This note is to clarify what you have to know about PDE for the exam. You should know what heat and wave equations are and what initial/boundary value problems for these equations mean:

**Heat equation:** \( \partial_t u = \partial_x^2 u, \ u(x,0) = f(x) \), is the initial value problem, and the boundary conditions can be

- a) \( u(0,t) = u_1, \ u(L,t) = u_2, \ 0 < x < L \) (temperature fixed at both boundaries); typically \( u_1 = u_2 = 0 \);
- b) \( \partial_x u(0,t) = \partial_x u(L,t) = 0, \ 0 < x < L \) (no heat flow through the boundaries).

**Wave equation:** \( \partial_t^2 u = \partial_x^2 u, \ u(x,0) = f(x), \ \partial_x u(x,0) = g(x) \) is the initial value problem (note that you need two initial conditions), and the boundary conditions can be

- a) \( u(0,t) = 0, \ u(L,t) = 0, \ 0 < x < L \) (an oscillating string is fixed at both ends);
- b) \( \partial_x u(0,t) = \partial_x u(L,t) = 0, \ 0 < x < L \) (a string can move up and down at the ends).

You may be asked to solve the heat equation with any boundary conditions but I will not ask you to solve the wave equation.

To solve the heat equation with boundary conditions a) proceed as follows:

- Note that \( w(x) = u_1 + x(u_2 - u_1)/L \) solves the heat equation (it does not depend on \( t \) and \( w''(x) = 0 \)), and \( w(0) = u_1, \ w(L) = u_2 \).
- Hence \( u(x,t) = w(x) + v(x,t) \) where \( v \) solves the initial/boundary value problem
  \[
  \partial_t v = \partial_x^2 v, \ \ v(x,0) = f(x) - w(x), \ \ v(0,t) = v(L,t) = 0.
  \]
- To find \( v \) we expand \( f(x) - w(x) \) in sine series,
  \[
  f(x) - w(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \ 0 < x < L,
  \]
  and get
  \[
  v(x,t) = \sum_{n=1}^{\infty} e^{-(n\pi/L)^2 t} b_n \sin \frac{n\pi x}{L}.
  \]
- Returning to the original problem we get as our solution
  \[
  u(x,t) = u_1 + x(u_2 - u_1)/L + \sum_{n=1}^{\infty} e^{-(n\pi/L)^2 t} b_n \sin \frac{n\pi x}{L}.
  \]

For boundary condition b) there is no \( w(x) \), so expand \( f \) in cosine series and get

\[
 u(x,t) = \frac{a_0}{2} \sum_{n=1}^{\infty} e^{-(n\pi/L)^2 t} a_n \cos \frac{n\pi x}{L}.
\]