\kappa(\mathrm{neigh}_{\Lambda_{\Phi_0}}(0)) \subset \mathbb{C}^2$, where Φ_0 is the standard weight function given in (5.9), is of the form

$$\kappa(\mathrm{neigh}_{\Lambda_{\Phi_0}}(0)) = \mathrm{neigh}_{\Lambda_\Phi}(0). \tag{5.19}$$

Here Φ is real analytic near $0 \in \mathbb{C}$ and we may assume that $\Phi(0) = 0$. It follows therefore that $\Phi(x) = \mathcal{O}(|x|^2)$ as $x \to 0$, and using also that

$$T_0\Lambda_\Phi = d\kappa(0)(\Lambda_{\Phi_0}) = \Lambda_{\Phi_0},$$

we obtain the more precise description of the deformed weight,

$$\Phi(x) = \Phi_0(x) + \mathcal{O}(x^3). \tag{5.20}$$

We then define, using the notation in the proof of Proposition 5.1,

$$\widehat{\Phi}(x) = \chi_\varepsilon\left(e^{2/\varepsilon}x/\eta\right)\Phi(x) + \left(1 - \chi_\varepsilon\left(e^{2/\varepsilon}x/\eta\right)\right)\Phi_1(x), \tag{5.21}$$

for $\varepsilon > 0$, $\eta > 0$ small enough fixed. It follows from (5.11), (5.20) that the function $\widehat{\Phi} \in C^\infty(\mathbb{C})$ is strictly convex, satisfying $\widehat{\Phi}(x) = \Phi_1(x)$ for $|x| \geq \eta$, $\widehat{\Phi}(x) = \Phi(x)$ for $|x| < \eta e^{-1/\varepsilon}$, with $||\,\nabla(\widehat{\Phi} - \Phi_1)\,||_{L^\infty(\mathbb{C})}$ small enough, $\nabla^2\widehat{\Phi} \in L^\infty(\mathbb{C})$. We also get, using (5.8), (5.15), and (5.20), that $\forall\theta > 0\ \exists\delta > 0$ such that

$$(x, \xi) \in \Lambda_{\widehat{\Phi}},\ \ |(x, \xi)| \geq \theta \Longrightarrow |p_0(x, \xi)| \geq \delta m(x). \tag{5.22}$$

The discussion shows that

$$P^{\mathrm{w}}(x,hD_x;h)=\mathcal{O}(1):H_{\widehat{\Phi},m}(\mathbb{C})\to H_{\widehat{\Phi}}(\mathbb{C}),\tag{5.23}$$

and it follows from the ellipticity property (5.22) and the proof of Proposition 4.3 that for each $\varepsilon>0$ there exist $h_0,\eta,\gamma,C>0$ such that for all $0<h<h_0$, $|z|<\gamma$, we have

$$\|u\|_{L^2_{\widehat{\Phi},m}(|x|>\varepsilon)}\leq C\|(P^{\mathrm{w}}(x,hD_x;h)-z)u\|_{H_{\widehat{\Phi}}(\{|x|>\varepsilon/4\})}+Ce^{-\eta/h}\|u\|_{H_{\widehat{\Phi},m}(\mathbb{C})},\tag{5.24}$$

for $u\in H_{\Phi,m}(\mathbb{C})$.

Let $U\Subset\mathbb{C}$ be a small open neighbourhood of the origin such that $\widehat{\Phi}=\Phi$ in a neighbourhood of $\overline{U}$. There exists a classical analytic h–pseudodifferential operator

$$P=P_U=\mathcal{O}(1):H_\Phi(U)\to H_\Phi(U),\tag{5.25}$$

with the leading symbol p_0 in (5.8), realized using a good contour of the form (4.32), with $\Psi=\Phi$, such that for some $\delta>0$, we have for all $u\in H_{\widehat{\Phi},m}(\mathbb{C})$,

$$\|P^{\mathrm{w}}(x,hD_x;h)u-P_U(u|_U)\|_{H_\Phi(U)}\leq\mathcal{O}(e^{-\delta/h})\|u\|_{H_{\widehat{\Phi},m}(\mathbb{C})}.\tag{5.26}$$

We may view P_U as a local realization of the globally defined operator $P^{\mathrm{w}}(x,hD_x;h)$ in (5.23), and the key point in the proof of Theorem 3 is the following precise normal for $P=P_U$.

Proposition 5.2. *Suppose that $P=P_U$ is defined above, so that in particular (5.25) holds. Then there exists an elliptic analytic Fourier integral operator $A=\mathcal{O}(1):H_{\Phi_0}(V)\to H_\Phi(U)$, of the form (4.39), with a microlocal inverse $B=\mathcal{O}(1):H_\Phi(U)\to H_{\Phi_0}(V)$, such that the conjugated operator $Q=BPA:H_{\Phi_0}(V)\to H_{\Phi_0}(V)$ is of the form*

$$Q=G(2ixhD_x+h;h)+R\tag{5.27}$$

where $G=G(w;h)$ is a classical analytic symbol in a neighbourhood of $0\in\mathbb{C}$, $G(2ixhD_x+h;h)$ is an analytic h–pseudodifferential operator defined as in the remark after the statement of Lemma 3.4, and for each $W\Subset V$ there exists $\delta=\delta_W>0$ such that

$$R=\mathcal{O}(e^{-\delta/h}):H_{\Phi_0}(V)\to H_{\Phi_0}(W).\tag{5.28}$$

Proof. Let $A_0=\mathcal{O}(1):H_{\Phi_0}(V)\to H_\Phi(U)$ be an elliptic analytic Fourier integral operator of the form (4.39), associated to the holomorphic canonical transformation $\kappa:\mathrm{neigh}_{\mathbb{C}^2}(0)\to\mathrm{neigh}_{\mathbb{C}^2}(0)$ introduced in Theorem A.1 with $p=p_0$ in (5.8), such that (5.19) holds. Let B_0 be a microlocal inverse of A_0 – see Proposition 4.5.

It follows from (4.45) that $Q_0:=B_0PA_0:H_{\Phi_0}(V)\to H_{\Phi_0}(V)$ is a classical analytic h–pseudodifferential operator with the principal symbol given by $q_0(x,\xi)=p_0(\kappa(x,\xi))=g(x\xi)$, for some $g\in\mathscr{O}(\mathrm{neigh}_{\mathbb{C}}(0))$, $g(0)=0$, $g'(0)\neq0$. Furthermore, Proposition 3.2 shows that there exists an analytic h–pseudodifferential operator of the form e^{A_1}, where A_1 is an analytic h–pseudodifferential operator of order 0, such that the conjugated operator $e^{A_1} \circ Q_0 \circ e^{-A_1} : H_{\Phi_0}(V) \to H_\Phi(V)$ is of the form (5.27), (5.28). We now obtain the desired Fourier integral operator by putting $A := A_0 \circ e^{A_1}$. $\square$

We now define

$$\varphi_j(x) = (\pi j!)^{-1/2} h^{-\frac{1}{2}(j+1)} x^j, \quad j = 0, 1, 2, \dots \tag{5.29}$$

noting that they form an orthonormal basis in the Bargmann space $H_{\Phi_0}(\mathbb{C})$. To φ_j's we associate quasi-eigenvalues of (5.27):

$$\{G\left((2j+1)h; h\right); \, j \in \mathbb{N}\} \cap \mathrm{neigh}_{\mathbb{C}}(0). \tag{5.30}$$

The first step is a localization result which shows that the spectrum of P^{w} in (5.23) lies near these quasi-eigenvalues:

Proposition 5.3. *There exist $r_0, c_0, h_0 > 0$ such that for*

$$z \in D(0, r_0) \quad |z - G((2j+1)h; h)| \geq c_0 h, \quad 0 < h < h_0, \tag{5.31}$$

the operator

$$P^{\mathrm{w}}(x, hD_x; h) - z : H_{\widehat{\Phi}, m}(\mathbb{C}) \to H_{\widehat{\Phi}}(\mathbb{C}) \tag{5.32}$$

is invertible. In particular, considering P^{w} as a closed densely defined operator on $H_{\widehat{\Phi}}(\mathbb{C})$ with the domain given by $H_{\widehat{\Phi}, m}(\mathbb{C})$, we have

$$\mathrm{Spec}(P^{\mathrm{w}}(x, hD_x; h)) \cap D(0, r_0) \subset \bigcup_j D(G((2j+1)h, h), c_0 h). \tag{5.33}$$

The proof is quite involved and we start with a few technical lemmas. The first comes from the work of Gérard–Sjöstrand [GeSj87, Lemma 4.5] and is natural when studying approximation by polynomials (5.29). Since the proof is very simple we recall it for the convenience of the reader.

Lemma 5.4. *Suppose that u is holomorphic in a neighbourhood of $\overline{D(0,1)}$ satisfying $u^{(j)}(0) = 0$ for $j < N$. Then for $0 < \rho < 1$ we have*

$$\|u\|_{L^\infty(D(0,\rho))} \leq N(1-\rho)^{-1} \rho^N \|u\|_{L^\infty(D(0,1))}. \tag{5.34}$$

Proof. We have by Taylor's formula,

$$u(x) = \int_0^1 \frac{(1-t)^{N-1}}{(N-1)!} \left(\frac{d}{dt}\right)^N u(tx) \, dt.$$

To estimate $u(x)$ for $|x| \leq \rho$, we may assume that $|x| = \rho$. Cauchy's estimates applied to the holomorphic function $D\left(0, 1/\rho\right) \ni \zeta \mapsto u(\zeta x)$ give

$$|u^{(N)}(tx)| \leq \frac{N!}{(1/\rho - t)^N} \|u\|_{L^\infty(B(0,1))}.$$

Since

$$N\int_0^1 \frac{(1-t)^{N-1}}{(1/\rho - t)^N}\,dt = N\rho^N \int_0^1 \frac{(1-t)^{N-1}}{(1-\rho t)^N} dt \leq \frac{N}{1-\rho}\rho^N,$$

we obtain (5.34). □

The next lemma provides estimates on elements of $H_{\Phi_0}(V)$ orthogonal to x^j's:

Lemma 5.5. *Put $V = D(0,r)$ and define $N(h)$ as the least integer such that*

$$N(h) \geq (\tfrac{1}{8}r^2 + \delta)h^{-1}. \tag{5.35}$$

Then for $u \in H_{\Phi_0}(V)$ and φ_j in (5.29)

$$(u, \varphi_j)_{H_{\Phi_0}(V)} = 0, \quad j < N(h) \implies \|u\|_{H_{\Phi_0}(D(0,r/16))} \leq Ce^{-\delta/h}\|u\|_{H_{\Phi_0}(D(0,r))}. \tag{5.36}$$

Proof. The monomials

$$f_j = \varphi_j/\|\varphi_j\|_{H_{\Phi_0}(V)}, \quad j = 0,1,2\ldots, \tag{5.37}$$

form an orthonormal basis in $H_{\Phi_0}(V)$, and we have for $u \in H_{\Phi_0}(V)$ (recall that V is a disc),

$$(u, f_j)_{H_{\Phi_0}(V)} = u^{(j)}(0)/j!\,(x^j, f_j)_{H_{\Phi_0}(V)}, \quad j = 0,1,2\ldots. \tag{5.38}$$

If $(u, f_j)_{H_{\Phi_0}(V)} = 0$, $j < N$, and $0 < r_0 < r_1 < r_2 < r$, then Lemma 5.4 (applied $\rho = r_0/r_1$) and (5.38) give,

$$\begin{aligned}
\|u\|^2_{H_{\Phi_0}(D(0,r_0))} &\leq \|u\|^2_{L^2(D(0,r_0))} \leq Cr_0^2\|u\|^2_{L^\infty(D(0,r_0))} \\
&\leq Cr_1^2N^2(1-r_0/r_1)^{-2}(r_0/r_1)^{2N}\|u\|^2_{L^\infty(D(0,r_1))} \\
&\leq CN^2r_1^2(r_2-r_1)^{-2}(1-r_0/r_1)^{-2}(r_0/r_1)^{2N}\|u\|^2_{L^2(D(0,r_2))} \\
&\leq CN^2r_1^2(r_2-r_1)^{-2}(1-r_0/r_1)^{-2}(r_0/r_1)^{2N}e^{r_2^2/h}\|u\|^2_{H_{\Phi_0}(D(0,r_2))}.
\end{aligned} \tag{5.39}$$

Taking $r_0 = r/16$, $r_1 = er/16$, $r_2 = r/2$, gives

$$\|u\|_{H_{\Phi_0}(D(0,r/16))} \leq CN^2e^{-2N}e^{r^2/4h}\|u\|_{H_{\Phi_0}(V)}. \tag{5.40}$$

Choosing N satisfying (5.35) gives (5.36). □

The final preparatory lemma concern exponential decay of φ_j's away from 0:

Lemma 5.6. *For $\rho > 0$ and φ_j given in (5.29),*

$$j + 1 \leq \frac{\rho^2}{4h} \implies \int_{|x|>\rho} |\varphi_j(x)|^2\, e^{-2\Phi_0(x)/h}\, L(dx) \leq e^{-\rho^2/4h}. \tag{5.41}$$

Proof. The integral in (5.41) is equal to

$$\frac{1}{\pi j!}\frac{1}{h^{1+j}}\int_{|x|>\rho}|x|^{2j}\,e^{-|x|^2/h}\,L(dx)=\frac{1}{j!}\frac{1}{h^{1+j}}\int_\rho^\infty 2r^{2j+1}e^{-r^2/h}\,dr$$
$$=\frac{1}{j!}\frac{1}{h^{1+j}}\int_{\rho^2}^\infty y^j e^{-y/h}\,dy\le e^{-\rho^2/2h}\frac{1}{j!}\frac{1}{h^{1+j}}\int_0^\infty y^j e^{-y/2h}\,dy=e^{-\rho^2/2h}\,2^{j+1},\quad (5.42)$$

which gives the estimate in (5.41). □

We now move to

Proof of Proposition 5.3. We consider

$$(P^{\rm w}(x,hD_x;h)-z)u=v,\quad u\in H_{\widehat{\Phi},m}(\mathbb{C}),\ v\in H_{\widehat{\Phi}}(\mathbb{C}),\quad z\in\mathrm{neigh}_{\mathbb{C}}(0).\qquad (5.43)$$

Restricting the attention to the neighbourhood U in (5.25), we rewrite this using (5.26) as

$$(P-z)u=v+w\quad\text{in } U,\quad \|w\|_{H_\Phi(U)}\le Ce^{-\delta/h}\|u\|_{H_{\widehat{\Phi},m}(\mathbb{C})}.\qquad (5.44)$$

Following Proposition 5.2 we apply the operator B to (5.44) to get

$$(Q-z)\widetilde{u}=\widetilde{v}\quad\text{in } V,\quad \widetilde{u}=Bu\in H_{\Phi_0}(V),\qquad (5.45)$$

where

$$\widetilde{v}=B(v+w)-BP(1-AB)u\qquad (5.46)$$

satisfies

$$\|\widetilde{v}\|_{H_{\Phi_0}(V)}\le C\|v\|_{H_{\widehat{\Phi}}(\mathbb{C})}+Ce^{-\delta/h}\|u\|_{H_{\widehat{\Phi},m}(\mathbb{C})}+C\|u\|_{H_\Phi(U\setminus\widetilde{U})},\qquad (5.47)$$

with $\widetilde{U}\Subset U$ being a small open neighbourhood of $0\in\mathbb{C}$.

It is convenient to take

$$V=D(0,r),\quad 0<r\ll 1.$$

and $W=D(0,r/16)$. In the notation of Lemma 5.5, we define

$$\Pi_{N(h)}:H_{\Phi_0}(V)\to H_{\Phi_0}(V),\quad \Pi_{N(h)}u:=\sum_{j=0}^{N(h)-1}u^{(j)}(0)x^j/j!,\qquad (5.48)$$

which in view of (5.38) is the orthogonal projection onto the span of f_j's, $0\le j<N(h)$ in (5.37). Lemma 5.5 then gives

$$1-\Pi_{N(h)}=\mathcal{O}(1)\,e^{-\delta/h}:H_{\Phi_0}(V)\to H_{\Phi_0}(W),\quad W:=D(0,r/16).\qquad (5.49)$$

Lemma 5.6 and (5.35) show that $\|\varphi_j\|_{H_{\Phi_0}(V)}=1+Ce^{-r^2/4h}$, $j<N(h)$, uniformly with respect to j. Hence, in the notation of (5.37),

$$f_j=\varphi_j+\mathcal{O}(1)_{H_{\Phi_0}(V)}\,e^{-r^2/4h},\quad j<N(h).\qquad (5.50)$$

Another application of Lemma 5.6 (with $\rho = \alpha r$) gives

$$\|f_j\|_{H_{\Phi_0}(\alpha r<|x|<r)} = \mathcal{O}(1)\, e^{-\delta/h}, \quad \alpha = \left(\tfrac{2}{3}\right)^{\frac{1}{2}}, \quad \delta > 0. \tag{5.51}$$

Applying the orthogonal projection (5.48) to (5.45), we get

$$\Pi_{N(h)}(Q-z)\Pi_{N(h)}\widetilde{u} = \Pi_{N(h)}\widetilde{v} - \Pi_{N(h)}(Q-z)(1-\Pi_{N(h)})\widetilde{u}. \tag{5.52}$$

We first estimate the norm on the second term on the right hand side of (5.52). To this end, let $\widetilde{V} = D(0, r')$ with $r/16 < r' < r$ sufficiently close to $r/16$ so that we still have

$$1 - \Pi_{N(h)} = \mathcal{O}(1)\, e^{-\delta/h} : H_{\Phi_0}(V) \to H_{\Phi_0}(\widetilde{V}). \tag{5.53}$$

Then, (see Proposition 5.2 for the mapping properties of $Q - z$)

$$\begin{aligned}
\|\Pi_{N(h)}(Q-z)(1-\Pi_{N(h)})\widetilde{u}\|_{H_{\Phi_0}(V)} &\leq C\|(1-\Pi_{N(h)})\widetilde{u}\|_{H_{\Phi_0}(V)} \\
&\leq C\| 1\!\!1_{\widetilde{V}}(1-\Pi_{N(h)})\widetilde{u}\|_{L^2_{\Phi_0}(V)} + C\|(1-1\!\!1_{\widetilde{V}})(1-\Pi_{N(h)})\widetilde{u}\|_{L^2_{\Phi_0}(V)} \\
&\leq C\|\widetilde{u}\|_{H_{\Phi_0}(V\setminus\widetilde{V})} + Ce^{-\delta/h}\|\widetilde{u}\|_{H_{\Phi_0}(V)}.
\end{aligned} \tag{5.54}$$

It follows from (5.27) that the matrix $\mathcal{D}_{N(h)}$ of the operator $\Pi_{N(h)}(Q-z)\Pi_{N(h)}$ is of the form ($\delta > 0$)

$$(G((2j+1)h;h) - z)\delta_{j,k})_{0\leq j,k\leq N(h)-1} + \mathcal{O}(e^{-\delta/h}) = \mathcal{O}(1) : \mathbb{C}^{N(h)} \to \mathbb{C}^{N(h)}. \tag{5.55}$$

Here we equip $\mathbb{C}^{N(h)}$ with the Euclidean norm.

For z satisfying (5.31) we also have

$$\mathcal{D}_{N(h)}^{-1} = \mathcal{O}(h^{-1}) : \mathbb{C}^{N(h)} \to \mathbb{C}^{N(h)}. \tag{5.56}$$

Hence, for z satisfying (5.31), (5.52) and (5.54) give

$$\begin{aligned}
\|\Pi_{N(h)}\widetilde{u}\|_{H_{\Phi_0}(V)} &\leq Ch^{-1}\left(\|\Pi_{N(h)}\widetilde{v}\|_{H_{\Phi_0}(V)} + \|\Pi_{N(h)}(Q-z)(1-\Pi_{N(h)})\widetilde{u}\|_{H_{\Phi_0}(V)}\right) \\
&\leq Ch^{-1}\left(\|\widetilde{v}\|_{H_{\Phi_0}(V)} + \|\widetilde{u}\|_{H_{\Phi_0}(V\setminus\widetilde{V})}\right) + Ce^{-\delta/h}\|\widetilde{u}\|_{H_{\Phi_0}(V)}.
\end{aligned}$$

This and (5.49) give

$$\begin{aligned}
\|\widetilde{u}\|_{H_{\Phi_0}(W)} &\leq \|\Pi_{N(h)}\widetilde{u}\|_{H_{\Phi_0}(W)} + \|(1-\Pi_{N(h)})\widetilde{u}\|_{H_{\Phi_0}(W)} \\
&\leq Ch^{-1}\left(\|\widetilde{v}\|_{H_{\Phi_0}(V)} + \|\widetilde{u}\|_{H_{\Phi_0}(V\setminus\widetilde{V})}\right) + Ce^{-\delta/h}\|\widetilde{u}\|_{H_{\Phi_0}(V)}.
\end{aligned} \tag{5.57}$$

Recalling (5.47) and the fact that $\widetilde{u} = Bu$, we get

$$\begin{aligned}
\|Bu\|_{H_{\Phi_0}(W)} &\leq Ch^{-1}\left(\|v\|_{H_{\widehat{\Phi}}(\mathbb{C})} + \|u\|_{H_{\Phi}(U\setminus\widehat{U})} + \|Bu\|_{H_{\Phi_0}(V\setminus\widetilde{V})}\right) + Ce^{-\delta/h}\|u\|_{H_{\widehat{\Phi},m}(\mathbb{C})} \\
&\leq Ch^{-1}\left(\|v\|_{H_{\widehat{\Phi}}(\mathbb{C})} + \|u\|_{H_{\Phi}(U\setminus\widehat{U})}\right) + C\,e^{-\delta/h}\|u\|_{H_{\widehat{\Phi},m}(\mathbb{C})}.
\end{aligned} \tag{5.58}$$

Here $\widehat{U} \Subset \widetilde{U} \Subset U$ and we have used the fact that

$$\|Bu\|_{H_{\Phi_0}(V\setminus\widetilde{V})} \leq C\|u\|_{H_\Phi(U\setminus\widehat{U})} + Ce^{-\delta/h}\|u\|_{H_\Phi(U)}, \tag{5.59}$$

for $\widehat{U} \Subset U$. Writing $u = ABu + (1-AB)u$ we obtain from (5.58) for $U_1 \Subset \kappa(W) \Subset U$,

$$\|u\|_{H_\Phi(U_1)} \leq Ch^{-1}\left(\|v\|_{H_{\widehat{\Phi}}(\mathbb{C})} + \|u\|_{H_\Phi(U\setminus\widehat{U})}\right) + Ce^{-\delta/h}\|u\|_{H_{\widehat{\Phi},m}(\mathbb{C})}. \tag{5.60}$$

Combining (5.60) with (5.24) applied for $\varepsilon > 0$ sufficiently small but fixed, we obtain that for z satisfying (5.31), equation (5.43) implies that

$$\|u\|_{H_{\widehat{\Phi},m}(\mathbb{C})} \leq Ch^{-1}\|v\|_{H_{\widehat{\Phi}}(\mathbb{C})} + Ce^{-\delta/h}\|u\|_{H_{\widehat{\Phi},m}(\mathbb{C})}. \tag{5.61}$$

The operator (5.32) is therefore injective, for $h > 0$ small enough, and hence bijective, as it is Fredholm of index 0. □

The main part of Theorem 3 follows from the next proposition about the spectrum of P^{w}: it follows from (5.21) that $\widehat{\Phi} - \Phi_1$ is compactly supported so that the spectrum of P^{w} on $H_{\widehat{\Phi}}(\mathbb{C})$ is that same as the spectrum on $H_{\Phi_1}(\mathbb{C})$ (the two spaces are the same but the norms are different). We also recall that H^{w} on $L^2(\mathbb{R})$ and P^{w} on H_{Φ_1} are unitarily equivalent using an FBI transform adapted to Φ_1, see (5.6),(5.7).

Proposition 5.7. *Suppose that*

$$|z - G((2j+1)h;h)| < h/C, \quad hj \leq \tfrac{1}{4}\rho^2, \quad 0 < \rho \ll 1. \tag{5.62}$$

Then

$$z \in \mathrm{Spec}_{H_{\widehat{\Phi}}}(P^{\mathrm{w}}(x,hD,h)) \iff z = G((2j+1)h;h) + Ce^{-\delta/h}, \quad \delta > 0, \tag{5.63}$$

and the eigenvalue is simple.

Proof. We will use the Schur complement formula and a globally well-posed Grushin problem for the operator $P^{\mathrm{w}}(x,hD_x;h) - z : H_{\widehat{\Phi},m}(\mathbb{C}) \to H_{\widehat{\Phi}}(\mathbb{C})$, see [Zw12, §D.1] for a general introduction to this method and references.

It is convenient for us to assume, as we may, that the order function m introduced in (4.3) satisfies $m \in C^\infty(\mathbb{C})$ with

$$|\partial^\alpha m| \leq C_\alpha m, \quad \forall \alpha \in \mathbb{N}^2, \tag{5.64}$$

see [Zw12, Lemma 8.7].

As before $V = D(0,r)$ and $\rho \leq r/C_0$, where $C_0 > 0$ is sufficiently large. This guarantees that V is much larger than the region where φ_j in (5.29) is not exponentially small – see Lemma 5.6. Let $\chi \in C_0^\infty(\mathbb{C};[0,1])$ satisfy

$$\chi(x) = 1, \quad |x| \leq \rho, \qquad \chi(x) < 1, \quad |x| > \rho,$$

and define

$$\widetilde{\Phi}_0(x) = \Phi_0(x) - \tfrac{1}{8}\rho^2(1-\chi(x)). \tag{5.65}$$

We then have

$$\widetilde{\Phi}_0(x) = \Phi_0(x), \quad |x| \leq \rho, \quad \widetilde{\Phi}_0(x) < \Phi_0(x), \quad |x| > \rho. \tag{5.66}$$

It follows from Lemma 5.6 that for $hj \leq \frac{1}{4}\rho^2$,

$$\|\varphi_j\|^2_{H_{\widetilde{\Phi}_0}(V)} \leq \int_{\mathbb{C}} |\varphi_j(x)|^2 \, e^{-2\widetilde{\Phi}_0(x)/h} \, L(dx) \leq 2. \tag{5.67}$$

Using (4.37), (4.39), and (5.67) we see that

$$\|A\varphi_j\|_{H_{\widetilde{\Phi}}(U)} = \mathcal{O}(1), \quad hj \leq \tfrac{1}{4}\rho^2, \tag{5.68}$$

where $\widetilde{\Phi} \in C(U;\mathbb{R})$ is a suitable weight satisfying

$$\widetilde{\Phi} \leq \Phi \quad \text{in } U, \quad \widetilde{\Phi} < \Phi \quad \text{in } U \backslash \overline{U_1}. \tag{5.69}$$

Indeed, we can take, in the notation of (4.37) and (4.39),

$$\widetilde{\Phi}(x) = \Phi(x) - \inf_{y \in V} \left(\frac{|x - \widetilde{\kappa}(y)|^2}{2C} + \frac{\rho^2}{8}(1 - \chi(y)) \right).$$

Here $U_1 \Subset U \Subset \mathbb{C}$ can be chosen arbitrarily small provided that $\rho > 0$ is small enough. Let $\Pi_{\Phi_m} : L^2_{\Phi,m}(\mathbb{C}) \to H_{\Phi,m}(\mathbb{C})$ be the orthogonal projection and

$$\psi \in C_0^\infty(U;[0,1]), \quad \psi(x) = 1, \quad x \in \operatorname{neigh}_{\mathbb{C}}(\overline{U_1}). \tag{5.70}$$

We define a continuous uniformly bounded map

$$R_- = \mathcal{O}(1) : \mathbb{C} \ni u_- \longmapsto u_- \Pi_{\Phi_m}(\psi A \varphi_j) \in H_{\Phi,m}(\mathbb{C}). \tag{5.71}$$

Using (5.64) we have

$$\begin{aligned} &\Phi_m := \Phi - h\log m \in C^\infty(\mathbb{C}), \quad \partial^\alpha \Phi_m \in L^\infty(\mathbb{C}), \quad |\alpha| \geq 2, \\ &\Delta\Phi_m \geq \Delta\Phi - \mathcal{O}(h) \geq 1/C - \mathcal{O}(h) \geq 1/2C, \quad 0 < h < h_0, \end{aligned} \tag{5.72}$$

and $L^2_{\Phi,m}(\mathbb{C}) = L^2_{\Phi_m}(\mathbb{C})$, $H_{\Phi,m}(\mathbb{C}) = H_{\Phi_m}(\mathbb{C})$.

It is important know that $R_- u_-$ is exponentially small away from the origin:

Lemma 5.8. *There exists $\delta > 0$ such that*

$$\|R_- u_- - u_- \psi A \varphi_j\|_{L^2_{\Phi,m}(\mathbb{C})} \leq C e^{-\delta/h} \, |u_-|, \quad hj \leq \tfrac{1}{4}\rho^2 \tag{5.73}$$

Proof. Definition (5.71) gives

$$R_- u_- - u_- \psi A \varphi_j = -u_-(1 - \Pi_{\Phi_m})(\psi A \varphi_j), \tag{5.74}$$

where $u := (1 - \Pi_{\Phi_m})(\psi A \varphi_j) \in L^2_{\Phi_m}(\mathbb{C})$ is the minimal norm solution of

$$\bar{\partial} u = \bar{\partial}(\psi A \varphi_j) = (\bar{\partial}\psi) A \varphi_j = \mathcal{O}(e^{-\delta/h})_{L^2_{\Phi_m}},$$

where we use (5.68), (5.69), and (5.70) to obtain the last estimate ($\bar{\partial}\psi$ is supported in the set where we have estimates with better weights).

An application of L^2-estimates for the $\overline{\partial}$ equation in the exponentially weighted space $L^2_{\Phi,m}(\mathbb{C}) = L^2_{\Phi_m}(\mathbb{C})$ (in the elementary one-dimensional case, see [Hö94, §4.2]) shows that $\|u\|_{L^2_{\Phi_m}} = \mathcal{O}(e^{-\delta/h})$, concluding the proof. □

Using R_- from (5.71) and defining

$$R_+ : H_\Phi(\mathbb{C}) \ni u \mapsto (B(u|_U), \varphi_j)_{H_{\Phi_0}(V)} \in \mathbb{C}, \tag{5.75}$$

where $U \Subset \mathbb{C}$ is the open neighbourhood of 0 given in (5.25), we pose the following global Grushin problem for $z \in \mathbb{C}$ satisfying (5.62):

$$\mathcal{P}(z) := \begin{pmatrix} P^{\mathrm{w}} - z & R_- \\ R_+ & 0 \end{pmatrix} : H_{\Phi,m}(\mathbb{C}) \times \mathbb{C} \to H_\Phi(\mathbb{C}) \times \mathbb{C}, \;\; \mathcal{P}(z) \begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix}. \tag{5.76}$$

Restricting u to U, using (5.26), Lemma 5.8, (5.68), (5.69) as well as the fact that $\|(\psi - 1)A\varphi_j\|_{H_\Phi(U)} = \mathcal{O}(e^{-\delta/h})$ in view of (5.70)

$$\begin{aligned} &(P - z)u + u_- A\varphi_j = v + w \quad \text{in } U, \quad R_+ u = v_+, \\ &\|w\|_{H_\Phi(U)} \le Ce^{-\delta/h}(\|u\|_{H_{\Phi,m}(\mathbb{C})} + |u_-|), \quad \delta > 0. \end{aligned} \tag{5.77}$$

Applying the operator B to the first equation in (5.77) we get, similarly to (5.45),

$$(Q - z)\widetilde{u} + u_- \varphi_j = \widetilde{v} \quad \text{in } V, \quad (\widetilde{u}, \varphi_j)_{H_{\Phi_0}(V)} = v_+. \tag{5.78}$$

Here $\widetilde{u} = B(u|_U) \in H_{\Phi_0}(V)$. With $\widetilde{U} \Subset U$ close to U, and similarly to (5.46), (5.47), we have

$$\begin{aligned} &\widetilde{v} = B(v + w) - BP(1 - AB)u + u_-(1 - BA)\varphi_j, \\ &\|\widetilde{v}\|_{H_{\Phi_0}(V)} \le C\|v\|_{H_{\widehat{\Phi}}(\mathbb{C})} + Ce^{-\delta/h}\left(\|u\|_{H_{\widehat{\Phi},m}(\mathbb{C})} + |u_-|\right) + C\|u\|_{H_\Phi(U \setminus \widetilde{U})}. \end{aligned} \tag{5.79}$$

Recalling (5.50), we rewrite the microlocal Grushin problem (5.78) as follows,

$$(Q - z)\widetilde{u} + u_- f_j = \widetilde{v} \quad \text{in } V, \quad (\widetilde{u}, f_j)_{H_{\Phi_0}(V)} = \widetilde{v}_+. \tag{5.80}$$

Here $f_j \in H_{\Phi_0}(V)$ is given by (5.37). The function $\widetilde{v}$ in (5.80) is modified by an exponentially small quantity compared to the $\widetilde{v}$ in (5.79) to take into account the difference between f_j and φ_j, see (5.50). It still satisfies the estimate in (5.79) and we also have

$$|\widetilde{v}_+| \le |v_+| + Ce^{-\delta/h}\|u\|_{H_{\Phi,m}(\mathbb{C})}. \tag{5.81}$$

The analysis of (5.80) proceeds similarly to (5.45), applying the orthogonal projection $\Pi_{N(h)}$ in (5.48) to the first equation in (5.80) and using that $\Pi_{N(h)} f_j = f_j$, for $hj \le \rho^2/4$. We get

$$\Pi_{N(h)}(Q - z)\Pi_{N(h)}\widetilde{u} + u_- f_j = \Pi_{N(h)}\widetilde{v} - \Pi_{N(h)}(Q - z)(1 - \Pi_{N(h)})\widetilde{u}, \tag{5.82}$$

$$(\Pi_{N(h)}\widetilde{u}, f_j)_{H_{\Phi_0}(V)} = \widetilde{v}_+. \tag{5.83}$$

Recalling that the matrix of the operator $\Pi_{N(h)}(Q-z)\Pi_{N(h)}$ acting on the range of $\Pi_{N(h)}$, with respect to the orthonormal basis f_j, $0 \leq j \leq N(h)-1$, is given by (5.55) and using (5.62) we conclude that the problem (5.82), (5.83) enjoys the estimate

$$\begin{aligned} h\|\Pi_{N(h)}\widetilde{u}\|_{H_{\Phi_0}(V)} + |u_-| &\leq C\|\Pi_{N(h)}\widetilde{v} - \Pi_{N(h)}(Q-z)(1-\Pi_{N(h)})\widetilde{u}\|_{H_{\Phi_0}(V)} \\ &\quad + Ch\,|\widetilde{v}_+| + C\,e^{-\delta/h}\|\widetilde{u}\|_{H_{\Phi_0}(V)} \\ &\leq C\|\widetilde{v}\|_{H_{\Phi_0}(V)} + C\|\Pi_{N(h)}(Q-z)(1-\Pi_{N(h)})\widetilde{u}\|_{H_{\Phi_0}(V)} \\ &\quad + Ch\,|\widetilde{v}_+| + C\,e^{-\delta/h}\|\widetilde{u}\|_{H_{\Phi_0}(V)}. \end{aligned}$$

Combining this with (5.54), (5.79) and (5.81) gives

$$\begin{aligned} h\|\Pi_{N(h)}\widetilde{u}\|_{H_{\Phi_0}(V)} + |u_-| &\leq C\|v\|_{H_{\widehat{\Phi}}(\mathbb{C})} + C\|u\|_{H_{\Phi}(U\setminus\widetilde{U})} + C\|\widetilde{u}\|_{H_{\Phi_0}(V\setminus\widetilde{V})} \\ &\quad + Ch\,|v_+| + Ce^{-\delta/h}\left(\|u\|_{H_{\widehat{\Phi},m}(\mathbb{C})} + |u_-|\right). \end{aligned} \tag{5.84}$$

Here $\widetilde{V} \Subset V$ and $\widetilde{U} \Subset U$. We obtain therefore, in view of (5.49) and (5.84),

$$\begin{aligned} h\|\widetilde{u}\|_{H_{\Phi_0}(W)} + |u_-| &\leq h\|\Pi_{N(h)}\widetilde{u}\|_{H_{\Phi_0}(W)} + |u_-| + h\|(1-\Pi_{N(h)})\widetilde{u}\|_{H_{\Phi_0}(W)} \\ &\leq C\left(\|v\|_{H_{\widehat{\Phi}}(\mathbb{C})} + \|u\|_{H_{\Phi}(U\setminus\widetilde{U})} + \|\widetilde{u}\|_{H_{\Phi_0}(V\setminus\widetilde{V})} + h\,|v_+|\right) \\ &\quad + C\,e^{-\delta/h}\left(\|u\|_{H_{\widehat{\Phi},m}(\mathbb{C})} + |u_-|\right). \end{aligned} \tag{5.85}$$

We get from (5.85), using that $\widetilde{u} = B(u|_U)$, similarly to (5.58), (5.59), (5.60),

$$\begin{aligned} h\|u\|_{H_{\Phi}(U_1)} + |u_-| &\leq C\left(\|v\|_{H_{\widehat{\Phi}}(\mathbb{C})} + \|u\|_{H_{\Phi}(U\setminus\widehat{U})} + h\,|v_+|\right) \\ &\quad + C\,e^{-\delta/h}\left(\|u\|_{H_{\widehat{\Phi},m}(\mathbb{C})} + |u_-|\right). \end{aligned} \tag{5.86}$$

Here $U_1 \Subset \widehat{U} \Subset U$. We have using the elliptic estimate (5.24) and the first equation in (5.76),

$$\|u\|_{H_{\Phi}(U\setminus\widehat{U})} \leq C\left(\|v\|_{H_{\widehat{\Phi}}(\mathbb{C})} + e^{-\delta/h}\left(\|u\|_{H_{\widehat{\Phi},m}(\mathbb{C})} + |u_-|\right)\right). \tag{5.87}$$

Here we have also used that, in view of Lemma 5.8, the term R_-u_- is exponentially small away from an arbitrarily small neighbourhood of 0, provided that $\rho > 0$ in (5.62) is small enough. We get, injecting (5.87) into (5.86),

$$h\|u\|_{H_{\Phi}(U_1)} + |u_-| \leq C\left(\|v\|_{H_{\widehat{\Phi}}(\mathbb{C})} + h\,|v_+|\right) + C\,e^{-\delta/h}\left(\|u\|_{H_{\widehat{\Phi},m}(\mathbb{C})} + |u_-|\right). \tag{5.88}$$

Using the elliptic estimate again,

$$\|u\|_{H_{\widehat{\Phi},m}(\mathbb{C}\setminus U_1)} \leq C\left(\|v\|_{H_{\widehat{\Phi}}(\mathbb{C})} + e^{-\delta/h}\left(\|u\|_{H_{\widehat{\Phi},m}(\mathbb{C})} + |u_-|\right)\right), \tag{5.89}$$

valid for $\rho > 0$ small enough, we get, adding (5.88) and (5.89),

$$h\|u\|_{H_{\widehat{\Phi},m}(\mathbb{C})} + |u_-| \leq C\left(\|v\|_{H_{\widehat{\Phi}}(\mathbb{C})} + h\,|v_+|\right) + Ce^{-\delta/h}\left(\|u\|_{H_{\widehat{\Phi},m}(\mathbb{C})} + |u_-|\right). \tag{5.90}$$

We have proved therefore that for all $h > 0$ small enough, $\mathcal{P}(z)$ in (5.76) is injective, for $z \in \mathbb{C}$ satisfying (5.62) for $\rho > 0$ sufficiently small but fixed. Since $\mathcal{P}(z)$ is a Fredholm operator of index zero (it is a finite rank perturbation of the operator in which $R_\pm$ are replaced by 0 and that operator has that property thanks to Proposition 5.3), it is consequently bijective.

Let

$$\mathcal{E}(z) = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix} : H_\Phi(\mathbb{C}) \times \mathbb{C} \to H_{\Phi,m}(\mathbb{C}) \times \mathbb{C}, \tag{5.91}$$

be the inverse of $\mathcal{P}(z)$ in (5.76), and let us recall that $z \in \mathbb{C}$ in the set (5.62) belongs to spectrum of $P^{\mathrm{w}}(x, hD_x; h)$ precisely when $E_{-+}(z) = 0$. We shall now compute $E_{-+}(z)$ up to an exponentially small term. For that we set

$$\widetilde{E}_+ = R_-, \quad \widetilde{E}_{-+}(z) = z - G((2j+1)h; h). \tag{5.92}$$

Using (5.71), (5.75), (5.77), (5.69), and Lemma 5.8, we see that uniformly in j, we have for $v_+ \in \mathbb{C}$,

$$\begin{aligned} R_+\widetilde{E}_+ v_+ &= R_+(v_+\Pi_\Phi(\psi A\varphi_j)) = R_+(v_+\psi A\varphi_j) + Ce^{-\delta/h}v_+ \\ &= R_+(v_+ A\varphi_j) + Ce^{-\delta/h}v_+ \\ &= v_+ + Ce^{-\delta/h}v_+, \end{aligned} \tag{5.93}$$

that is, $R_+\widetilde{E}_+ = 1 + \mathcal{O}(e^{-\delta/h})$.

Next, we consider

$$(P^{\mathrm{w}} - z)\,\widetilde{E}_+\, v_+ + R_-\widetilde{E}_{-+}v_+ = (P^{\mathrm{w}} - z)\, R_-\, v_+ + R_-\,(z - G((2j+1)h; h))\, v_+. \tag{5.94}$$

We first observe that in view of Lemma 5.8,

$$\|R_- v_+\|_{L^2_\Phi(\mathbb{C}\setminus U)} = \mathcal{O}(e^{-\delta/h})\, |v_+|\,. \tag{5.95}$$

Restricting the attention to the neighbourhood U, we see that modulo an error term of the form $\mathcal{O}(e^{-\delta/h})\,|v_+|$, the holomorphic function in (5.94) is equal to

$$\begin{aligned} v_+\left((P - z)A\varphi_j + A\varphi_j\,(z - G((2j+1)h; h))\right) &= v_+\,(PA - AQ)\,\varphi_j \\ &= \mathcal{O}(e^{-\delta/h}), \end{aligned} \tag{5.96}$$

in $H_\Phi(U)$. It follows that

$$(P^{\mathrm{w}}(x, hD_x; h) - z)\,\widetilde{E}_+ + R_-\widetilde{E}_{-+} = \mathcal{O}(e^{-\delta/h}) : \mathbb{C} \to H_\Phi(\mathbb{C}). \tag{5.97}$$

Combining (5.93) and (5.97) with the well-posedness of the Grushin problem, we see that we have computed $E_{-+}(z)$ in (5.91) up to an exponentially small error,

$$E_{-+}(z) = z - G((2j+1)h; h) + \mathcal{O}(e^{-\delta/h}), \tag{5.98}$$

uniformly in j satisfying the condition in (5.62), for $\rho > 0$ small enough but fixed. $\square$

We finish by indicating the vanishing of G_1 when the subprincipal symbol of H in Theorem 3 vanishes. For this we can work in the setting of [Hi04] and consider expansions modulo $\mathcal{O}(h^\infty)$ and C^∞ Fourier integral operators. In the proof of Proposition 5.2, we can follow the proof of [HelSj89, (a.3.11)] which shows that the conjugation by A_0 gives a normal form up to $\mathcal{O}(h^2)$ terms. The pseudodifferential conjugation does not introduce any $\mathcal{O}(h)$ terms and that shows that $G_1 \equiv 0$.

Appendix: Vey's Morse Lemma in dimension 2

For the reader's convenience we present a self-contained account of Vey's Morse Lemma in dimension two:

Theorem A.1. *Suppose that*

$$p \in \mathscr{O}(\mathrm{neigh}_{\mathbb{C}^2}(0,0)), \quad p(x,\xi) = q(x,\xi) + \mathcal{O}((x,\xi)^3)),$$

where q is a quadratic form such that $\{q(x,\xi) : (x,\xi) \in \mathbb{R}^2\} \neq \mathbb{C}$ and $q|_{\mathbb{R}^2\setminus 0} \neq 0$. Then there exists a biholomorphic map $F : \mathrm{neigh}_{\mathbb{C}^2}(0) \to \mathrm{neigh}_{\mathbb{C}^2}(0)$ and $f \in \mathscr{O}(\mathrm{neigh}_{\mathbb{C}}(0))$, $f(0) = 1$, such that

$$F^* p(z,\zeta) = q(z,\zeta), \quad F^*(d\xi \wedge dx) = f(q(z,\zeta)) d\zeta \wedge dz, \tag{A.1}$$

$F(0) = 0$, $dF(0) = I_{\mathbb{C}^2}$. Equivalently, we can find a biholomorphic map $\kappa : \mathrm{neigh}_{\mathbb{C}^2}(0) \to \mathrm{neigh}_{\mathbb{C}^2}(0)$ such that

$$\begin{gathered} \kappa^* p(z,\zeta) = g(q(z,\zeta)), \qquad \kappa^*(d\xi \wedge dx) = d\zeta \wedge dz, \\ \kappa(0) = 0, \quad d\kappa(0) = I_{\mathbb{C}^2}, \quad g'(t) = f(g(t))^{-1}, \quad g(0) = 0. \end{gathered} \tag{A.2}$$

If $p|_{\mathbb{R}^2}$ is real valued and q is positive definite, then F, κ and f, g are real valued in the real domain. We also note that while F and κ are not unique, f and g are.

We start by recalling the following fact (see [PS06, Proposition 2.1.10] for a slightly different version and more general statements):

Lemma A.1. *Any holomorphic quadratic form on $\mathbb{C}^2$ such that $\{q(x,\xi) : (x,\xi) \in \mathbb{R}^2\} \neq \mathbb{C}$ and $q|_{\mathbb{R}^2\setminus 0} \neq 0$ is complex symplectically equivalent to $\frac{1}{2i}\mu(x^2+\xi^2)$, $\mu \in \mathbb{C}\setminus\{0\}$, (or equivalently to $\mu x\xi$).*

Proof. If $q(x,\xi) = ax^2 + 2bx\xi + c\xi^2$ and, since $q|_{\mathbb{R}^2\setminus 0} \neq 0$, we can rescale and assume $a = 1$. If $a_\pm$ are roots of $z^2 + 2bz + c = 0$ then $q(x,\xi) = (x - a_+\xi)(x - a_-\xi)$, and again, $a_+, a_- \notin \mathbb{R}$. By a real linear change of variables we can assume that $a_+ = i$ and $a_- \notin \mathbb{R}$. Since the range of q is not all of $\mathbb{C}$, $a_- \neq i$. Writing $z = x + i\xi$, we then get

$$q(x,\xi) = \alpha \bar{z}(z + \beta \bar{z}), \quad \alpha := -i(i - a_-), \quad \beta = \frac{i + a_-}{i - a_-}, \quad |\beta| \neq 1.$$

If $|\beta| < 1$ then $\mathrm{Re}(\bar{\alpha} q) = |\alpha|^2(|z|^2 - \mathrm{Re}(\beta\bar{z}^2)) > 0$, $z \neq 0$. Otherwise, $q(x,\xi) = \alpha\beta(\bar{z}^2 + |z|^2/\beta) = \alpha|\beta|r^2(e^{i\theta} - 1/|\beta|)$, $\theta = -\frac{1}{2}\arg z - \arg\beta$, takes call complex values.

Hence (by multiplying q by a complex number) we can assume that $\mathrm{Re}\, q|_{\mathbb{R}^2\setminus 0} > 0$. We can then take a real linear canonical transformation such that $\mathrm{Re}\, q|_{\mathbb{R}^2} = \lambda(x^2+\xi^2)$, $\lambda \in \mathbb{R}\setminus 0$. Now we use an orthogonal transformation (with determinant 1; it preserves our new real part) to diagonalize $\mathrm{Im}\, q|_{\mathbb{R}^2} = \mu_1 x^2 + \mu_2\xi^2$, $\mu_j \in \mathbb{R}$. Complexifying these $\mathbb{R}$-linear transformations gives us a complex symplectic reduction to $(\lambda + i\mu_1)x^2 + (\lambda + i\mu_2)\xi^2$. A complex symplectic scaling finally produces $\frac{1}{2i}\mu(x^2+\xi^2)$, $\mu \in \mathbb{C}\setminus\{0\}$. ☐

In our setting the proof is a simple adaptation of the elegant argument presented by Colin de Verdière and Vey [CdVV79]. An alternative argument was given (independently of [Ve77],[CdVV79]) by Helffer–Sjöstrand [HelSj89]. The advantage we stress here is the algorithmic nature of the method which allows computation of the functions f and g.

We also record the following simple but useful

Theorem A.2. *In the notation of Theorem A.1, suppose that* $q(x,\xi) = \frac{1}{2}\mu(x^2+\xi^2)$, $\mu \in \mathbb{C}\setminus\{0\}$ *and that for* $t \in \mathrm{neigh}_{\mathbb{R}}(0)$,

$$S(t\mu) := \int_{\gamma(t\mu)} \xi dx$$

where $\gamma(t\mu)$ *is a closed cycle on the complex curve* $p(x,\xi) = t\mu$ *close to the positively oriented (real) circle* $\frac{1}{2}(x^2+\xi^2) = t$. *Then* $S \in \mathscr{O}(\mathrm{neigh}_{\mathbb{C}}(0))$ *and in the notation of* (A.1) *and* (A.2)

$$\mu S'(w) = 2\pi f(w), \quad S(0) = 0, \quad \mu S(g(w)) = 2\pi w. \tag{A.3}$$

Proof. In coordinates given by F in (A.1), $p = \frac{1}{2}\mu(z^2+\zeta^2)$ and $\gamma(t\mu)$ is the positively oriented real circle. By Stokes's theorem we have (in the real domain and using polar coordinates)

$$S(t\mu) = \int_{\frac{1}{2}(x^2+\xi^2)\le t} f(\tfrac{1}{2}\mu(x^2+\xi^2))dxd\xi = 2\pi\int_{\frac{1}{2}r^2\le t} f(\tfrac{1}{2}\mu r^2) r dr = 2\pi\int_0^t f(\mu\tau)d\tau.$$

This gives $\mu S'(t\mu) = 2\pi f(\mu t)$ and the first statement in (A.3) follows. Since f is holomorphic near 0 so is S. The relation between S and g now follows from the relation between g and f given in (A.2). ☐

Proof of Theorem A.1. We start by showing how (A.2) follows from (A.1). If $p(x,\xi) = q(x,\xi)$ and $\omega := f(q(x,\xi))d\xi\wedge dx$, we need to find $(x,\xi) = H(z,\zeta)$ such that $H^*q = g(q)$ and $H^*\omega = d\zeta\wedge dz$. Then $\kappa = F\circ H$ gives (A.2).

We define H by

$$x = \sqrt{a(q(z,\zeta))}z, \quad \xi = \sqrt{a(q(z,\zeta))}\zeta, \quad a(0) = 1.$$

so that

$$H^*q = qa(q), \quad H^*\omega = f(a(q)q)(a(q) + qa'(q))d\zeta \wedge dz, \quad q = q(z,\zeta).$$

We then put $g(t) = a(t)t$ where a is chosen so that,

$$g'(t) = a(t) + ta'(t) = \frac{1}{f(a(t)t)} = \frac{1}{f(g(t))},$$

which is the condition in (A.2). We also note that $dH(0) = I_{\mathbb{C}}$ and hence $d\kappa(0) = dF(0) = I_{\mathbb{C}^2}$.

Following [CdVV79] we first establish the following fact: for any $R \in \mathscr{O}(\mathrm{neigh}_{\mathbb{C}^2}(0,0))$ there exists $f \in \mathscr{O}(\mathrm{neigh}_{\mathbb{C}}(0))$ and $\eta \in \mathscr{O}(\mathrm{neigh}_{\mathbb{C}^2}(0,0))$ such that

$$R(z,\zeta)d\zeta \wedge dz = f(q(z,\zeta))d\zeta \wedge dz + dq \wedge d\eta, \quad \eta(0,0) = 0. \tag{A.4}$$

In dimension two and for holomorphic R (and unlike in the general case considered in [CdVV79]) this is very simple. We can assume (by a complex linear symplectic change of variables) that $q(z,\zeta) = i\mu z\zeta$ so that (A.4) becomes

$$R(z,\zeta) = f(i\mu z\zeta) + i\mu(z\partial_z\eta(z,\zeta) - \zeta\partial_\zeta\eta(z,\zeta)), \quad \eta(0,0) = 0.$$

If $R(z,\zeta) = \sum_{n,m} R_{nm} z^n \zeta^m$ then the solutions are given by

$$f(w) = \sum_{n=0}^{\infty} (-i)^n \mu^{-n} R_{nn} w^n, \quad \eta(z,\zeta) = (i\mu)^{-1} \sum_{n \neq m} \frac{R_{nm} z^n w^m}{n-m}. \tag{A.5}$$

To obtain (A.1) we first apply the standard holomorphic Morse lemma to obtain a biholomorphism $\widetilde{F}$ such that

$$\widetilde{F}^* p(z,\zeta) = q(z,\zeta), \quad \widetilde{F}^*(d\xi \wedge dx) = R(z,\zeta)d\zeta \wedge dz.$$

We then use (A.4) and define

$$\omega_t := f(q(z,\zeta))d\zeta \wedge dz + t dq \wedge d\eta.$$

We now search for a family of biholomorphic maps F_t such that $F_0(z,\zeta) = (z,\zeta)$ and

$$F_t^*\omega_t = \omega_0, \quad F_t^* q = q. \tag{A.6}$$

Finding F_t is equivalent to finding a holomorphic family of vector fields X_t which define F_t by $X_t(F_t(x)) = \partial_t F_t(x)$. Then (A.6) is equivalent to

$$\mathscr{L}_{X_t}\omega_t + \partial_t\omega_t = 0, \quad X_t q = 0. \tag{A.7}$$

Cartan's formula and the definition of ω_t show that that this is equivalent to

$$d(\iota_{X_t}\omega_t) = dq \wedge d\eta = -d(\eta dq).$$

Since $dq \wedge d\eta$ vanishes at $(0,0)$, ω_t is nondegenerate in a neighbourhood of $(0,0)$ and we can find X_t such that $\iota_{X_t}\omega_t = -\eta dq$. We have

$$0 = \omega_t(X_t, X_t) = -\eta dq(X_t) = -\eta X_t q.$$

If $\eta \equiv 0$ then $X_t \equiv 0$. Otherwise, $X_t q$ vanishes on a open set and analyticity shows that $X_t q \equiv 0$. This means that (A.7), and hence (A.6), hold and we obtain (A.1) with $F = \widetilde{F} \circ F_1$.

Finally, we note that once we have F satisfying (A.1), we can can compose it with a linear symplectic transformation so that $dF(0) = I_{\mathbb{C}^2}$. In fact, once (A.1) holds, we have the $dF(0)$ is symplectic (we have $f(0) = 1$) and $dF(0)^* q = q$. But then then $\widetilde{F} := dF(0)^{-1} \circ F$ satisfies (A.1) and $d\widetilde{F}(0) = I_{\mathbb{C}^2}$. $\square$

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