SOLITON INTERACTION WITH SLOWLY VARYING POTENTIALS

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Abstract. We study the Gross-Pitaevskii equation with a slowly varying smooth potential, $V(x) = W(hx)$. We show that up to time $\log(1/h)/h$ and errors of size h^2 in H^1, the solution is a soliton evolving according to the classical dynamics of a natural effective Hamiltonian, $(\xi^2 + \operatorname{sech}^2 * V(x))/2$. This provides an improvement ($h \to h^2$) compared to previous works, and is strikingly confirmed by numerical simulations – see Fig.1.

1. Introduction

The Gross-Pitaevskii equation is the nonlinear Schrödinger equation with an external potential:

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u - V(x)u + u|u|^2 = 0 \\ u(x,0) = e^{iv_0 x}\operatorname{sech}(x - a_0)\,, \end{cases} \tag{1.1}$$

In [10] we investigated the case of $V(x) = h^2 W(x)$ where $0 < h \ll 1$ and $W \in H^{-1}(\mathbb{R};\mathbb{R})$. Because of our earlier work on high velocity scattering [8],[9], which is potential specific, the paper presented the case of $W(x) = \delta_0(x)$, but as was pointed out there the method applies verbatim to the more general case, $W \in H^{-1}$. Motivated by the approach of [10] we now revisit the case of slowly varying potentials, $V = W(hx)$, and show that using the effective Hamiltonian approach we can describe the evolution of the soliton with errors of size h^2. In particular, in this setting, we improve the results of Fröhlich-Gustafson-Jonsson-Sigal [5].

Theorem 1. *Suppose that in* (1.1), $V(x) = W(hx)$, *where* $W \in \mathcal{C}^3(\mathbb{R};\mathbb{R})$. *Let* $\delta \in (0, 1/2)$. *Then on the time interval*

$$0 \le t \le \frac{\delta \log(1/h)}{Ch}\,,$$

we have

$$\|u(t,\bullet) - e^{i(\bullet - a(t))v(t)}e^{i\gamma(t)}\operatorname{sech}(\bullet - a(t))\|_{H^1(\mathbb{R})} \le Ch^{2-\delta}\,, \quad 0 < h < h_0\,, \tag{1.2}$$

where a, v, γ, *solve the following system of equations*

$$\begin{gathered} \dot a = v\,, \quad \dot v = -\frac{1}{2}\operatorname{sech}^2 * V'(a)\,, \\ \dot\gamma = \frac{1}{2} + \frac{v^2}{2} - \operatorname{sech}^2 * V(a) + (x\operatorname{sech}^2 x \tanh x) * V(a)\,, \end{gathered} \tag{1.3}$$

with initial data $(a_0, v_0, 0)$. *The constants* C *and* h_0 *depend on* $\|W^{(k)}\|_{L^\infty}$, $0 \le k \le 3$, *and* $|v_0|$ *only.*

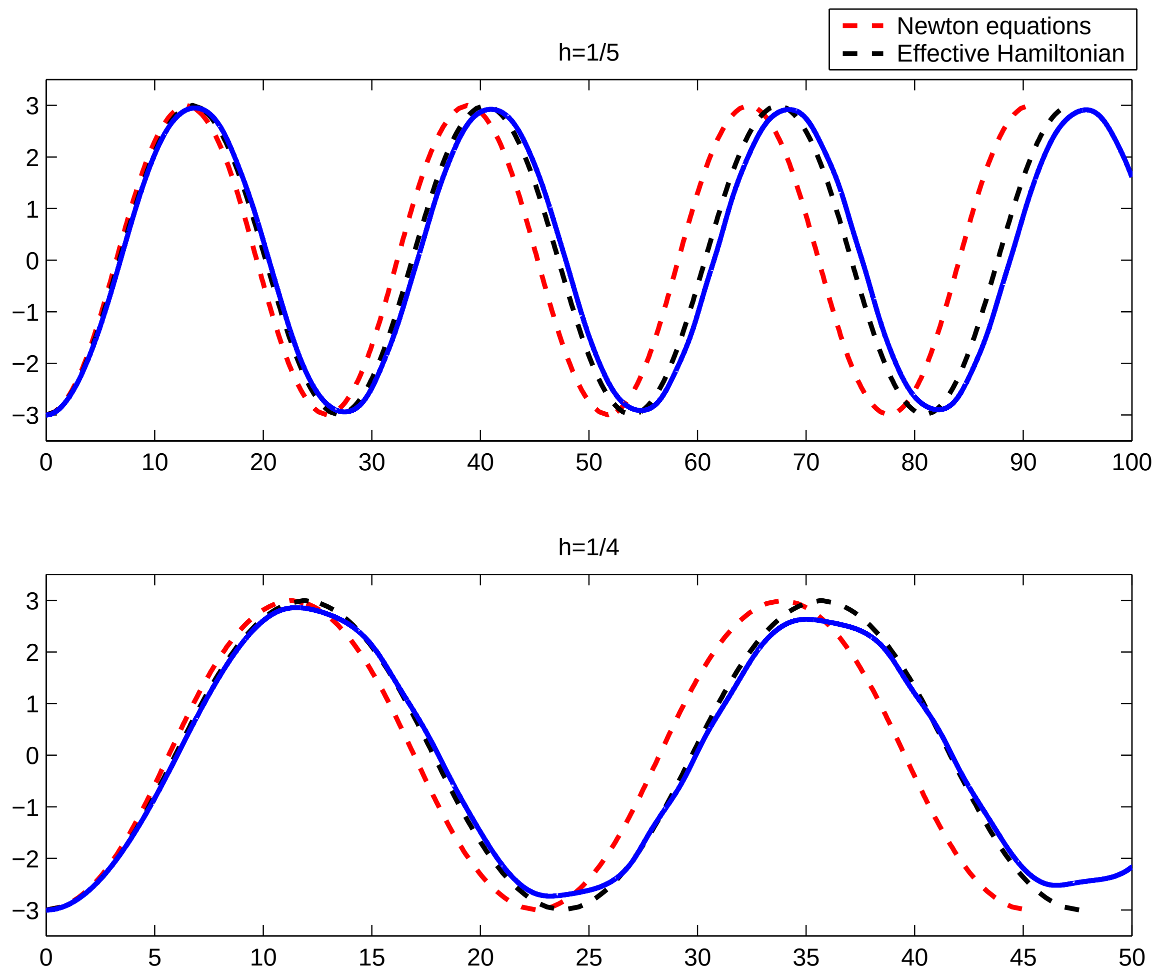

FIGURE 1. Comparison of the dynamics of the center of motion of the soliton for the Gross-Pitaevskii equation with a slowly varying potential,

$$iu_t = -\frac{1}{2}u_{xx} - |u|^2 u - \operatorname{sech}^2(hx)u\,, \quad h = 1/5\,, \quad h = 1/4\,,$$

and initial condition in (1.1) with $v_0 = 0$, $a_0 = -3$. The dashed red curve shows the solution to Newton's equations used in [1] and [5], the blue curve shows the center of the approximate soliton u, and the black dashed curve is given by the equations of motion of the effective Hamiltonian

$$\frac{1}{2}\left(v^2 + \operatorname{sech}^2(h\bullet) * \operatorname{sech}^2(a)\right)\,.$$

The improvement of the approximation given by the effective Hamiltonian is remarkable even in the case of $h = 1/4$ in which we already see radiative dissipation in the first cycle.

As in [10] the proof of our theorem follows the long tradition of the study of stability of solitons which started with the work of M.I. Weinstein [15]. The interaction of solitons with external potentials was studied in the stationary semiclassical setting by Floer and A. Weinstein [4] and Oh [13], and the first dynamical result belongs to Bronski and Jerrard [1], see also [2],[12]. The semiclassical regime is equivalent to considering slowly varying potentials, and the dynamics in that case was studied in [5],[6],[7],[14] (see also numerous references given there).

The results of [5] in the special case of (1.1) give

$$\|u(t,\bullet) - e^{i(\bullet - a(t))v(t)} e^{i\gamma(t)} \operatorname{sech}(\bullet - a(t))\|_{H^1(\mathbb{R})} \leq C h^{1-\delta}\,, \quad 0 \leq t \leq \delta \log(1/h)/h\,, \tag{1.4}$$

where

$$\begin{gathered} \dot a = v + \mathcal{O}(h^2)\,, \quad \dot v = -V'(a) + \mathcal{O}(h^2)\,, \\ \dot\gamma = 1/2 + v^2/2 - V(a) + \mathcal{O}(h^2)\,, \end{gathered} \tag{1.5}$$

with initial data $(a_0, v_0, 0)$. In other words, the motion of the soliton is approximately given by Newton's equations, $\dot a = v$, $\dot v = -V'(a)$. However it is not clear if, as in (1.3), the errors $\mathcal{O}(h^2)$ in (1.5) can be removed without affecting (1.4) – see §6. Since (1.3) imply equations (1.5), Theorem 1 shows that we can replace $h^{1-\delta}$ by $h^{2-\delta}$ in (1.4), keeping (1.5).

When $V(x) = h^2 W(x)$, and $W \in H^{-1}(\mathbb{R};\mathbb{R})$ [10], Newton's equations are clearly not applicable. To describe evolution up to time $\log(1/h)/h$ we introduced a natural effective Hamiltonian. A numerical experiment shown in [10, Fig.2] and reproduced here in Fig.1 suggested that the effective Hamiltonian approach gives a dramatic improvement for slowly varying potentials. Theorem 1 (and a more precise Theorem 2 below) quantify that improvement by giving the dynamics without any error terms $\mathcal{O}(h^2)$.

We also mention an interesting observation of Eboli-Marques[3][1]: if

$$-\frac{1}{2}\eta'' - \eta^3 + \frac{1}{2}h^2 x^2 = \frac{\mu^2}{2}\,,$$

then

$$u(t,x) = \eta(x - a(t)) \exp(iv(t)(x - a(t)) + \gamma(t))\,,$$

with

$$\dot a(t) = v(t)\,, \quad \dot v(t) = -ha(t)\,,$$

and a suitably chosen γ, solves (1.1) for $V(x) = h^2 x^2$. If η is the ground state then the soliton-like profile moves according to classical equations of motion for the Hamiltonian $(v^2 + h^2 a^2)/2$.

To describe the natural effective Hamiltonian we recall that the Gross-Pitaevski equation (1.1) is the equation for the Hamiltonian flow of

$$H_V(u) \stackrel{\text{def}}{=} \frac{1}{4}\int (|\partial_x u|^2 - |u|^4)dx + \frac{1}{2}\int V|u|^2\,, \tag{1.6}$$

[1]We thank Enno Lenzmann for informing us of this fact and providing the reference.

with respect to the symplectic form on $H^1(\mathbb{R},\mathbb{C})$ (considered as a real Hilbert space):

$$\omega(u,v) = \operatorname{Im} \int u\bar{v}\,, \quad u,v \in H^1(\mathbb{R},\mathbb{C})\,. \tag{1.7}$$

When $V \equiv 0$, $\eta = \operatorname{sech}$ is a minimizer of H_0 with the prescribed L^2 norm ($\|\eta\|^2_{L^2} = 2$):

$$d\mathcal{E}_\eta = 0\,, \quad \mathcal{E}(u) \stackrel{\text{def}}{=} H_0(u) + \frac{1}{4}\|u\|^2_{L^2}\,. \tag{1.8}$$

Here $d\mathcal{E}_\eta$ is the differential of $\mathcal{E} \;:\; H^1 \to \mathbb{R}$, see §2.1.

The flow of H_0 is tangent to the manifold of solitons,

$$M = \{e^{i\gamma}e^{iv(x-a)}\mu\operatorname{sech}(\mu(x-a))\,, \quad a,v,\gamma \in \mathbb{R}\,, \quad \mu \in \mathbb{R}_+\}\,,$$

which of course corresponds to the fact that the solution of (1.1) with $V = 0$ and $u_0(x,0) = e^{i\gamma_0 + iv_0(x-a_0)}\mu\operatorname{sech}(\mu(x-a_0))$, is

$$\begin{aligned} u(x,t) = g(t)\cdot\eta &\stackrel{\text{def}}{=} e^{i\gamma + iv_0(x-a_0) + i(\mu^2 - v_0^2)t/2}\mu\operatorname{sech}(\mu(x-a_0-v_0t))\,, \\ g(t) &\stackrel{\text{def}}{=} (a_0 + v_0t, v_0, \gamma_0 + (\mu^2 - v_0^2)t/2, \mu)\,. \end{aligned} \tag{1.9}$$

The symplectic form (1.7) restricted to M is

$$\omega\restriction_M = \mu dv \wedge da + v d\mu \wedge da + d\gamma \wedge d\mu\,, \tag{1.10}$$

see §2.3. The evolution of the parameters (a,v,γ,μ) in the solution (1.9) follows the Hamilton flow of

$$H_0\restriction_M = \frac{\mu v^2}{2} - \frac{\mu^3}{6}\,,$$

with respect to the symplectic form $\omega\restriction_M$.

The system of equations (1.3) is obtained using the following basic idea: if a Hamilton flow of H, with initial condition on a symplectic submanifold, M, stays close to M, then the flow is close to the Hamilton flow of $H\restriction_M$. In our case M is the manifold of solitons and H is given by (1.6)

$$H_V\restriction_M (a,v,\gamma,\mu) = \frac{\mu v^2}{2} - \frac{\mu^3}{6} + \frac{1}{2}\mu^2(V * \operatorname{sech}^2)(\mu a)\,. \tag{1.11}$$

The equations (1.3) are simply the equations of the flow of $H_V\restriction_M$ – see §2.4. They are easily seen to imply (1.5) but some h corrections are built into the classical motion leading to the improvement in Theorem 1, see also Fig.1.

As in previous works all of this hinges on the proximity of $u(x,t)$ to the manifold of solitons, M. In [10] we followed [5] and used Weinstein's Lyapunov function [15],

$$L(w) \stackrel{\text{def}}{=} \mathcal{E}(w+\eta) - \mathcal{E}(\eta)\,,$$

where $\mathcal{E}$ is given by (1.8). Here, in the notation similar to (1.9),

$$u(t) = g(t)\cdot(\eta + w(t))\,,$$

for an optimally chosen $g(t) = (a(t), v(t), \gamma(t), \mu(t))$ – see Lemma 2.3.

The use of $L(w)$ seems essential for the all-time orbital stability of solitons. Up to times $h^{-1}\log(1/h)$ we found that it is easier to use its quadratic approxition

$$L_0 \stackrel{\text{def}}{=} \langle \mathcal{L}w, w\rangle\,, \quad \mathcal{L}w \stackrel{\text{def}}{=} \mathcal{E}_\eta''(w) = -\frac{1}{2}\partial_x^2 w - 2\eta^2 w - \eta^2\bar{w} + \frac{1}{2}w\,. \tag{1.12}$$

Rather than use conservation of energy, $H_V(u)$, we use the nonlinear equation for $w(t)$ in estimating $L_0(w)$ – see §5. That involves solving a nonhomogeneous linear equation approximately using the spectral properties of $\mathcal{L}$ – see (5.2) and Proposition 4.2. The fact that the solution of that equation is of size h^2 in H^1 gives the first indication of the improvement based on using the effective Hamiltonian. Ultimately, this makes the argument simpler than in the case of $W \in H^{-1}$ (or the special case of $W = \delta_0$).

The paper is organized as follows. In §2 we recall the Hamiltonian structure of the nonlinear flow of (1.1) and describe the manifold of solitons. As in [10], its identification with the Lie group $G = H_3 \ltimes \mathbb{R}_+$, where H_3 is the Heisenberg group, provides useful notational shortcuts. In §3 we describe the reparametrized evolution. The starting point there is an application of the implicit function theorem and a decomposition of the solution into symplectically orthogonal components. That method has a long tradition in soliton stability and we learned it from [5]. The analysis of the orthogonal component using an approximate solution to a linear equation and a bootstrap argument are presented in §5. This results in a somewhat more precise version of Theorem 1 – see Theorem 2. The ODE estimates needed for the exact evolution (1.3) are given in §6. Finally we show how Theorem 2 implies Theorem 1. Except for basic material such as properties of Sobolev spaces or elementary symplectic geometry, and a reference to the proof of Proposition 4.1, the paper is meant to be self contained.

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2. The Hamiltonian structure and the manifold of solitons

In this section we recall the well known facts about the Hamiltonian structure of the nonlinear Schrödinger equation. The manifold of solitons is given as an orbit of a semidirect product of the Heisenberg group and $\mathbb{R}_+$.

2.1. **Symplectic structure.**

In our work, we consider

$$V \stackrel{\text{def}}{=} H^1(\mathbb{R}, \mathbb{C}) \subset L^2(\mathbb{R}, \mathbb{C})\,,$$

viewed as a *real* Hilbert space. The inner product and the symplectic form are given by

$$\langle u, v\rangle \stackrel{\text{def}}{=} \operatorname{Re}\int u\bar{v}\,, \quad \omega(u,v) \stackrel{\text{def}}{=} \langle iu, v\rangle = \operatorname{Im}\int u\bar{v}\,, \tag{2.1}$$

Let $H : V \to \mathbb{R}$ be a function, a Hamiltonian. The associated Hamiltonian vector field is a map $\Xi_H : V \to TV$, which means that for a particular point $u \in V$, we have $(\Xi_H)_u \in T_u V$. The vector field Ξ_H is defined by the relation

$$\omega(v, (\Xi_H)_u) = d_u H(v)\,, \tag{2.2}$$

where $v \in T_u V$, and $d_u H : T_u V \to \mathbb{R}$ is defined by

$$d_u H(v) = \frac{d}{ds}\Big|_{s=0} H(u + sv)\,.$$

In the notation above

$$dH_u(v) = \langle dH_u, v\rangle\,, \quad (\Xi_H)_u = \frac{1}{i} dH_u\,. \tag{2.3}$$

If we take $V = H^1(\mathbb{R}, \mathbb{C})$ with the symplectic form (2.1), and

$$H(u) = \int \frac{1}{4}|\partial_x u|^2 - \frac{1}{4}|u|^4$$

then we can compute

$$\begin{aligned} d_u H(v) &= \operatorname{Re} \int ((1/2)\partial_x u \partial_x \bar{v} - |u|^2 u \bar{v}) \\ &= \operatorname{Re} \int (-(1/2)\partial_x^2 u - |u|^2 u)\bar{v}\,. \end{aligned}$$

Thus, in view of (2.3) and (2.2),

$$(\Xi_H)_u = \frac{1}{i}\left(-\frac{1}{2}\partial_x^2 u - |u|^2 u\right)$$

The flow associated to this vector field (Hamiltonian flow) is

$$\dot{u} = (\Xi_H)_u = \frac{1}{i}\left(-\frac{1}{2}\partial_x^2 u - |u|^2 u\right)\,. \tag{2.4}$$

For future reference we state two general lemmas of symplectic geometry. The simple proofs can be found in [10, §2].

Lemma 2.1. *Suppose that $g : V \to V$ is a diffeomorphism such that $g^*\omega = \mu(g)\omega$, where $\mu(g) \in C^\infty(V; \mathbb{R})$. Then for $f \in C^\infty(V, \mathbb{R})$,*

$$(g^{-1})_* \Xi_f(g(\rho)) = \frac{1}{\mu(g)} \Xi_{g^* f}(\rho)\,, \quad \rho \in V\,. \tag{2.5}$$

Suppose that $f \in C^\infty(V; \mathbb{R})$ and that $df(\rho_0) = 0$. Then the Hessian of f at ρ_0, $f''(\rho_0) : T_\rho V \mapsto T^*_\rho V$, is well defined. We can identify $T_\rho V$ with $T^*_\rho V$ using the innner product, and define the Hamiltonian map $F : T_\rho V \to T_\rho V$ by

$$F = \frac{1}{i} f''(\rho_0)\,, \quad \langle f''(\rho_0)X, Y\rangle = \omega(Y, FX)\,. \tag{2.6}$$

In this notation we have

Lemma 2.2. *Suppose that $N \subset V$ is a finite dimensional symplectic submanifold of V, and $f \in C^\infty(V, \mathbb{R})$ satisfies*

$$\Xi_f(\rho) \in T_\rho N \subset T_\rho V \,, \quad \rho \in N \,.$$

If at $\rho_0 \in N$, $df(\rho_0) = 0$, then the Hamiltonian map defined by (2.6) *satisfies*

$$F(T_\rho N) \subset T_\rho N \,.$$

The Hamiltonian map, F, is simply the linearization of Ξ_f at a critical point of f. An example relevant to this paper is

$$f(u) = \mathcal{E}(u) \stackrel{\text{def}}{=} H_0(u) + \frac{1}{4}\|u\|^2 \,,$$

see (1.8). The soliton η is a critical point of $\mathcal{E}$ and the Hessian of E is given by $\mathcal{L}$ in (1.12). The Hamiltonian map $F = (1/i)\mathcal{L}$ is the linearization of Ξ_f at η. In other words, $(1/i)(\mathcal{L} - 1/2)$ is the linearization of (1.1) (with $V = 0$) at η. The $1/2$ term comes from $\|u\|^2/4$ in the definition of $\mathcal{E}$.

In Lemma 2.2 we can take N to be the four dimensional manifold of solitons and $\rho = \eta$. It then says that $(1/i)\mathcal{L}$ preserves the symplectic orthogonality of $w \in T_\eta V$ to $T_\eta M$.

2.2. **Manifold of solitons as an orbit of a group.** For $g = (a, v, \gamma, \mu) \in \mathbb{R}^3 \times \mathbb{R}_+$ we define the following map

$$H^1 \ni u \longmapsto g \cdot u \in H^1 \,, \quad (g \cdot u)(x) \stackrel{\text{def}}{=} e^{i\gamma} e^{iv(x-a)} \mu u(\mu(x-a)) \,. \tag{2.7}$$

This action gives a group structure on $\mathbb{R}^3 \times \mathbb{R}_+$ and it is easy to check that this transformation group is a semidirect product of the Heisenberg group H_3 and $\mathbb{R}_+$:

$$G = H_3 \ltimes \mathbb{R}_+ \,, \quad \mu \cdot (a, v, \gamma) = (\frac{a}{\mu}, \mu v, \gamma) \,.$$

Explicitly, the group law on G is given by

$$(a, v, \gamma, \mu) \cdot (a', v', \gamma', \mu') = (a'', v'', \gamma'', \mu'') \,,$$

where

$$v'' = v + v'\mu \,, \quad a'' = a + \frac{a'}{\mu} \,, \quad \gamma'' = \gamma + \gamma' + \frac{va'}{\mu} \,, \quad \mu'' = \mu\mu'$$

The action of G is not symplectic but it is *conformally symplectic* in the sense that

$$g^*\omega = \mu(g)\omega \,, \quad g = (h(g), \mu(g)) \,, \quad \mu(g) \in \mathbb{R}_+ \,, \tag{2.8}$$

as is easily seen from (2.1).

The Lie algebra of G, denoted by $\mathfrak{g}$, is generated by e_1, e_2, e_3, e_4,

$$\exp(te_1) = (t, 0, 0, 1), \quad \exp(te_2) = (0, t, 0, 1),$$
$$\exp(te_3) = (0, 0, t, 1), \quad \exp(te_4) = (0, 0, 0, e^t),$$

and the bracket acts as follows:

$$[e_1, e_4] = e_1, \quad [e_2, e_4] = -e_2, \quad [e_1, e_2] = -e_3, \quad [e_3, \bullet] = 0, \tag{2.9}$$

so e_3 is in the center. The infinitesimal representation obtained from (2.7) is given by

$$e_1 = -\partial_x, \quad e_2 = ix, \quad e_3 = i, \quad e_4 = \partial_x \cdot x. \tag{2.10}$$

It acts, for instance on $\mathcal{S}(\mathbb{R}) \subset H^1$, and by $X \in \mathfrak{g}$ we will denote a linear combination of the operators e_j.

The proof of the following standard lemma can be again found in [10, §2]:

Lemma 2.3. *Suppose $\mathbb{R} \ni t \mapsto g(t)$ is a C^1 function and that $u \in \mathcal{S}(\mathbb{R})$. Then, in the notation of* (2.7),

$$\frac{d}{dt} g(t) \cdot u = g(t) \cdot (X(t)u),$$

where $X(t) \in \mathfrak{g}$ is given by

$$X(t) = \dot{a}(t)\mu(t)e_1 + \frac{\dot{v}(t)}{\mu(t)} e_2 + (\dot{\gamma}(t) - \dot{a}(t)v(t))e_3 + \frac{\dot{\mu}(t)}{\mu(t)} e_4, \tag{2.11}$$

where $g(t) = (a(t), v(t), \gamma(t), \mu(t))$.

The manifold of solitons is an orbit of this group, $G \cdot \eta$, to which Ξ_H, defined in (2.2), is tangent. In view of (2.4) that means that

$$i\left(\frac{1}{2}\partial_x^2 \eta + |\eta|^2 \eta\right) = X \cdot \eta,$$

for some $X \in \mathfrak{g}$. The simplest choice is given by taking $X = \lambda i$, $\lambda \in \mathbb{R}$, so that η solves a nonlinear elliptic equation

$$-\frac{1}{2}\eta'' - \eta^3 + \lambda\eta = 0.$$

This has a solution in H^1 if $\lambda = \mu^2/2 > 0$ and it then is $\eta(x) = \mu \operatorname{sech}(\mu x)$. We will fix $\mu = 1$ so that

$$\eta(x) = \operatorname{sech} x.$$

Using Lemma 2.1 we can check that $G \cdot \eta$ is the *only* orbit of G to which Ξ_H is tangent.

We define the submanifold of solitons, $M \subset H_1$, as the orbit of η under G,

$$M = G \cdot \eta \subset H_1$$

and thus we have the identifications

$$M = G \cdot \eta \simeq G/\mathbb{Z}, \quad T_\eta M = \mathfrak{g} \cdot \eta \simeq \mathfrak{g}. \tag{2.12}$$

The quotient corresponds to the $\mathbb{Z}$-action

$$(a, v, \gamma, \mu) \mapsto (a, v, \gamma + 2\pi k, \mu)\,, \quad k \in \mathbb{Z}$$

2.3. **Symplectic structure on the manifold of solitons.** We first compute the symplectic form $\omega\restriction_M$ on $T_\eta M$ using the identification (2.12):

$$(\omega\restriction_M)_\eta(e_i, e_j) = \operatorname{Im} \int (e_i \cdot \eta)(x)(\overline{e_j \cdot \eta})(x)\,.$$

Since

$$\int \eta^2(x)dx = 2\,, \quad \int \eta(x)\partial_x\eta(x) = 0\,, \quad \int \partial_x\eta(x)x\eta(x)dx = -1\,,$$

we obtain from (2.10) that

$$(\omega\restriction_M)_\eta(e_2, e_1) = 1\,, \quad (\omega\restriction_M)_\eta(e_3, e_4) = 1\,, \tag{2.13}$$

and all the other $(\omega\restriction_M)_\eta(e_i, e_j)$'s vanish. In other words,

$$(\omega\restriction_M)_\eta = (dv \wedge da + d\gamma \wedge d\mu)_{(0,0,0,1)} = (d(vda + \gamma d\mu))_{(0,0,0,1)}\,.$$

Using (2.12) we conclude that

$$\omega\restriction_M = \mu dv \wedge da + vd\mu \wedge da + d\gamma \wedge d\mu\,, \tag{2.14}$$

see [10, Lemma 2.3].

Now let f be a function defined on M, $f = f(a, v, \gamma, \mu)$. The associated Hamiltonian vectorfield, Ξ_f, is defined by

$$\omega(\cdot, \Xi_f) = df = f_a da + f_v dv + f_\mu d\mu + f_\gamma d\gamma\,.$$

Using (2.14) we obtain

$$\Xi_f = \frac{f_v}{\mu}\partial_a + \left(-\frac{f_a}{\mu} - \frac{vf_\gamma}{\mu}\right)\partial_v + f_\gamma\partial_\mu + \left(v\frac{f_v}{\mu} - f_\mu\right)\partial_\gamma\,. \tag{2.15}$$

The Hamilton flow is obtained by solving

$$\dot v = -\frac{f_a}{\mu} - \frac{vf_\gamma}{\mu}\,, \quad \dot a = \frac{f_v}{\mu}\,, \quad \dot\mu = f_\gamma\,, \quad \dot\gamma = v\frac{f_v}{\mu} - f_\mu\,.$$

The restriction of

$$H(u) = \frac{1}{4}\int |\partial_x u|^2 - \frac{1}{4}\int |u|^4$$

to M is given by computing by

$$f(a, v, \gamma, \mu) = H(g \cdot \eta) = \frac{\mu v^2}{2} - \frac{\mu^3}{6}\,, \quad g = (a, v, \gamma, \mu)\,. \tag{2.16}$$

The flow of (2.15) for this f describes the evolution of a soliton.

2.4. **The Gross-Pitaevski Hamiltonian restricted to the manifold of solitons.** We now consider the Gross-Pitaevski Hamiltonian,

$$H_q(u) \stackrel{\text{def}}{=} \frac{1}{4}\int (|\partial_x u|^2 - |u|^4)dx + \frac{1}{2}\int V(x)|u|^2 dx\,, \tag{2.17}$$

and its restriction to $M = G \cdot \eta$:

$$H_q\restriction_M = f(a, v, \gamma, \mu) = \frac{\mu v^2}{2} - \frac{\mu^3}{6} + \frac{\mu^2}{2} V * (\operatorname{sech}^2(\mu \bullet))(a)\,. \tag{2.18}$$

The flow of $(H_q)\restriction_M$ can be read off from (2.15):

(2.19)
$$\begin{gathered}\dot{v} = -\frac{f_a}{\mu} - \frac{v f_\gamma}{\mu} = -\frac{\mu^2}{2} V' * (\operatorname{sech}^2(\mu\bullet))(a)\,, \quad \dot{a} = \frac{f_v}{\mu} = v\,, \quad \dot{\mu} = f_\gamma = 0\,, \\ \dot{\gamma} = v\frac{f_v}{\mu} - f_\mu = \frac{1}{2}v^2 + \frac{1}{2}\mu^2 - \mu V * (\operatorname{sech}^2(\mu\bullet))(a) + \mu V * (x \operatorname{sech}^2(x)\tanh(x)\restriction_{x=\mu\bullet})(a)\,.\end{gathered}$$

This are the same equations as (1.3). The evolution of a and v is simply the Hamiltonian evolution of $(v^2 + \mu^2 V * \operatorname{sech}^2(\mu\bullet)(a))/2$, $\mu = \text{const}$. The more mysterious evolution of the phase γ is now explained by (2.18).

We can also rewrite (2.19) as follows:

$$f = \frac{\mu v^2}{2} - \frac{\mu^3}{6} + \frac{\mu}{2}\int V\Big(\frac{x}{\mu} + a\Big)\eta^2(x)dx\,,$$

so that

$$\begin{aligned}\dot{\gamma} &= \frac{v^2}{2} + \frac{\mu^2}{2} - \frac{1}{2}\int V\Big(\frac{x}{\mu} + a\Big)\eta^2(x)\,dx + \frac{1}{2}\int V'\Big(\frac{x}{\mu} + a\Big)\frac{x}{\mu}\eta^2(x)\,dx \\ &= \frac{v^2}{2} + \frac{\mu^2}{2} - V(a) + \frac{V''(a)\pi^2}{24\mu^2} + \mathcal{O}(h^4)\end{aligned}$$

and

$$\dot{v} = -\frac{1}{2}\int V'\Big(\frac{x}{\mu} + a\Big)\eta^2(x)\,dx = -V'(a) - \frac{V'''(a)\pi^2}{24\mu^2} + \mathcal{O}(h^5)\,,$$

where we used that $\int \eta^2 = 2$ and $\int x^2\eta^2 = \pi^2/6$ in the Taylor expansions. Here for the purpose of presentation we assumed that $W \in \mathcal{C}^5$ but we will never use Taylor's formula with more than four terms, for which $\mathcal{C}^3$ is only needed.

3. Reparametrized evolution

To see the effective dynamics described in §2.4 we write the solution of (1.1) as

$$u(t) = g(t)\cdot(\eta + w(t))\,, \quad w(t) \in H^1(\mathbb{R}, \mathbb{C})\,,$$

where $w(t)$ satisfies

$$\omega(w(t), X\eta) = 0\,, \quad \forall X \in \mathfrak{g}\,.$$

To see that this decomposition is possible, initially for small times, we apply the following consequence of the implicit function theorem and the nondegeneracy of $\omega\restriction_M$ (see [5, Proposition 5.1] for a more general statement):

Lemma 3.1. *For $\Sigma \Subset G/\mathbb{Z}$ (where the topology on $G/\mathbb{Z}$ is given by the identification with $\mathbb{R} \times \mathbb{R} \times S^1 \times \mathbb{R}_+$) let*

$$U_{\Sigma,\delta} = \{\, u \in H_1 \,:\, \inf_{g\in\Sigma} \|u - g\cdot\eta\|_{H^1} < \delta \}\,.$$

If $\delta \leq \delta_0 = \delta_0(\Sigma)$ then for any $u \in U_{\Sigma,\delta}$, there exists a unique $g(u) \in \Sigma$ such that

$$\omega(g(u)^{-1}\cdot u - \eta, X\cdot\eta) = 0 \quad \forall X \in \mathfrak{g}\,. \tag{3.1}$$

Moreover, the map $u \mapsto g(u)$ is in $C^1(U_{\Sigma,\delta}, \Sigma)$.

Proof. We define the transformation

$$F \;:\; H^1(\mathbb{R},\mathbb{C}) \times G \;\longrightarrow\; \mathfrak{g}^*\,, \quad [F(u,h)](X) \stackrel{\text{def}}{=} \omega(h\cdot u - \eta, X\cdot\eta)\,,$$

and want to solve $F(u,h) = 0$ for $h = h(u)$. By the implicit fuction theorem that follows for u near $G\cdot\eta$ if for any $g_0 \in G$ the linear transformation

$$d_hF(g_0\cdot\eta, g_0) \;:\; T_{g_0}G \longrightarrow \mathfrak{g}^*\,,$$

is invertible. Clearly we only need to check it for $g_0 = e$, that is that $d_hF(\eta, e) \;:\; \mathfrak{g} \to \mathfrak{g}^*$, is invertible. But as an element of $\mathfrak{g}^*\otimes\mathfrak{g}^*$, $d_hF(\eta,e) = (\omega\restriction_M)_\eta$, which is nondegenerate. □

From §§2.1 and 2.4 we recall that the equation for u (1.1) can be written as

$$\partial_t u = \Xi_{H_q}(u)\,, \quad H_q(u) \stackrel{\text{def}}{=} \frac{1}{4}\int (|\partial_x u|^2 - |u|^4)dx + \frac{1}{2}\int V(x)|u|^2 dx\,. \tag{3.2}$$

Using Lemma 3.1 we define

$$g(t) \stackrel{\text{def}}{=} g(u(t))\,, \quad \tilde{u} \stackrel{\text{def}}{=} g(t)^{-1}u(t)\,, \quad w(t) \stackrel{\text{def}}{=} \tilde{u} - \eta\,, \tag{3.3}$$

and we want to to derive an equation for $w(t)$.

Let

$$\begin{aligned} \alpha = \alpha(a,\mu) &\stackrel{\text{def}}{=} \frac{1}{2}\int V\Big(\frac{x}{\mu} + a\Big)\eta^2(x)dx - \frac{1}{2}\int V'\Big(\frac{x}{\mu} + a\Big)\frac{x}{\mu}\eta^2(x)\,dx\,, \\ \beta = \beta(a,\mu) &\stackrel{\text{def}}{=} \frac{1}{2\mu}\int V'\Big(\frac{x}{\mu} + a\Big)\eta^2(x)\,dx\,. \end{aligned} \tag{3.4}$$

Note that dependence on a, μ makes α, β into time-dependent parameters. They are however independent of x. The proper motivation for the choice of α and β will come in Lemma 3.3 below.

Set

$$X = (-\dot{a} + v)\mu e_1 + \Big(-\frac{\dot{v}}{\mu} - \beta\Big) e_2 + \Big(-\dot{\gamma} + \dot{a}v - \frac{1}{2}v^2 + \frac{1}{2}\mu^2 - \alpha \Big) e_3 - \frac{\dot{\mu}}{\mu} e_4 \,, \tag{3.5}$$

where e_j's are given by (2.10). Let also

$$\mathcal{N} w = 2|w|^2 \eta + \eta w^2 + |w|^2 w \,. \tag{3.6}$$

We now have

Lemma 3.2. *The equation for w, defined by* (3.3), *is*

$$\begin{aligned} \partial_t w = {} & X\eta + i\Big[-V\Big(\frac{x}{\mu} + a\Big) + \alpha + \beta x \Big]\eta \\ & + Xw + i\Big[-V\Big(\frac{x}{\mu} + a\Big) + \alpha + \beta x\Big] w - i\mu^2 \mathcal{L} w + i\mu^2 \mathcal{N} w \,, \end{aligned}$$

where X is given by (3.5), *$\mathcal{L}$ by* (1.12), *$\mathcal{N}$ by* (3.6).

Proof. We compute, by the chain rule

$$\partial_t u = \partial_t (g \cdot (\eta + w)) = g \cdot Y(\eta + w) + g \cdot \partial_t w$$

where

$$Y = \dot{a}\mu e_1 + \frac{\dot{v}}{\mu} e_2 + (\dot{\gamma} - \dot{a}v) e_3 + \frac{\dot{\mu}}{\mu} e_4 \,.$$

From this, and the fact that the equation for u can be written $\partial_t u = \Xi_H(u)$, we get

$$\partial_t w = -Y\tilde{u} + g^{-1}\partial_t u = -Y\tilde{u} + g^{-1}\Xi_H g \tilde{u}$$

We apply Lemma 2.1 to obtain

$$\partial_t w = -Y\tilde{u} + \frac{1}{\mu}\Xi_{g^* H}\tilde{u}$$

We compute

$$\begin{aligned} (g^*H)(\tilde{u}) = H(g\tilde{u}) = {} & \frac{1}{4} v^2 \mu \|\tilde{u}\|_{L^2}^2 + \frac{1}{4}\mu^3 \|\partial_x \tilde{u}\|_{L^2}^2 + \frac{1}{2} v\mu^2 \operatorname{Im} \int \bar{\tilde{u}} \partial_x \tilde{u} \, dx \\ & - \frac{1}{4}\mu^3 \|\tilde{u}\|_{L^4}^4 + \frac{1}{2}\mu \int V\Big(\frac{x}{\mu} + a\Big) |\tilde{u}(x)|^2 \, dx \end{aligned}$$

Therefore,

$$\Xi_{g^* H}\tilde{u} = \frac{1}{i}\Big(\frac{1}{2} v^2 \mu \tilde{u} - \frac{1}{2}\mu^3 \partial_x^2 \tilde{u} - \mu^3 |\tilde{u}|^2 \tilde{u} + \mu V\Big(\frac{x}{\mu} + a\Big)\tilde{u}\Big) - v\mu^2 \partial_x \tilde{u}$$

Substituting and expanding the cubic term,

$$\begin{aligned} \partial_t w = {} & -Y(\eta + w) - \frac{1}{2} i v^2 (\eta + w) + \frac{1}{2} i\mu^2 \partial_x^2 (\eta + w) + i\mu^2 [\eta^3 + 2\eta^2 w + \eta^2 \bar{w} \\ & + 2|w|^2 \eta + \eta w^2 + |w|^2 w] - iV\Big(\frac{x}{\mu} + a\Big)(\eta + w) - v\mu \partial_x (\eta + w) \end{aligned}$$

Using that $-\frac{1}{2}\eta + \frac{1}{2}\eta'' + \eta^3 = 0$, we obtain

$$\begin{aligned}\partial_t w = \; & -Y\eta - \frac{1}{2}iv^2\eta + \frac{1}{2}i\mu^2\eta - iV\Big(\frac{x}{\mu}+a\Big)\eta - v\mu\partial_x\eta \\ & - Yw - \frac{1}{2}iv^2 w + \frac{1}{2}i\mu^2 w - i\mu^2\mathcal{L}w + i\mu^2\mathcal{N}w - iV\Big(\frac{x}{\mu}+a\Big)w - v\mu\partial_x w\end{aligned}$$

Now set

$$X = -Y + v\mu e_1 - \beta e_2 + \Big[-\frac{1}{2}v^2 + \frac{1}{2}\mu^2 - \alpha\Big]e_3$$

From this, we get the claimed equation. □

Let us make some remarks about the lemma. First, note that if $X = 0$, then

$$\dot{v} = -\beta\mu\,, \quad \dot{\gamma} = \dot{a}v - \frac{1}{2}v^2 + \frac{1}{2}\mu^2 - \alpha\,, \quad \dot{a} = v\,, \quad b\dot{\mu} = 0\,,$$

which are exactly the equations of motion of the effective Hamiltonian – see §2.4.

Second, note that the term

$$i\Big[-V\Big(\frac{x}{\mu}+a\Big) + \alpha + \beta x\Big]\eta$$

projects symplectically to 0 (used in the next lemma), in the sense that

$$P\left(i(-V(\bullet/\mu + a) + \alpha + \beta\bullet)\eta\right) = 0\,, \tag{3.7}$$

where $P : \mathcal{S}'(\mathbb{R},\mathbb{C}) \to \mathfrak{g}$ is defined by the condition that

$$\omega(u - P(u)\eta, Y\eta) = 0\,, \qquad \forall\, Y \in \mathfrak{g}$$

see [10, (3.7)] for an explicit expression of P. To see (3.7) we use the following simple

Lemma 3.3. *Let P be the symplectic projection defined above and $f \in \mathcal{S}'(\mathbb{R};\mathbb{R})$ (that is f is real valued tempered distribution). Then,*

$$\begin{gathered} P(if(x)\eta) = \alpha e_3\cdot\eta + \beta e_2\cdot\eta = i\alpha\eta + i\beta x\eta\,, \\ \alpha = \frac{1}{2}\int f(x)\eta^2(x) - \frac{1}{2}\int f'(x)x\eta^2(x)dx\,, \quad \beta = \frac{1}{2}\int f'(x)\eta^2(x)dx\,. \end{gathered} \tag{3.8}$$

Proof. This follows from a straightforward calculation based on (2.13). If $f(x) = V(x/\mu + a)$, we obtain (3.7) with α and β given by (3.8). □

Finally, note the following Taylor expansions:

$$\begin{aligned} V\Big(\frac{x}{\mu}+a\Big) &= V(a)+V'(a)\frac{x}{\mu}+V''(a)\frac{x^2}{2\mu^2}+\mathcal{O}(h^3)\,, \\ \alpha &= V(a)-\frac{V''(a)\pi^2}{24\mu^2}+\mathcal{O}(h^3)\,, \\ \beta &= \frac{V'(a)}{\mu}+\mathcal{O}(h^3)\,, \end{aligned}$$

and thus

$$-V\Big(\frac{x}{\mu}+a\Big)+\alpha+\beta x = -\frac{V''(a)\pi^2}{24\mu^2}-\frac{V''(a)}{2\mu^2}x^2+\mathcal{O}(h^3)=\mathcal{O}(h^2)\,, \tag{3.9}$$

where all the errors are polynomially bounded in x. If we assumed that $W \in \mathcal{C}^4$ then he expansions for α would be valid with an error $\mathcal{O}(h^4)$.

Lemma 3.4. *Let w be given by* (3.3)*, with g obtained from Lemma 3.1, and X by* (3.5)*. Suppose $\frac{1}{2} \leq \mu \leq 1$. Then*

$$|X| \leq c(h^2\|w\|_{H^1}+\|w\|_{H^1}^2+\|w\|_{H^1}^3)$$

Proof. We use the symplectic orthogonality of $Y\eta$, $Y \in \mathfrak{g}$ and w.

Since $Pw_t = \partial_t P w = 0$, Lemma 3.2 gives

$$\begin{aligned} X = {} & P\left(i(V(\bullet/\mu+a)-\alpha-\beta\bullet)\eta\right)+P(i(V(\bullet/\mu+a)-\alpha-\beta\bullet)w)-P(Xw) \\ & +\ \mu^2P(i\mathcal{N}w)+\mu^2P(i\mathcal{L}w)\,. \end{aligned}$$

We recall from [10, Lemma 3.3] the following straightforward estimates:

$$\|P(Yw)\| \leq C|Y|\|w\|_{L^2}\,, \quad \|P(i\mathcal{N}u)\| \leq C\|w\|_{L^2}^2\left(1+\|w\|_{H^1}^{\frac12}\|w\|_{L^2}^{\frac12}\right)\,. \tag{3.10}$$

We already observed in (3.7) that the first term on the right hand side vanishes[2]. From (3.9) we see that the second term is $\mathcal{O}(h^2)\|w\|_{L^2}$. The third and fourth term are estimated using (3.10) by

$$C(|X|\|w\|_{L^2}+\|w\|_{H^1}^2+\|w\|_{H^1}^3)\,.$$

The last term vanishes: the linear operator $\mathcal{L}$ is the Hessian of $\mathcal{E}$, given in (1.8), at the critical point η. The fact that $\Xi_{\mathcal{E}}$ is tangent to M and Lemma 2.2 (or a direct computation) show that

$$P(i\mathcal{L}w)=0\,,$$

Summarizing,

$$|X| \leq C\|w\|_{L^2}|X|+C(h^2\|w\|_{H^1}+\|w\|_{H^1}^2+\|w\|_{H^1}^3)\,.$$

The smallness of $\|w\|_{L^2}$ concludes the proof. $\square$

[2]This is the significant bonus of using the effective Hamiltonian.

We conclude this section which two lemmas which effectively eliminate μ from the coefficients of X.

Lemma 3.5. *Suppose that* $w \in H^1(\mathbb{R}, \mathbb{C})$ *and that* $\omega(w, X \cdot \eta) = 0$, *for every* $X \in \mathfrak{g}$. *Then*

$$\|w\|_{L^2}^2 = 2(1-\mu)/\mu \,. \tag{3.11}$$

Proof. We first compute

$$\|\eta + w\|_{L^2}^2 = \|g^{-1}u\|_{L^2}^2 = \|u\|_{L^2}^2/\mu(g) = 2/\mu(g) \,,$$

where we used the conservation of the L^2 norm. By the symplectic orthogonality assumption statement of the lemma $\mathrm{Re}\langle w, \eta \rangle = 0$ and hence

$$\|\eta + w\|_{L^2}^2 = 2 + \|w\|_{L^2}^2 \,,$$

from which the conclusion follows. □

From this we immediately deduce the following

Lemma 3.6. *Suppose* $1 - \mu \ll 1$ *and* $0 < h \leq 1$. *Let* $X_0 = X|_{\mu=1}$ *where* X *is given by* (3.5)*:*

$$X_0 \stackrel{\text{def}}{=} (-\dot{a} + v)e_1 + \Big(- \dot{v} - \beta \Big) e_2 + \Big(- \dot{\gamma} + \dot{a} v - \frac{1}{2} v^2 + \frac{1}{2} - \alpha \Big) e_3 - \dot{\mu} e_4 \,, \tag{3.12}$$

where α *and* β *are given by* (3.4) *(and depend on* μ*). The the conclusions of Lemma 3.4 hold for* X_0*:*

$$|X_0| \leq c(h^2 \|w\|_{H^1} + \|w\|_{H^1}^2 + \|w\|_{H^1}^3) \,.$$

4. Spectral estimates

In this section we will recall the now standard estimates on the operator $\mathcal{L}$ which arises as Hessian of $\mathcal{E}$ at η:

$$\mathcal{L} w = -\frac{1}{2} \partial_x^2 w - 2\eta^2 w - \eta^2 \bar{w} + \frac{1}{2} w \,,$$

or

$$\mathcal{L} w = \begin{bmatrix} L_+ & 0 \\ 0 & L_- \end{bmatrix} \begin{bmatrix} \mathrm{Re}\, w \\ \mathrm{Im}\, w \end{bmatrix} \,, \quad L_\pm = -\frac{1}{2} \partial_x^2 - (2 \pm 1)\eta^2 + \frac{1}{2} \,.$$

In our special case we can be more precise than in the general case (see [15], and also [5, Appendix D]). The self-adjoint operators $L_\pm$ belong to the class of Schrödinger operators with *Pöschl-Teller* potentials and their spectra can be explicitly computed:

$$\sigma(L_-) = \{0\} \cup [1/2, \infty) \,, \quad \sigma(L_+) = \{0, -3/2\} \cup [1/2, \infty) \,.$$

The eigenfuctions can computed by the same method but a straightforward verification is sufficient to see that

$$L_- \eta = 0 \,, \quad L_+(\partial_x \eta) = 0 \,, \quad L_+(\eta^2) = -\frac{3}{2} \eta^2 \,.$$

From §[10, §4] we recall the following

Proposition 4.1. *Let $w \in H^1(\mathbb{R}, \mathbb{C})$ and suppose that for any $X \in \mathfrak{g}$, $\omega(w, X \cdot \eta) = 0$. Then,*

$$\langle \mathcal{L}w, w \rangle \geq \frac{2\rho_0}{7 + 2\rho_0} \|w\|^2_{H^1} \simeq 0.0555 \|w\|^2_{H^1}, \quad \rho_0 = \frac{9}{2(12 + \pi^2)}. \tag{4.1}$$

The next proposition will be useful in solving a linear equation for an approximation of w.

Proposition 4.2. *The equation*

$$L_+ f = \left(\frac{\pi^2}{12} + x^2 \right) \eta(x) \tag{4.2}$$

has a unique solution in $L^2(\mathbb{R})$. In addition,

$$e^{(1-\epsilon)|x|} f^{(k)}(x) \in L^\infty(\mathbb{R}), \quad \forall\, \epsilon > 0, \ k \in \mathbb{N},$$

and

$$\omega(f, X\eta) = 0, \quad \forall\, X \in \mathfrak{g}. \tag{4.3}$$

Proof. Since 0 is an isolated point of the spectrum of L_+ and L_+ is self-adjoint, L_+ has a bounded inverse on

$$(\ker L_+)^\perp = (\operatorname{span}\{\partial_x \eta\})^\perp.$$

Hence,

$$(\ker L_+)^\perp \ni \left(\frac{\pi^2}{12} + x^2 \right) \eta \overset{L_+^{-1}}{\longmapsto} f \in (\ker L_+)^\perp.$$

By elliptic regularity, f is smooth. Moreover, since L_+ commutes with $g(x) \mapsto g(-x)$ and $(\frac{\pi^2}{12} + x^2)\eta(x)$ is even, we conclude that f is even.

Next we argue that f has exponential decay. For that we conjugate the equation by $e^{\sigma x}$:

$$e^{\sigma x} L_+ e^{-\sigma x} = L_+ + \sigma \partial_x - \sigma^2/2,$$

and apply both sides to $e^{\sigma x} f(x)$:

$$e^{\sigma x} \Big(\frac{\pi^2}{12} + x^2 \Big) \eta = \Big(-\partial_x^2 + 2\sigma\partial_x - \frac{\sigma^2}{2} + \frac{1}{2} \Big) e^{\sigma x} f + 3\eta^2 e^{\sigma x} f.$$

Taking the Fourier transform this means that we have

$$\left(\frac{1}{2}\xi^2 + \frac{1}{2} - \frac{1}{2}\sigma^2 + i\sigma\xi \right) \widehat{e^{\sigma x} f}(\xi) = \Big[3e^{\sigma x}\eta^2 f + e^{\sigma x} \Big(\frac{\pi^2}{12} + x^2 \Big) \eta \Big]^{\widehat{}}(\xi),$$

which makes sense if $|\sigma| < 1$ as then the right hand side is in L^2. Since the multiplier on the left-hand side is bounded away from 0, we can invert the expression to obtain that $e^{\sigma x} f \in H^2$. From this, we deduce by Sobolev embedding that $e^{\sigma x} f \in L^\infty$. By applying derivatives to (4.2) and then repeating this argument, we in fact obtain that all derivatives are pointwise exponentially localized in space.

Finally, we prove the orthogonality condition (4.3), that is, $\omega(f, e_j \cdot \eta) = 0$ where e_j's are given by (2.10). Since f is real, we clearly have

$$\omega(f, \eta') = \omega(f, (x\eta)') = 0\,.$$

Since f is even, we also have $\omega(f, ix\eta) = 0$. It remains to show that $\omega(f, i\eta) = 0$, that is that $\int f\eta = 0$. Note that $L_+[(x\eta)'] = \eta$ (by direct computation or from the structure of the generalized kernel of $i\mathcal{L}$). Hence

$$\int f\eta = \int f L_+[(x\eta)']\, dx = \int L_+ f \; (x\eta)'\, dx = \int \Big(\frac{\pi^2}{12} + x^2\Big)\eta(\eta + x\eta')\, dx\,.$$

Integration by parts and $\int \eta^2 = 2$, $\int x^2\eta^2 = \pi^2/6$, show that this is 0. □

5. Proof of the main estimate

In the arguments that follow, we will assume that $g(u)$ satisfying (3.1) is always defined on any time interval under consideration, and thus w given by (3.3) is also always defined. This is, however, not known *a priori* and Lemma 3.1 must be considered as part of the bootstrap argument, together with the lemmas that follow. However, since this aspect of the argument is standard in papers on this subject, we will not make mention of it.

Recall from Lemma 3.2 that the equation for w is

$$\begin{aligned}\partial_t w = {} & X\eta + i\Big[-V\Big(\frac{x}{\mu} + a\Big) + \alpha + \beta x\Big]\eta \\ & + Xw + i\Big[-V\Big(\frac{x}{\mu} + a\Big) + \alpha + \beta x\Big]w - i\mu^2\mathcal{L}w + i\mu^2\mathcal{N}w\,,\end{aligned} \tag{5.1}$$

where α and β are time dependent parameters given by (3.4). Note that by Taylor's theorem the forcing term in this equation has second-order expansion

$$-i\Big[V\Big(\frac{x}{\mu} + a\Big) + \alpha + \beta x\Big]\eta = -i\frac{V''(a)}{2\mu^2}\Big(\frac{\pi^2}{12} + x^2\Big)\eta + h^3\langle x\rangle^3\eta f_2$$

where, provided[3] $1/2 \le \mu \le 1$, we have $|f_2(x,t)| \le c$.

We first consider the forced linear evolution obtained from (5.1) by discarding all terms we expect will be of order h^3 or higher:

$$\partial_t \tilde{w} = -i\mu^2\mathcal{L}\tilde{w} - i\frac{V''(a)}{2\mu^2}\Big(\frac{\pi^2}{12} + x^2\Big)\eta \tag{5.2}$$

We now introduce a natural approximate solution to this forced linear equation. Let

$$\tilde{w} = -\frac{V''(a)}{2\mu^4} f\,, \quad f \overset{\text{def}}{=} L_+^{-1}\left(\frac{\pi^2}{12} + x^2\right)\eta\,, \tag{5.3}$$

[3]This will follow easily from the bootstrap assumption.

where f is given in Proposition 4.2. Then $\tilde{w}$ satisfies (5.2) to second-order, i.e.

$$\partial_t \tilde{w} = -i\mu^2 \mathcal{L}\tilde{w} - i\frac{V''(a)}{2\mu^2}\Big(\frac{\pi^2}{12} + x^2\Big)\eta + h^3\theta f \tag{5.4}$$

where

$$\theta(t) \stackrel{\text{def}}{=} \frac{1}{h^3}\Big[-\frac{V'''(a)\dot{a}}{2\mu^4} + \frac{2V''(a)\dot{\mu}}{\mu^5}\Big]$$

Lemma 5.1. *Suppose there is a constant c_1 such that*

$$\|w\|_{L^\infty_{[t_1,t_2]}H^1_x} \le c_1 h^{3/2} \tag{5.5}$$

Then provided $|t_2 - t_1| \le h^{-2}$, we have

$$\sup_{t_1\le t\le t_2} |\theta(t)| \le c\,,$$

where c is a constant depending only upon c_1, $\|W^{(k)}\|_{L^\infty}$ for $k = 0,1,2,3$ and $|v(t_1)|$.

Proof. By (5.5), Lemma 3.6 and Taylor expansions (see (3.9)), we have

$$|\dot{v} + V'(a)| + |\dot{\mu}| + |-\dot{a} + v| \le ch^3 \tag{5.6}$$

By integrating the first bound, we obtain the following rough estimate

$$\sup_{t_1\le t\le t_2} |v(t)| \le |v(t_1)| + ch\|W'\|_{L^\infty}(t_2 - t_1) + ch^3(t_2 - t_1)\,. \tag{5.7}$$

We have

$$|\dot{v}v + V'(a)\dot{a}| \le |\dot{v} + V'(a)||v| + |V'(a)||\dot{a} - v|$$

and thus, by (5.6) and (5.7),

$$\sup_{t_1\le t\le t_2} |\dot{v}v + V'(a)\dot{a}| \le ch^3|v(t_1)| + ch^4\|W'\|_{L^\infty}\langle t_2 - t_1\rangle + ch^6(t_2 - t_1)$$

Integrating this bound, we obtain a near conservation of classical energy,

$$\begin{aligned}&\Big|\Big(\frac{v^2}{2} + V(a)\Big) - \Big(\frac{v(t_1)^2}{2} + V(a(t_1))\Big)\Big| \\ &\quad \le ch^3|v(t_1)|(t_2 - t_1) + ch^4\|W'\|_{L^\infty}\langle t_2 - t_1\rangle(t_2 - t_1) + ch^6(t_2 - t_1)^2\end{aligned}$$

from which we obtain $\sup_{t_1\le t\le t_2} |v(t)| \le c$. This and (5.6) imply that

$$\sup_{t_1\le t\le t_2} |\dot{a}(t)| \le c\,,$$

and thus $|\theta(t)| \le c$. □

This approximate solution $\tilde{w}$ provides heuristic evidence that w should be of order h^2, but it will also play a key rôle in our rigorous argument establishing this fact.

Lemma 5.2 (Lyapunov energy estimate)**.** *Suppose that, for some constant* c_1*,*

$$\|w\|_{L^\infty_{[t_1,t_2]}H^1_x} \leq c_1 h^{3/2} \tag{5.8}$$

Then, provided

$$|t_2 - t_1| \leq \frac{c}{h}\,, \tag{5.9}$$

we have,

$$\|w\|_{L^\infty_{[t_1,t_2]}H^1} \leq 4\sqrt{c_2}\|w(t_1)\|_{H^1} + ch^2 \tag{5.10}$$

The constants c *in* (5.9) *and* (5.10) *depend upon* c_1*,* $\|W^{(k)}\|_{L^\infty}$ *for* $k = 0, 1, 2, 3$*, and* $|v(t_1)|$*, although are independent of* $\|w(t_1)\|_{H^1}$*. The constant* $c_2 \overset{\text{def}}{=} (7 + 2\rho_0)/(2\rho_0) \approx 18.02$*, with* ρ_0 *the absolute constant appearing in* (4.1)*.*

We will postpone the proof of Lemma 5.2 to the end of the section. In the next theorem we iterate the above bound, and close the bootstrap argument.

Theorem 2. *Let* $0 \leq \delta < \frac{1}{4}$*. Let* $w_0 = w(0)$*, and suppose that there is a constant* c_1 *such that*

$$\|w_0\|_{H^1} \leq c_1 h^{\frac{3}{2}+3\delta}\,.$$

Then, provided

$$|t| \leq \frac{c(1 + \delta|\log h|)}{h} \quad \text{and} \quad 0 < h \leq \epsilon\,, \tag{5.11}$$

we have

$$\|w\|_{L^\infty_{[0,t]}H^1_x} \leq 4\sqrt{c_2}h^{-2\delta}\|w_0\|_{H^1} + ch^{2(1-\delta)} \tag{5.12}$$

The constant c *in* (5.11) *and* (5.12) *and the constant* ϵ *in* (5.11) *depend upon* c_1*,* $\|W^{(k)}\|_{L^\infty}$ *for* $k = 0, 1, 2, 3$*, and* $|v(0)|$*, although are independent of* δ *and* $\|w_0\|_{H^1}$*. The constant* $c_2 \overset{\text{def}}{=} (7 + 2\rho_0)/(2\rho_0) \approx 18.02$*, with* ρ_0 *the absolute constant appearing in* (4.1)*.*

Proof. We apply Lemma 5.2 k times on successive intervals each of size c/h (where c is as given in Lemma 5.2) to obtain the bound

$$\|w\|_{L^\infty_{[0,ck/h]}H^1_x} \leq (4\sqrt{c_2})^k\|w_0\|_{H^1} + \left(\sum_{j=0}^{k-1}(4\sqrt{c_2})^j\right)ch^2$$

This is only valid provided that the hypothesis of Lemma 5.2 is satisfied over the whole collection of time intervals:

$$(4\sqrt{c_2})^k\|w_0\|_{H^1} + \left(\sum_{j=0}^{k-1}(4\sqrt{c_2})^j\right)ch^2 \leq c_1 h^{3/2}\,,$$

By taking

$$k = 1 + \frac{2\delta|\log h|}{\log(4\sqrt{c_2})}$$

we obtain that

$$(4\sqrt{c_2})^k\|w_0\|_{H^1} + \left(\sum_{j=0}^{k-1}(4\sqrt{c_2})^j\right) ch^2 \leq (4\sqrt{c_2})h^{-2\delta}\|w_0\|_{H^1} + ch^{2(1-\delta)}$$

Thus, it suffices to ensure that

$$(4\sqrt{c_2})h^{-2\delta}\|w_0\|_{H^1} + ch^{2(1-\delta)} \leq c_1 h^{3/2}\,,$$

and this is accomplished provided $h \leq \epsilon$, for a suitable ϵ with the dependence as stated in the proposition. We note that the constant c from Lemma 5.2 did not change from one iteration to the next, since $|v(t_1)|$ remains uniformly bounded by Lemma 5.1, which applies on a time interval of size h^{-2}. $\square$

We now conclude the section with the proof of Lemma 5.2.

Proof of Lemma 5.2. For the proof, we shall assume, in place of (5.8), the bound

$$\|w\|_{L^\infty_{[t_1,t_2]}H^1} \leq c_1 h^{2(1-\delta')}$$

We will conclude at the end that it suffices to take $\delta' = \frac{1}{4}$. Let

$$w_1 \stackrel{\text{def}}{=} w - \tilde{w}\,,$$

where $\tilde{w}$ is an approximate solution to (5.2) given by (5.3).

From (5.1) and (5.4), we derive the equation satisfied by w_1:

$$\begin{aligned}\partial_t w_1 = &\; - i\mu^2\mathcal{L}w_1 + X\eta + i\Big[-V\Big(\frac{x}{\mu}+a\Big) + \alpha + \beta x + \frac{V''(a)}{2\mu^2}\Big(\frac{\pi^2}{12}+x^2\Big)\Big]\eta - h^3\theta f \\ &+ Xw + i\Big[-V\Big(\frac{x}{\mu}+a\Big) + \alpha + \beta x\Big]w + i\mu^2\mathcal{N}w\,.\end{aligned}$$

By grouping forcing terms of size $\mathcal{O}(h^3)$, we rewrite the above as

$$\partial_t w_1 = -i\mu^2\mathcal{L}w_1 + h^3 f_1 + Xw + i\Big[-V\Big(\frac{x}{\mu}+a\Big) + \alpha + \beta x\Big]w + i\mu^2\mathcal{N}w$$

where $\|f_1(x,t)\|_{H^1_x} \leq c$ (uniformly in t). Note that here we have applied Lemma 5.1 to conclude that $|\theta(t)| \leq c$.

We recall that $\mathcal{L}$ is self-adjoint with respect to

$$\langle u, v\rangle = \operatorname{Re}\int u\bar{v}\,,$$

and hence, writing $Xw = Xw_1 + X\tilde{w}$, we compute

$$\begin{aligned}\frac{1}{2}\partial_t\langle \mathcal{L}w_1, w_1\rangle &= \langle \mathcal{L}w_1, \partial_t w_1\rangle \\ &= -\mu^2\langle \mathcal{L}w_1, i\mathcal{L}w_1\rangle + \langle \mathcal{L}w_1, h^3 f_1\rangle + \langle \mathcal{L}w_1, Xw_1\rangle + \langle \mathcal{L}w_1, X\tilde{w}\rangle \\ &\quad + \langle \mathcal{L}w_1, i\Big[-V\Big(\frac{x}{\mu}+a\Big)+\alpha+\beta x\Big]w_1\rangle \\ &\quad + \langle \mathcal{L}w_1, i\Big[-V\Big(\frac{x}{\mu}+a\Big)+\alpha+\beta x\Big]\tilde{w}\rangle + \mu^2\langle \mathcal{L}w_1, i\mathcal{N}w\rangle \\ &= \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}+\mathrm{VI}+\mathrm{VII}\end{aligned}$$

Now we analyse these terms one-by-one. First,

$$\mathrm{I} = 0. \tag{5.13}$$

For II, we use integration by parts for the ∂_x^2 term to move one ∂_x onto f_1 and then apply the Cauchy-Schwarz inequality; for the other terms we use a direct application of the Cauchy-Schwarz inequality, and together these give

$$|\mathrm{II}| \le 4h^3\|w_1\|_{H^1}\|f_1\|_{H^1_x} \le ch^3\|w_1\|_{H^1} \tag{5.14}$$

The next term, III, requires more care:

$$\begin{aligned}\mathrm{III} &= \langle \mathcal{L}w_1, Xw_1\rangle \\ &= \frac{1}{2}\langle (w_1 - \partial_x^2 w_1 - 4\eta^2 w_1 - 2\eta^2\bar{w}_1), (-a_1\partial_x w_1 + a_2 ixw_1 + a_3 iw_1 + a_4\partial_x(xw_1))\rangle\end{aligned}$$

where a_j, the components of X, are time dependent but space independent. By the bootstrap assumption (5.8) and Lemma 3.4, $|a_j| \le ch^{4-4\delta'}$. To proceed, we do some further calculations:

$$\langle w_1, Xw_1\rangle = a_4\langle w_1, (w_1 + x\partial_x w_1)\rangle = \frac{1}{2}a_4\|w_1\|^2_{L^2}$$

$$\begin{aligned}\langle \partial_x^2 w_1, Xw_1\rangle &= a_2\langle \partial_x^2 w_1, ixw_1\rangle + a_4\langle \partial_x^2 w_1, w_1 + x\partial_x w_1\rangle \\ &= a_2\langle \partial_x w_1, iw_1\rangle - \frac{3}{2}a_4\|\partial_x w_1\|^2_{L^2}\end{aligned}$$

and thus the above two terms are bounded by $ch^{4-4\delta'}\|w_1\|^2_{H^1}$. For the terms involving η, we use the Cauchy-Schwarz inequality and the fact that $\eta \in \mathcal{S}$:

$$|\langle \eta^2 w_1, Xw_1\rangle| + |\langle \eta^2 w_1, X\bar{w}_1\rangle| \le (\max|a_j|)\|w_1\|^2_{H^1}$$

Altogether then, we have

$$|\mathrm{III}| \le ch^{4-4\delta'}\|w_1\|^2_{H^1} \tag{5.15}$$

Now we move on to IV:

$$\begin{aligned}\mathrm{IV} &= \langle \mathcal{L}w_1, X\tilde{w}\rangle \\ &= \frac{1}{2}\langle (w_1 - \partial_x^2 w_1 - 4\eta^2 w_1 - 2\eta^2 \bar{w}_1), (-a_1\partial_x\tilde{w} + a_2 ix\tilde{w} + a_3 i\tilde{w} + a_4\partial_x(x\tilde{w}))\rangle\,.\end{aligned}$$

For this term, we are forced to directly estimate by the Cauchy-Schwarz inequality (although for the $\partial_x^2 w_1$ term, we integrate by parts one ∂_x factor). The bound that we get is

$$|\mathrm{IV}| \leq ch^{4-4\delta'}\|w_1\|_{H^1}(\|\langle x\rangle\tilde{w}\|_{L^2} + \|\langle x\rangle\partial_x\tilde{w}\|_{L^2} + \|\langle x\rangle\partial_x^2\tilde{w}\|_{L^2})$$

From the definition (5.3) of $\tilde{w}$ and Proposition 4.2, we obtain that all the norms involving $\tilde{w}$ are bounded by h^2. Thus,

$$|\mathrm{IV}| \leq ch^{6-4\delta'}\|w_1\|_{H^1}\,. \tag{5.16}$$

Next, we move on to V:

$$\begin{aligned}\mathrm{V} &= \langle \mathcal{L}w_1, i\Big[-V\Big(\frac{x}{\mu}+a\Big)+\alpha+\beta x\Big]w_1\rangle \\ &= \frac{1}{2}\langle (w_1 - \partial_x^2 w_1 - 4\eta^2 w_1 - 2\eta^2\bar{w}_1), i\Big[-V\Big(\frac{x}{\mu}+a\Big)+\alpha+\beta x\Big]w_1\rangle \\ &= \frac{1}{2}\langle (-\partial_x^2 w_1 - \eta^2\bar{w}_1), i\Big[-V\Big(\frac{x}{\mu}+a\Big)+\alpha+\beta x\Big]w_1\rangle\,.\end{aligned}$$

To estimate the first term, we integrate by parts and use that

$$\Big| -\frac{1}{\mu}V'\Big(\frac{x}{\mu}+a\Big)+\beta\Big| \leq ch$$

(note that other estimates are available, like cxh^2, but we do not want an x coefficient here). For the second term, we use (3.9), the Cauchy-Schwarz inequality and the rapid decay of η^2:

$$\Big|\Big[-V\Big(\frac{x}{\mu}+a\Big)+\alpha+\beta x\Big]\eta^2\Big| \leq ch^2$$

This gives the bound[4]

$$|\mathrm{V}| \leq ch\|w_1\|_{H^1}^2\,. \tag{5.17}$$

Now we move on to the next term, VI.

$$\mathrm{VI} = \langle \mathcal{L}w_1, i\Big[-V\Big(\frac{x}{\mu}+a\Big)+\alpha+\beta x\Big]\tilde{w}\rangle$$

[4]It is unlikely that we can do better than h as a coefficient here, and thus this seems to be what limits us ultimately to time $1/h$.

In the ∂_x^2 term of $\mathcal{L}$, we integrate by parts one ∂_x, and then estimate by the Cauchy-Schwarz inequality. All other terms, are estimated by a direct application of the Cauchy-Schwarz inequality. The bound obtained is

$$\begin{aligned} |\mathrm{VI}| &\le \|w_1\|_{H^1} \left\| \langle \partial_x \rangle \Big[-V\Big(\frac{x}{\mu}+a\Big) + \alpha + \beta x \Big] \tilde{w} \right\|_{L^2} \\ &\le ch^2 \|w_1\|_{H^1} (\|\langle x\rangle^2 \tilde{w}\|_{L^2} + \|\langle x\rangle^2 \partial_x \tilde{w}\|_{L^2}) \\ &\le ch^4 \|w_1\|_{H^1} \end{aligned} \tag{5.18}$$

by the localization of $\tilde{w}$. For the last term, VII, we use integration by parts once for the ∂_x^2 term, and then apply the Cauchy-Schwarz inequality to all terms. Since we are in one-dimension, we have the embedding $\|w\|_{L^\infty} \le c\|w\|_{H^1}$.

$$\mathrm{VII} = -\mu^2 \langle \mathcal{L} w_1, i\mathcal{N} w\rangle$$

$$\Longrightarrow |\mathrm{VII}| \le \|w_1\|_{H^1} (\|w\|_{H^1}^2 + \|w\|_{H^1}^3) \le ch^{4-4\delta'} \|w_1\|_{H^1} \tag{5.19}$$

by the bootstrap assumption.

This completes the step-by-step estimation process, and the bound we get from (5.13), (5.14), (5.15), (5.16), (5.17), (5.18), (5.19) is

$$|\partial_t \langle \mathcal{L} w_1, w_1\rangle| \le c(h^{4-4\delta'} + h^3)\|w_1\|_{H^1} + c(h^{4-4\delta'} + h)\|w_1\|_{H^1}^2$$

We see that it suffices to take $\delta' = \frac{1}{4}$. Integrating in time, we get

$$\begin{aligned} \langle \mathcal{L} w_1(t), w_1(t)\rangle \le\ & \langle \mathcal{L} w_1(t_1), w_1(t_1)\rangle + c(t-t_1)h^3 \|w_1\|_{L^\infty_{[t_1,t_2]} H^1} \\ & + c(t-t_1)h\|w_1\|^2_{L^\infty_{[t_1,t_2]} H^1} \end{aligned} \tag{5.20}$$

By (3.1) and (4.3), $w_1(t)$ satisfies the hypothesis of Proposition 4.1, and we have

$$\frac{1}{c_2}\|w_1(t)\|_{H^1}^2 \le \langle \mathcal{L} w_1(t), w_1(t)\rangle \tag{5.21}$$

By direct estimation, we have the upper bound

$$|\langle \mathcal{L} w_1(t_1), w_1(t_1)\rangle| \le 4\|w_1(t_1)\|_{H^1}^2$$

Combining this with (5.20) we get the bound

$$\begin{aligned} \|w_1(t)\|_{H^1}^2 \le\ & 4c_2 \|w_1(t_1)\|_{H^1}^2 + c(t-t_1)h^3\|w_1\|_{L^\infty_{[t_1,t_2]} H^1} \\ & + c(t-t_1)h\|w_1\|^2_{L^\infty_{[t_1,t_2]} H^1} \end{aligned}$$

From this, we infer from the monotonicity of the right side that

$$\begin{aligned} \|w_1\|^2_{L^\infty_{[t_1,t_2]} H^1} \le\ & 4c_2 \|w_1(t_1)\|_{H^1}^2 + c(t_2-t_1)h^3\|w_1\|_{L^\infty_{[t_1,t_2]} H^1} \\ & + c(t_2-t_1)h\|w_1\|^2_{L^\infty_{[t_1,t_2]} H^1} \end{aligned}$$

Requiring that $t_2 - t_1 \leq c/h$ implies

$$\begin{aligned}\|w_1\|^2_{L^\infty_{[t_1,t_2]}H^1} &\leq 8c_2\|w_1(t_1)\|^2_{H^1} + ch^2\|w_1\|_{L^\infty_{[t_1,t_2]}H^1} \\ &\leq 16c_2\|w_1(t_1)\|^2_{H^1} + ch^4 .\end{aligned}$$

Since $w = w_1 + \tilde{w}$ and $\|\tilde{w}\|_{H^1} \leq ch^2$,

$$\|w\|_{L^\infty_{[t_1,t_2]}H^1} \leq 4\sqrt{c_2}\|w(t_1)\|_{H^1} + ch^2 , \tag{5.22}$$

which is the claimed estimate. □

6. ODE analysis

To pass from an approximate equations for the parameters of the soliton, (a, v, γ, μ) given in Lemma 3.6, to the ODEs (1.3) we need some elementary estimates which we present in this section. They are similar to those in [10, §7].

Lemma 6.1. *Suppose that $0 < h \ll 1$, and $a = a(t)$, $v = v(t)$, $\epsilon_1 = \epsilon_1(t)$, $\epsilon_2 = \epsilon_2(t)$ are C^1 real-valued functions. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a C^2 mapping such that $|f|$ and $|f'|$ are uniformly bounded. Suppose that on $[0, T]$,*

$$\begin{cases} \dot{a} = v + \epsilon_1 & a(0) = a_0 \\ \dot{v} = hf(ha) + \epsilon_2 \,, & v(0) = v_0 \end{cases} \tag{6.1}$$

Let $\bar{a} = \bar{a}(t)$ and $\bar{v} = \bar{v}(t)$ be the C^1 real-valued functions satisfying the exact equations

$$\begin{cases} \dot{\bar{a}} = \bar{v} & \bar{a}(0) = a_0 \\ \dot{\bar{v}} = hf(h\bar{a}) \,, & \bar{v}(0) = v_0 \end{cases}$$

with the same initial data. Suppose that on $[0, T]$, we have $|\epsilon_j| \leq h^{4-\delta}$ for $j = 1, 2$. Then provided $T \leq \delta h^{-1} \log(1/h)$, we have on $[0, T]$ the estimates

$$|a - \bar{a}| \leq h^{2-2\delta} \log(1/h), \qquad |v - \bar{v}| \leq h^{3-2\delta} \log(1/h)$$

Before proceeding to the proof, we recall some basic tools.

Gronwall estimate. Suppose $b = b(t)$ and $w = w(t)$ are C^1 real-valued functions, h is a constant, and (b, w) satisfy the differential inequality:

$$\begin{cases} |\dot{b}| \leq |w| & b(0) = b_0 \\ |\dot{w}| \leq h^2|b| \,, & w(0) = w_0 \end{cases} \tag{6.2}$$

Let $x(t) = hb(t/h)$, $y(t) = w(t/h)$. Then

$$\begin{cases} |\dot{x}| \leq |y| & x(0) = x_0 = hb_0 \\ |\dot{y}| \leq |x| \,, & y(0) = y_0 = w_0 \end{cases}$$

Let $z(t) = x^2 + y^2$. Then $|\dot{z}| = |2x\dot{x} + 2y\dot{y}| \le 2|x||y| + 2|x||y| \le 2(x^2 + y^2) = 2z$, and hence $z(t) \le z(0)e^{2t}$. Thus

$$|x(t)| \le \sqrt{2}\max(|x_0|, |y_0|)\exp(t)$$
$$|y(t)| \le \sqrt{2}\max(|x_0|, |y_0|)\exp(t)$$

Converting from (x, y) back to (b, w), we obtain the Gronwall estimate

$$\begin{aligned} |b(t)| &\le \sqrt{2}\max(h|b_0|, |w_0|)\frac{\exp(ht)}{h} \\ |w(t)| &\le \sqrt{2}\max(h|b_0|, |w_0|)\exp(ht) \end{aligned} \tag{6.3}$$

Duhamel's formula. For a two-vector function $X(t) : \mathbb{R} \to \mathbb{R}^2$, a two-vector $X_0 \in \mathbb{R}^2$, and a 2×2 matrix function $A(t) : \mathbb{R} \to (2\times 2 \text{ matrices})$, let $X(t) = S(t, t')X_0$ denote the solution to the ODE system $\dot{X}(t) = A(t)X(t)$ with $X(t') = X_0$:

$$\frac{d}{dt}S(t, t')X_0 = A(t)S(t, t')X_0\,, \quad S(t', t')X_0 = X_0\,.$$

Then, for a given two-vector function $f(t) : \mathbb{R} \to \mathbb{R}^2$, the solution to the inhomogeneous ODE system

$$\dot{X}(t) = A(t)X(t) + F(t) \tag{6.4}$$

with initial condition $X(0) = 0$ is given by Duhamel's formula

$$X(t) = \int_0^t S(t, t')F(t')dt' \tag{6.5}$$

Proof of Lemma 6.1. Let $\tilde{a} = a - \bar{a}$ and $\tilde{v} = v - \bar{v}$; these perturbative functions satisfy

$$\begin{cases} \dot{\tilde{a}} = \tilde{v} + \epsilon_1 & \tilde{a}(0) = 0 \\ \dot{\tilde{v}} = h^2 g\tilde{a} + \epsilon_2\,, & \tilde{v}(0) = 0 \end{cases}$$

where $g = g(t)$ is given by

$$g = \begin{cases} \dfrac{f(ha) - f(h\bar{a})}{h(a - \bar{a})} & \text{if } \bar{a} \ne a \\ f'(ha) & \text{if } a = \bar{a} \end{cases}$$

which is C^1 (in particular, uniformly bounded). Set

$$A(t) = \begin{bmatrix} 0 & 1 \\ h^2 g(t) & 0 \end{bmatrix}, \quad F(t) = \begin{bmatrix} \epsilon_1(t) \\ \epsilon_2(t) \end{bmatrix}, \quad X(t) = \begin{bmatrix} \tilde{a}(t) \\ \tilde{v}(t) \end{bmatrix}$$

in (6.4), and appeal to Duhamel's formula (6.5) to obtain

$$\begin{bmatrix} \tilde{a}(t) \\ \tilde{v}(t) \end{bmatrix} = \int_0^t S(t, t') \begin{bmatrix} \epsilon_1(t') \\ \epsilon_2(t') \end{bmatrix} dt' \tag{6.6}$$

Apply the Gronwall estimate (6.3) with

$$\begin{bmatrix} b(t) \\ w(t) \end{bmatrix} = S(t+t',t') \begin{bmatrix} \epsilon_1(t') \\ \epsilon_2(t') \end{bmatrix}, \quad \begin{bmatrix} b_0 \\ w_0 \end{bmatrix} = \begin{bmatrix} \epsilon_1(t') \\ \epsilon_2(t') \end{bmatrix}$$

to conclude that

$$\left| S(t,t') \begin{bmatrix} \epsilon_1(t') \\ \epsilon_2(t') \end{bmatrix} \right| \leq \sqrt{2} \begin{bmatrix} h^{-1}\exp(h(t-t')) \\ \exp(h(t-t')) \end{bmatrix} \max(h|\epsilon_1(t')|, |\epsilon_2(t')|)$$

Feed this into (6.6) to obtain that on $[0,T]$

$$|\tilde{a}(t)| \leq \sqrt{2}\, T \frac{\exp(hT)}{h} \sup_{0\leq s\leq T} \max(h|\epsilon_1(s)|, |\epsilon_2(s)|)$$

$$|\tilde{v}(t)| \leq \sqrt{2}\, T \exp(hT) \sup_{0\leq s\leq T} \max(h|\epsilon_1(s)|, |\epsilon_2(s)|)$$

Taking $T \leq \delta h^{-1}\log(1/h)$, we obtain the claimed bounds. □

7. Proof of Theorem 1

We can now put all the components of the proof together. Lemma 3.6 and Theorem 2 show that on the time interval $0 < t < c\delta\log(1/h)/h$ we have (1.2) with the parameters satisfying

$$\dot{a} = v + \mathcal{O}(h^{4(1-\delta)}), \quad \dot{v} = -\mathrm{sech}^2 * V'(a)/2 + \mathcal{O}(h^{4(1-\delta)}), \quad \dot{\mu} = \mathcal{O}(h^{4(1-\delta)}),$$
$$\dot{\gamma} = 1/2 + v^2/2 - \mathrm{sech}^2 * V(a) + (x\,\mathrm{sech}^2 x \tanh x) * V(a) + \mathcal{O}(h^{4(1-\delta)}).$$

Lemma 6.1 can be applied to replace a and v with solutions of (1.3) and the direct integration of the error terms shows that the same is true for μ and γ. In particular we can drop μ altogether. □

We conclude the paper with some remarks. The proof above and Theorem 2 show that the conclusions of Theorem 1 remain unchanged if instead of taking $e^{ixv_0}\mathrm{sech}(x-x_0)$ as initial condition, we took

$$e^{ixv_0}\mathrm{sech}(x-x_0) + r(x), \quad \|r\|_{H^1} \leq Ch^{2-\delta}.$$

We could go down to $\|r\|_{H^1} \leq Ch^{3/2+3\delta}$ at the expense of complicating the final statement to (5.12). A more general condition on the initial value would make the bootstrap argument in §5 so unwieldy that we opted out of pursuing that technical issue.

In higher dimensions similar methods are clearly applicable for weaker nonlinearities and under further spectral assumptions – see [5] for examples. At this early stage we restrict ourselves to the physically relevant cubic nonlinearity which at the moment is tracktable only in dimension one.

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