1D SCATTERING THROUGH TIME DEPENDENT MEDIA WITH MEMORY

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With an appendix by Zhen Huang and Maciej Zworski

Abstract. We construct a scattering matrix with operator valued entries describing solutions to the 1+1 wave equation where permittivities has memory and depends on time and space. It is the analogue of the scattering matrix for spatially localised perturbations where the entries are functions of frequency and appear as Fourier multipliers in solutions of the wave equation. This provides a mathematical explanation of the numerical construction in the recent paper by Horsley et al [HGW23].

1. Introduction

There is a considerable interest in materials whose properties depend on time and which have memory. Having memory essentially means having a (causal) dependence on frequency – see for instance [Ga*22] for a survey and [Ga*26] for a recent experimental and theoretical study. Our motivation comes from a letter [HGW23] by Horsley, Galiffi, and Wang and we refer to it for more references to the literature and the physics background.

1.1. Scattering for permittivities with memory.

$$
\begin{gathered}
D_t^2 u(t,x) - a(x,t) \int_{-\infty}^t e^{-\gamma(t-t')} D_t u(t',x) dt' - D_x^2 u(t,x) = 0, , \\
a \in L^\infty(\mathbb{R}_x; C_c^\infty(\mathbb{R}_t)), \quad \operatorname{supp} a \subset [-R,R] \times [-T,T], \quad \gamma > 0, \\
u(t,x)|_{t \le 0} = g(x-t), \quad \operatorname{supp} g \subset (-\infty, -R).
\end{gathered}
\tag{1.1}
$$

(The compact support in time could be relaxed to superexponential decay but we restrict ourselves to the simplest case here).

At least formally this corresponds on the Fourier transform side to

$$
\begin{gathered}
P := D_x^2 - \omega^2 + A(x), \\
A(x) := a(x, D_\omega) \frac{\omega}{\omega + i\gamma} : H_r(\mathbb{R}_\omega) \to H_r(\mathbb{R}_\omega),
\end{gathered}
\tag{1.2}
$$

where

$$
H_\alpha := \left\{ f : f(\bullet) \in \mathscr{O}(\mathbb{C}_+), \quad \sup_{\sigma > 0} e^{-2\sigma\alpha} \int_{\mathbb{R}} |f(\lambda + i\sigma)|^2 d\lambda < +\infty \right\}, \tag{1.3}
$$

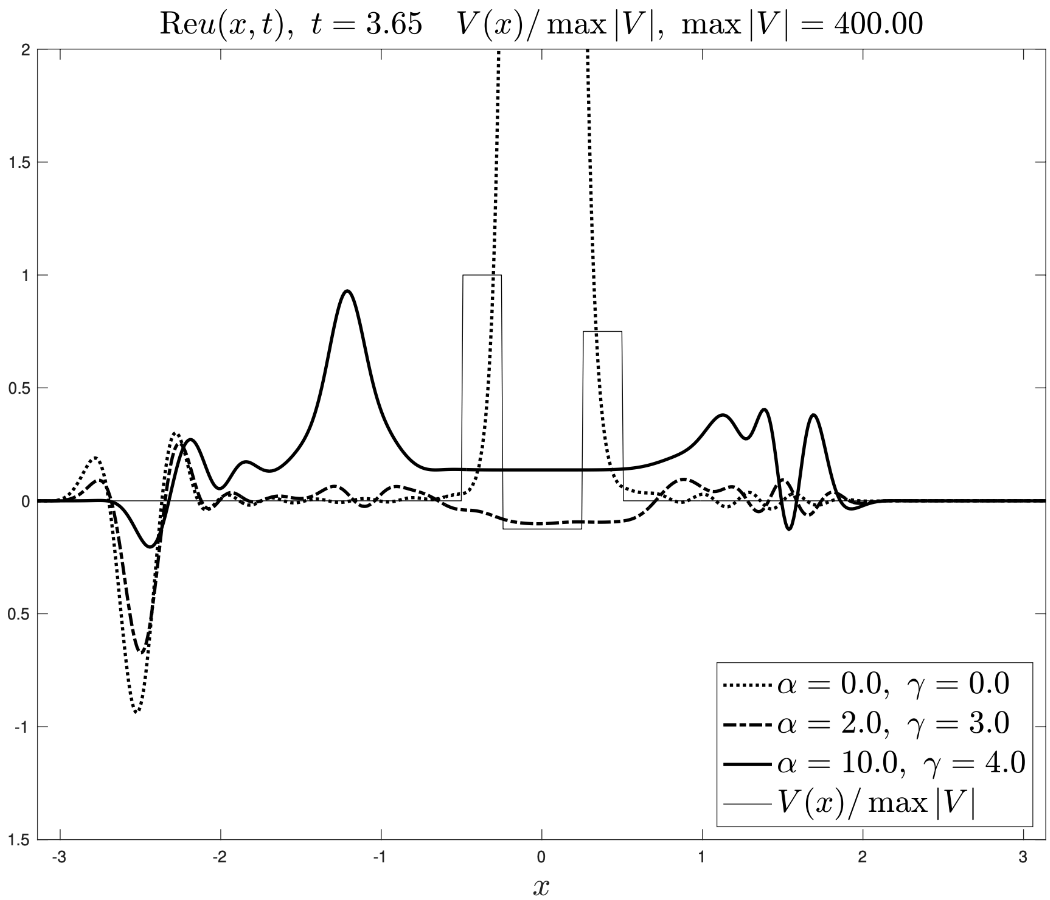

FIGURE 1. Comparisons of evolutions of a gaussian packet $g(t-x)$, $t \ll 1$, $g(y) = e^{-(x_0+y)^2)/\sigma - i\lambda(y-x_0))}$, $\sigma = 0.05$, $\lambda = 10$, $x_0 = -2$ for different values of the parameters in (2.9) (with $m = 1$) – this corresponds to the model considered in [HGW23]. An animated version is available at https://math.berkeley.edu/~zworski/wave_multi_one.mp4. The code for producing this movie and the figure is enclosed in the appendix.

and also write $H_\infty := \cup_\alpha H_\alpha$. The Hardy spaces H_α appear naturally in scattering theory for the wave equation since the work of Lax–Phillips [LP68] – see §3.1, and (3.6) specifically, for the review of the simplest case.

The action of $a(x, D_\omega)$ is given by

$$[a(x, D_\omega)u](x, \omega + i\sigma) := \frac{1}{2\pi} \int \widehat{a(x, \bullet)}(\omega - \lambda) u(x, \lambda + i\sigma) d\lambda. \tag{1.4}$$

We introduce the following Hilbert space of functions of position and frequency, $f : \mathbb{R} \times \mathbb{C}_+ \to \mathbb{C}$, $\mathbb{C}_+ := \{\omega \in \mathbb{C} : \operatorname{Im} \omega > 0\}$,

$$\mathscr{H}^s_\alpha := \left\{ f : f(x, \bullet) \in \mathscr{O}(\mathbb{C}_+), \ \sup_{\sigma>0} e^{-2\sigma\alpha} \int_{\mathbb{R}} \|f(\cdot, \lambda + i\sigma)\|^2_{H^s(\mathbb{R})} d\lambda < +\infty \right\}. \quad (1.5)$$

We write $\mathscr{H}_\alpha := \mathscr{H}^0_\alpha$, $\mathscr{H}_\infty = \cup_\alpha \mathscr{H}_\alpha$, and

$$\mathscr{H}^s_{\alpha,\mathrm{loc}} : \{u \, : \, u(x, \bullet) \in \mathscr{O}(\mathbb{C}_+) \text{ and } \chi(x)u(x, \omega) \in \mathscr{H}^s_\alpha \text{ for all } \chi \in C^\infty_c\}.$$

The goal now is to prove the following analogue of the existence of the scattering matrix (3.2):

Theorem 1. *Suppose that assumptions in* (1.1) *hold and* $f \in (\omega + i)^{-1} H_{-R}$. *Then there exist bounded operators*

$$T, R_+ : (\omega + i)^{-1} H_\alpha \to \omega^{-1} H_{\alpha+2R}, \quad \alpha \in \mathbb{R},$$

such that, for P *given in* (1.2), *there exists a unique solution,* $u(x, \omega) \in \omega^{-1}\mathscr{H}_{\infty,\mathrm{loc}}$, $(x, \omega) \in \mathbb{R}^2$, *of* $Pu = 0$ *such that*

$$u(x, \omega) = \begin{cases} f(\omega)e^{i\omega x} + R_+ f(\omega)e^{-i\omega x}, & x < -R, \\ Tf(\omega)e^{i\omega x}, & x > R. \end{cases} \quad (1.6)$$

The operators R_+ and T were constructed numerically in [HGW23] for the case when $a(x, t) = V_0 \, \mathbb{1}_{-R,R}(x)\chi(t)$. That was done by following the construction of the scattering matrix for step potentials (see §3.1 and [DyZw19, Exercise 2.10.6]) but with the exponentials

$$e^{\pm\omega x} \text{ for } |x| > R \quad \text{and} \quad e^{\pm i\sqrt{\omega^2 - V_0} x} \text{ for } |x| < R,$$

replaced Schrödinger propagators $e^{\pm ix\chi(D_\omega)\omega/(\omega+i\gamma)}$ for $|x| < R$. The needed inversion of operators was established numerically. The point of Theorem 1 is that, under the assumption of localisation in space and time, the operator exists even for a larger class of perturbations.

The proof has two parts: the first, in §3.2, is a functional analytic setup for constructing $u(x, \omega)$. The second one, in §3.3, is the proof of non-existence of purely outgoing solutions to $Pu = 0$, that is solutions satisfying (1.6) with $f \equiv 0$ but the terms corresponding to $R_+ f$ and Tf potentially nonzero.

1.2. **The wave equation.** We now describe how the operators constructed in Theorem 1 appear in the wave evolution. The theorem in the exact analogue of Proposition 4 in which the standard scattering matrix for compactly supported 1D potentials appears in the description of scattered waves.

Theorem 2. *Suppose that $g \in C_c^\infty((R,\infty))$ and that $u(t,x)$ is the unique solution of (1.1) satisfying $u(t,x) = g(t-x)$, $t < 0$. Then*

$$u(t,x) = \begin{cases} g(t-x) + \mathscr{R}_+ g(x+t), & x < -R, \\ \mathscr{T} g(t-x), & x > R, \end{cases} \tag{1.7}$$

where

$$\widehat{\mathscr{R}_+ g}(\omega) := R_+ \widehat{g}(\omega), \qquad \widehat{\mathscr{T} g}(\omega) := T\widehat{g}(\omega).$$

and T and R_+ are described in Theorem 1.

Existence of operators $\mathscr{R}_+$ and $\mathscr{T}$ (under some assumptions guaranteeing existence and uniqueness of solutions to the wave equation) is an elementary general fact – see Proposition 1. The point here is these operators are related to the stationary problem from Theorem 1 and have correct mapping properties. At this stage, we provide the codes for solving the wave equation (see the Appendix and Figure 1) but not the comparison with the scattering matrix. It is an interesting open question, relevant to physics problems considered in [HGW23] and references given there, to analyse quantitative properties of R_+ and T in asymptotic regimes of the parameters. The numerical experiments (which an interested reader is invited to perform using code in the Appendix) indicate interesting phenomena which should investigated.

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2. General theory

We discuss a general definition of a scattering operator for spatially localised perturbations of the 1+1 wave equation. We then define a general class of perturbations with time dependence and memory and prove existence and uniqueness for the corresponding wave equation. These preliminary results are not surprising but we not seem to have a ready to use reference covering existence and uniqueness of the Cauchy problem for operators described in §1 and for the more general ones given in (2.5). We present a detailed argument in our specific case noting that it generalises to higher dimensions.

2.1. **An abstract scattering operator.** The following elementary result shows that we can define the scattering matrix for very general spatially localised perturbations (see §3.1 for a review of the standard theory in our context).

Suppose that $P : \mathscr{D}'(\mathbb{R}^2) \to \mathscr{D}'(\mathbb{R}^2)$ has the property that

$$u \in \mathscr{D}'(\mathbb{R}^2), \ \operatorname{supp} u \cap (\mathbb{R} \times [-R, R]) = \emptyset \implies Pu = D_x^2 u. \tag{2.1}$$

More informally we can state this as

$$\forall\, t \in \mathbb{R} \ \operatorname{supp} u(t, \bullet) \cap [-R, R] = \emptyset \implies Pu(t, x) = D_x^2 u(t, x).$$

We do not assume that P is linear here. For a general class of linear operators P relevant to this note, see §2.2.

Under this assumption on P we have the following simple fact:

Proposition 1. *Suppose that $u \in \mathscr{D}'(\mathbb{R}^2)$ solves $(D_t^2 - P)u = 0$ and*

$$u|_{t<0} = \kappa_+^* g|_{t<0}, \quad \kappa_\pm : \mathbb{R}^2 \to \mathbb{R}, \quad \kappa_\pm(t, x) := t \mp x,$$
$$g \in \mathscr{D}'(\mathbb{R}), \quad \operatorname{supp} g \subset (-\infty, -R).$$

Then there exist

$$G, F \in \mathscr{D}'(\mathbb{R}), \quad \operatorname{supp} G, \operatorname{supp} F \subset (-R, \infty),$$

such that

$$u|_{x<-R} = \kappa_+^* g|_{x<-R} + \kappa_-^* G|_{x<-R}, \quad u|_{x>R} = \kappa_+^* F|_{x>R}. \tag{2.2}$$

Less formally the hypothesis reads as

$$u(t, x) = g(t - x), \quad t < 0, \quad \operatorname{supp} g \subset (-\infty, -R), \tag{2.3}$$

and the conclusion as

$$u(t, x) = \begin{cases} g(t - x) + G(t + x), & x < -R, \quad \operatorname{supp} G \subset (-R, \infty), \\ F(t - x), & x > R, \quad \operatorname{supp} F \subset (-\infty, R). \end{cases} \tag{2.4}$$

This means that under the assumption (2.1) and provided we have existence and uniqueness to solutions of $(D_t^2 - P)u = 0$, we have scattering maps:

$$g \mapsto \mathscr{T} g := F, \quad g \mapsto \mathscr{R}_+ g = G,$$

where $\mathscr{T}$ and $\mathscr{R}_+$ are transmission and reflection maps. (We have similar definitions for waves g approaching from the right.) As explained in §1 this paper describes mapping properties and basic structure of $\mathscr{T}$ and $\mathscr{R}_+$ for more specific perturbations and relates them to stationary scattering theory.

Proof of Proposition 1. We first proceed by pretending that u is a function and take $r > R$. If we write $z = t - x$ and $w = x + t$ so that for $v(z, w) = u(x, t)$ we have $\partial_z \partial_w v = 0$ for $\pm(w - z) > 2r$. In particular, $\partial_z \partial_v v(z, w) = 0$ for $-z > 2r - w$, that is

$\partial_w v(z,w) = f(w)$, $w > 2r + z$ for some f defined for all values of w. But that means that

$$v(z,w) = v(z, 2r+z) + \int_{2r+z}^{0} f(y)dy + \int_0^w f(y)dy, \quad \text{for } w - z > 2r,$$

that is $v(z,w) = G_+(z) + F_+(w)$ for $w - z > 2r$. Similarly, $v(z,w) = G_-(z) + F_-(w)$ for $w - z < -2r$. This argument applies to distributions by using [Hö03, Theorem 3.1.4′] and the fact that the restriction of v to $L := \{(z, 2r+z) : z \in \mathbb{R}\}$ is well defined as for $w - z > 2R$, $\mathrm{WF}(v) \subset \{(z,w;\zeta,\omega) : \zeta\omega = 0\}$ which (away from the zero section) is disjoint from $N^*L = \{(z, 2r - z, \zeta, \zeta) : z, \zeta \in \mathbb{R}\}$ – see [Hö03, Corollary 8.2.7]. Since $r > R$ is arbitrary we can replace r with R in our conclusion.

We now need to prove that $G_+ \equiv 0$ and $G_- = g$. To see the first claim we note that (2.3) gives $F_+(z) + G_+(w) = 0$ for $w - z = 2x > 2R$ and $w + z = 2t < 0$. In particular, $G_+'(w) = 0$ for $2R + z < w < -z$, and as z is arbitrary, this shows that G_+ is constant. We can absorb that constant into F_+ and hence have $G_+ \equiv 0$.

To see that $G_- = g$, we note that (2.3) gives $G_-(z) - g(z) + F_-(w) = 0$ for $w - z = 2x < -2R$ and $w + z = 2t < 0$. Hence $G_-'(z) - g'(z) = 0$ for $w < -z < -2R - w$. Since w is arbitrary, this means that $G_-(z) - g(z)$ is constant and we can make it 0 by changing F_-. □

2.2. **A class of permittivities.** The model we consider is the wave equation with permittivity which depends on time but also has memory in the sense of being an operator. In this section w do not assume spatial localisation and consider a generalisation of (1.1):

$$\begin{aligned} \mathscr{P} &:= \partial_t \varepsilon \partial_t - \partial_x^2, \quad x \in \mathbb{R}, \quad \varepsilon v(t,x) := v(t,x) + Bv(t,x), \\ Bv(t,x) &:= \int_{-\infty}^{t} B(x,t,t-t')v(t',x)dt'. \end{aligned} \tag{2.5}$$

We typically want to solve the problem

$$\mathscr{P} v = F(t,x), \quad \operatorname{supp} F \subset (-R,\infty) \times \mathbb{R}, \quad v(t,x)|_{t<-R} = 0. \tag{2.6}$$

We make the following assumptions on B:and that there exists m and m_0 such that

$$\forall\, k, \ell \,\exists\, C_{k\ell} \quad |\partial_t^k \partial_s^\ell B(x,t,s)| \le C_{k\ell} \langle t\rangle^{m_0} \langle s\rangle^m. \tag{2.7}$$

Later, but not in this section, we assume that we have a localisation in space, that is, for $R > 0$ independent of t and s

$$\operatorname{supp} B(\bullet, t, s) \subset (-R, R) \tag{2.8}$$

As an example we can take

$$B(x,t,s) = V(x) e^{-\alpha t^2} e^{-\gamma s} s^m, \quad \alpha, \gamma \ge 0, \quad V \in L^\infty(\mathbb{R}), \tag{2.9}$$

noting that the case of $\alpha = \gamma = 0$, $m = 1$, gives $\mathscr{P} = \partial_t^2 - \partial_x^2 + V(x)$. Another way of writing the operator B in (2.9) is as a pseudodifferential operator:

$$\begin{aligned} &B = b(x,t,D_t), \quad b(x,t,\tau) := V(x)e^{-\alpha t^2} i^m (\tau + i\gamma)^{-m}, \\ &a(t,D_t)h := \int_{\mathbb{R}} a(t,\tau)\widehat{h}(\tau)d\tau, \quad h(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{h}(\omega)e^{-i\omega t}d\omega. \end{aligned} \tag{2.10}$$

We note here that our convention for Fourier transform in time is non-standard but leads to cleaner formulas in our setting.

The most relevant case for us is (1.1). For that we take

$$\begin{aligned} &B(x,t,s) = \gamma^{-1}(e^{-s\gamma} - 1)a(x,t), \quad \gamma > 0, \\ &a \in L_x^\infty C_t^\infty, \quad \operatorname{supp} a \subset [-R,R] \times [-T,T]. \end{aligned} \tag{2.11}$$

In that case we have

$$m_0 = -\infty, \quad m = 1.$$

(This means that in (2.13) below we can take $\gamma(T) \equiv 1$.)

Since we do not know a ready to use reference covering existence and uniqueness of the Cauchy problem for $\mathscr{P}$ in (2.5) we present a detailed argument in our specific case noting that it generalises to higher dimensions.

2.3. **Existence and uniqueness.** The result of §2.1 is applicable to operators of the form (2.5) thanks to the following

Proposition 2. *Under the assumptions (2.5) and (2.7), for $F \in L^1([0,T];L^2(\mathbb{R}))$, there exists a unique*

$$u \in C((-\infty,T]);H^1(\mathbb{R})) \cap C^1((-\infty,T]);L^2(\mathbb{R})), \quad \operatorname{supp} u \subset [0,T], \tag{2.12}$$

such that

$$\mathscr{P}u = F \ \text{ on } (0,T) \times \mathbb{R}$$

Moreover, there exists $C_0, \lambda > 0$ (independent of T) such that for $0 \le t \le T$.

$$\begin{aligned} &\|u(t,\bullet)\|_{H^1(\mathbb{R})} + \|\partial_t u(t,\bullet)\|_{L^2(\mathbb{R})} \le C_0 \big\| e^{\lambda\gamma(T)(T-t')} F(t',x) \big\|_{L^2(0,T)_{t'} \times \mathbb{R}_x)}, \\ &\gamma(T) := 1 + \langle T \rangle^{m_0+1} + \langle T \rangle^{m_0+m+1}. \end{aligned} \tag{2.13}$$

To obtain uniqueness and (2.13) we will use

Lemma 3. *Suppose that (2.12) is satisfied and that in addition,*

$$F := \mathscr{P}u \in L^1([0,T];L^2(\mathbb{R})).$$

Then (2.13) holds.

Proof. This is done by an adaptation of the usual energy estimate based on the energy identity (see for instance [Hö85, (2.4.2)] for a very general version):

$$\begin{aligned}&\partial_t(e^{-2\lambda t}\tfrac{1}{2}(|u_t|^2+|\nabla_x u|^2+|u|^2))-\nabla_x\cdot(\operatorname{Re}\nabla_x u\bar{u}_t)+\lambda e^{-2\lambda t}(|u_t|^2+|\nabla_x u|^2+|u|^2)\\&=-2\operatorname{Im}e^{-2\lambda t}\mathscr{P}u\overline{D_t u}-e^{-2\lambda t}D_tBD_tu\overline{D_tu}-e^{-2\lambda t}\mu^2u\overline{D_tu}+e^{-2\lambda t}\operatorname{Re}u\bar{u}_t.\end{aligned}$$

Suppose $u\in H^1((-\infty,T)\times\mathbb{R})$ with $\operatorname{supp}u\subset[0,T]$ and $\mathscr{P}u\in L^2$. Then, integrating the energy identity on $(-\infty,T)\times\mathbb{R}$, using the divergence theorem (in higher dimensions), the fact that the form of B in (2.5) implies $\operatorname{supp}BD_tu\subset(-\infty,T]$, and that for $\operatorname{supp}v\subset[0,T]$,

$$\begin{aligned}&\int_{-\infty}^T\left\|e^{-\lambda t}D_t\int_0^t B(x,t,t-t')e^{\lambda t'}v(t',x)dt'\right\|_{L^2_x}^2dt\\&\le C\int_0^T\langle t\rangle^{2m_0}\|v(t,\bullet)\|^2_{L^2_x}dt+C\int_0^T\langle t\rangle^{2m_0}(1+\langle t\rangle^{2m})\|v\|^2_{L^1((0,t);L^2_x)}dt\\&\le C\gamma(T)^2\|v\|^2_{L^2},\end{aligned}$$

where $\gamma(T)$ is defined in (2.13). From this we obtain

$$\begin{aligned}&e^{-2\lambda T}\tfrac{1}{2}(\|u_t(T)\|^2_{L^2(\mathbb{R})}+\|u(T)\|^2_{H^1(\mathbb{R})})+\lambda(\|e^{-\lambda t}u\|^2_{L^2((-\infty,T);H^1(\mathbb{R}))}+\|e^{-\lambda t}u_t\|^2_{L^2((-\infty,T)\times\mathbb{R})})\\&=-2\operatorname{Im}\langle e^{-\lambda t}\mathscr{P}u,e^{-\lambda t}D_tu\rangle_{(-\infty,T)\times\mathbb{R}}-\langle e^{-\lambda t}D_tBe^{\lambda t}e^{-\lambda t}D_tu,e^{-\lambda t}D_tu\rangle_{L^2((-\infty,T)\times\mathbb{R})}\\&\quad-\langle e^{-\lambda t}\mu^2e^{\lambda t}e^{-\lambda t}u,e^{-\lambda t}D_tu\rangle_{(-\infty,T)\times\mathbb{R}}+\operatorname{Re}\langle e^{-\lambda t}u,e^{-\lambda t}\partial_tu\rangle\\&\le C\|e^{-\lambda t}f\|^2_{L^2((-\infty,T)\times\mathbb{R})}+(C_0\gamma(T)^2+1)\|e^{-\lambda t}D_tu\|^2_{L^2((-\infty,T)\times\mathbb{R})}\\&\quad+\tfrac{1}{2}\|e^{-\lambda t}u\|^2_{L^2((-\infty,T)\times\mathbb{R})}.\end{aligned}$$

Hence, taking λ large enough, and moving the two right-most terms to the left-hand side, we obtain (2.13) (with $\lambda\gamma(T)$ replacing λ). ☐

Proof of Proposition 2. Uniqueness is immediate from Lemma 3 and (2.13). It remains to show existence. For that we will use the free wave group, $U(t):H^s(\mathbb{R})\times H^{s-1}(\mathbb{R})\to H^s(\mathbb{R})\times H^{s-1}(\mathbb{R})$:

$$\begin{aligned}U(t)=:&\begin{pmatrix}\cos(D_xt)&\sin(D_xt)/D_x\\-D_x\sin(D_xt)&\cos(D_xt)\end{pmatrix}\\=&\,e^{tL},\qquad L:=\begin{pmatrix}0&I\\-D_x^2&0\end{pmatrix},\quad D_x:=(1/i)\partial_x.\end{aligned}\tag{2.14}$$

We fix $T_-\ge 0$ and for

$$\mathbf{w}\in C^0((-\infty,T_-);H^1\times L^2),\quad\operatorname{supp}\mathbf{w}\subset[0,T]\times\mathbb{R},\tag{2.15}$$

define a sequence

$$\mathbf{v}_n(t)\in C^0((-\infty,T);H^1\times L^2),\quad n=-1,0,\cdots,\quad\operatorname{supp}\mathbf{v}_n\subset[0,T]\times\mathbb{R},\tag{2.16}$$

inductively as follows:

$$\mathbf{v}_{-1} := \begin{cases} \mathbf{w}(t) & t \le T_-, \\ \mathbf{w}(T_-) & T_- \le t \le T. \end{cases}$$

Then for $n \ge 0$, we again define $\mathbf{v}_n$ differently in different ranges of t. For $t \le T_-$ we put $\mathbf{v}_n(t) := \mathbf{w}(t)$, while for $T_- \le t \le T$,

$$\mathbf{v}_n(t) := U(t-T_-)\mathbf{w}(T_-) + \int_{T_-}^{t} U(t-s)\left(\begin{pmatrix} 0 \\ F(s) \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \mu^2 & \partial_t B \end{pmatrix} \mathbf{v}_{n-1}(s) \right) ds. \quad (2.17)$$

Then (2.16) holds and

$$(\partial_t - L)\mathbf{v}_n = \begin{pmatrix} 0 \\ F \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \mu^2 & \partial_t B \end{pmatrix} \mathbf{v}_{n-1}, \quad t \in (T_-, T) \qquad \mathbf{v}_n(T_-) = \mathbf{w}(T_-).$$

In particular,

$$\begin{gathered} (\partial_t - L)(\mathbf{v}_n - \mathbf{v}_{n-1}) = -\begin{pmatrix} 0 & 0 \\ \mu^2 & \partial_t B \end{pmatrix} (\mathbf{v}_{n-1} - \mathbf{v}_{n-2}), \\ t \in (T_-, T), \quad \mathbf{v}_n(T_-) - \mathbf{v}_{n-1}(T_-) = 0. \end{gathered} \quad (2.18)$$

We now observe that (2.5) and (2.7) show that for $v \in C^0((-\infty, T); L^2)$, $\operatorname{supp} v \subset [0, T] \times \mathbb{R}$

$$\begin{aligned} \|\partial_t B v(s)\|^2_{L^2(\mathbb{R})} &\le \int_{\mathbb{R}} \left| \int_0^s \langle s \rangle^m \langle s - t' \rangle^{m_0} v(t', x) dt' \right|^2 dx \\ &\le \langle s \rangle^{2m} (1 + \langle s \rangle^{2m_0}) \|v\|^2_{L^1((0,s);L^2))}. \end{aligned} \quad (2.19)$$

Since $\mathbf{v}_{n-2}(t) - \mathbf{v}_{n-1}(t) = 0$ for $t < T_-$ it follows from (2.17) that

$$\begin{aligned} \|\mathbf{v}_n(t) - \mathbf{v}_{n-1}(t)\|_{H^1 \times L^2} \le & C \int_{T_-}^{t} (1 + |t-s|)(\|\mathbf{v}_{n-1}(s) - \mathbf{v}_{n-2}(s)\|_{H^1 \times L^2} \\ & + \langle s \rangle^m (1 + \langle s \rangle^{m_0}) \|\mathbf{v}_{n-1} - \mathbf{v}_{n-2}\|_{L^1(T_-, s); H^1 \times L^2}) ds, \end{aligned}$$

for some constant C.

Using $T_- + \delta \le T + 1$, we see that for $\delta < 1$, there is a new constant C_1 depending on T but not on T_-,

$$\begin{aligned} & \|\mathbf{v}_n - \mathbf{v}_{n-1}\|_{L^\infty((T_-, T_- + \delta); H^1 \times L^2)} \le \\ & \quad C(1 + \langle T_- + \delta \rangle^m (1 + \langle T_- + \delta \rangle^{m_0})\delta) \|\mathbf{v}_{n-1} - \mathbf{v}_{n-2}\|_{L^1((T_-, T_- + \delta); H^1 \times L^2)} \le \\ & \quad C_1 \delta \|\mathbf{v}_{n-1} - \mathbf{v}_{n-2}\|_{L^\infty((T_-, T_- + \delta); H^1 \times L^2)} \end{aligned}$$

Taking $\delta < 1/C_1$, we concluded that $\mathbf{v}_n$ is a Cauchy sequence in $L^\infty((T_-, T_- + \delta); H^1 \times L^2)$. Since

$$\mathbf{v}_n(t) = \mathbf{w}(t) \;\text{ for }\; t \le T_-, \quad \mathbf{v}_n \in C^0([T_-, T_- + \delta]; H^1 \times L^2), \quad \mathbf{v}_n(T_-) = \mathbf{w}(T_-),$$

$\mathbf{v}_n$ converges to $\mathbf{v} \in C^0((-\infty, T_- + \delta]; H^1 \times L^2)$ and $\mathbf{v}|_{(-\infty, T_-)} = \mathbf{w}|_{(-\infty, T_-)}$.

In particular we have convergence $\mathbf{v}_n(t) \to \mathbf{v}(t)$ in the sense of distributions on $(T_-, T_- + \delta)$, and hence, from (2.17), in the sense of distributions,

$$\partial_t \mathbf{v} = L\mathbf{v} - \begin{pmatrix} 0 & 0 \\ \mu^2 & \partial_t B \end{pmatrix} \mathbf{v} + \begin{pmatrix} 0 \\ F \end{pmatrix}, \quad \text{on } (T_-, T_- + \delta) \times \mathbb{R}.$$

Putting $\mathbf{v} = [v^1, v^2]^t$, this gives $\partial_t v^1 = v^2 \in C^0([T_-, T_- + \delta]; L^2)$. Hence $u := v^1$, satisfies

$$\begin{gathered} u \in C^0((-\infty, T_+ + \delta]; H^1) \cap C^1(-\infty, T_- + \delta]), \\ u(T_-) = w^1(T_-), \quad \partial_t u(T_-) = w^2(T_-), \quad \mathbf{w} =: [w^1, w^2]^t, \\ \mathscr{P}u = F \ \text{ in the sense of distributions on } (T_-, T_- + \delta). \end{gathered} \tag{2.20}$$

The only conditions here are (2.15) on w and and $\delta > 0$.

We can now pass from this small step procedure to finding

$$u \in C^0((-\infty, T); H^1(\mathbb{R})) \cap C^1((-\infty, T); L^2(\mathbb{R})), \quad \operatorname{supp} u \subset [0, T],$$

satisfying $\mathscr{P}u = F$ on $(0, T)$. To do this, we set $u_0 = 0$ and for $j \geq 1$ and use (2.20) to inductively obtain

$$u_j \in C^0((-\infty, j\delta]; H^1(\mathbb{R})) \cap C^1((-\infty, j\delta]; L^2(\mathbb{R})), \tag{2.21}$$

satisfying

$$\begin{gathered} \mathscr{P}u_j = F, \quad \text{distributionally on } ((j-1)\delta, j\delta) \times \mathbb{R}, \\ u_j((j-1)\delta) = u_{j-1}((j-1)\delta), \quad \partial_t u_j((j-1)\delta) = \partial_t u_{j-1}((j-1)\delta). \end{gathered} \tag{2.22}$$

We claim that

$$\mathscr{P}u_j = F \quad \text{distributionally on } (0, j\delta). \tag{2.23}$$

Indeed, suppose by induction that this holds for j replaced by $j-1$. Then, (2.21) holds and

$$\mathscr{P}u_j = F \quad \text{distributionally on } ((0, (j-1)\delta) \cup ((j-1)\delta, j\delta).$$

We we write $\mathscr{P} = \partial_t^2 - \mathscr{L}u$, $\mathscr{L}u := \partial_x^2 u - \partial_t B \partial_t u - \mu^2 u$, then

$$G := \mathscr{L}u \in C^0((-\infty, j\delta); H^{-1}),$$

and hence, with $H := F + G$,

$$\partial_t^2 u = H \in C^0((-\infty, j\delta); H^{-1}) \text{ distributionally on } ((0, (j-1)\delta) \cup ((j-1)\delta, j\delta).$$

But the continuity of u and $\partial_t u$ across $(j-1)\delta$ shows that the equation holds distributionally on $(0, j\delta)$. (The argument reduces to showing that if $g, G \in C^0$ and $\partial_t g = G$, on $\mathbb{R} \setminus \{0\}$ then $\partial_t g = G$ on $\mathbb{R}$. But that follows from testing against $\varphi(t)(1 - \chi(t/\varepsilon))$ where $\varphi, \chi \in C^\infty_{\rm c}(\mathbb{R})$, $\chi \equiv 1$ near 0, and using continuity of g and G while letting $\varepsilon \to 0$.) Taking $J \geq \delta^{-1} T$ and setting $u = u_J$ completes the construction of u. $\square$

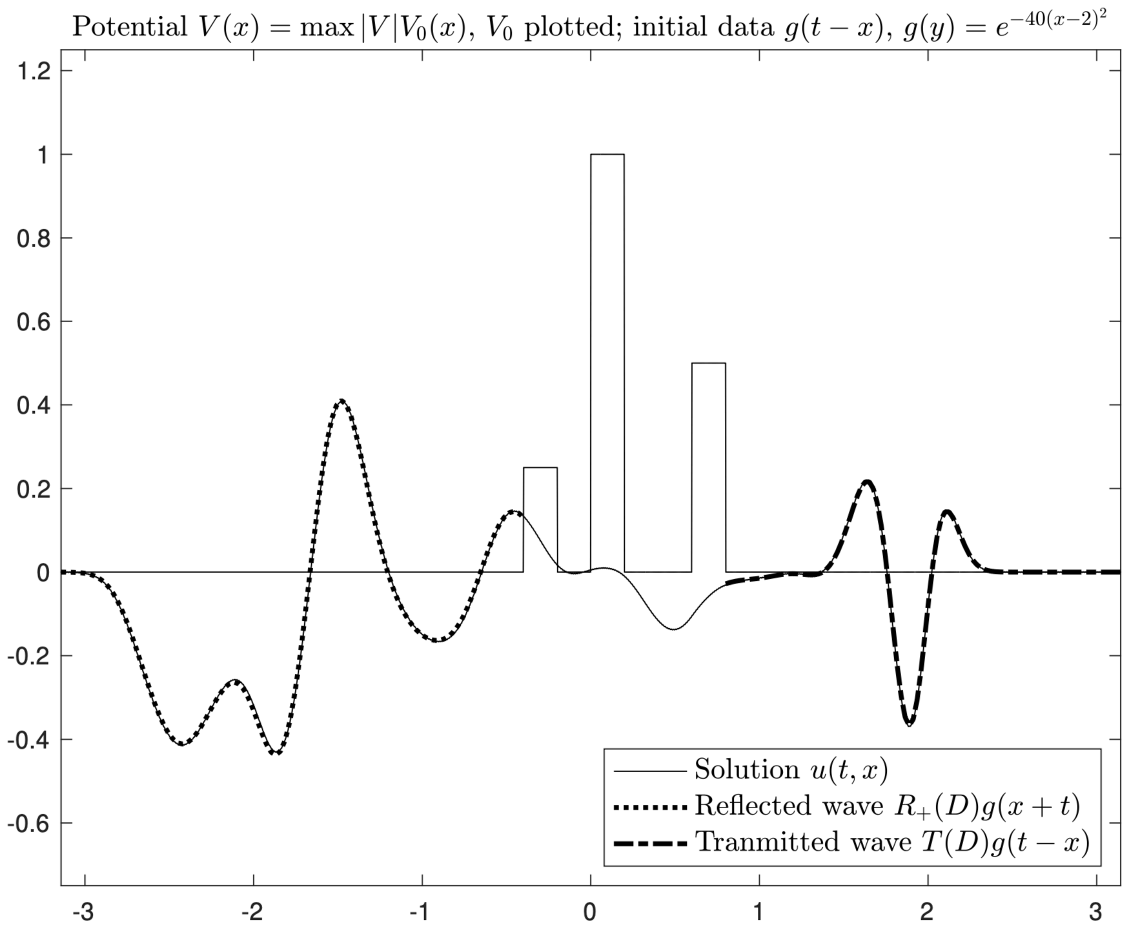

Figure 2. An illustration of Proposition 4: the solution to $(\partial_t^2 - \partial_x^2 + V(x))u = 0$, $\operatorname{supp} V \subset [-R, R]$, with $u(t,x) = g(t-x)$ for $t \ll -1$ is given by $g(t-x) + R_+(D)g(t+x)$ for $x \leq -R$ and $T(D)g(t-x)$ for $x \geq R$ (for all times; $t = 4$ shown). An animated version is available at https://math.berkeley.edu/~zworski/wave_pot.mp4.

3. An operator valued scattering matrix

In this section we prove Theorem 1. To motivate the construction and to link it to standard theory we first review the case compactly supported time independent potentials (the case $\alpha = \gamma = 0$ and $m = 1$ in §2.2) . The wave equation setting (unlike the more common Schrödinger equation setting) makes the the scattering matrix appear in a very clean way.

3.1. **Review of standard scattering matrix.** Suppose that $V \in L^\infty(\mathbb{R}; \mathbb{R})$ satisfies $\operatorname{supp} V \subset [-R, R]$. Any solution to

$$(D_x^2 + V(x) - \lambda^2)u = 0, \tag{3.1}$$

satisfies

$$u(x) = \begin{cases} A_+ e^{i\lambda x} + B_- e^{-i\lambda x}, & x > R; \\ A_- e^{i\lambda x} + B_+ e^{-i\lambda x}, & x < -R. \end{cases}$$

Taking the Wronskian of u and $\bar{u}$ for $\lambda \in \mathbb{R}\setminus 0$ shows that $|A_-|^2 + |B_-|^2 = |A_+|^2 + |B_+|^2$, and hence

$$S(\lambda) : \begin{pmatrix} A_- \\ B_- \end{pmatrix} \mapsto \begin{pmatrix} A_+ \\ B_+ \end{pmatrix}, \quad S(\lambda) = \begin{pmatrix} T(\lambda) & R_-(\lambda) \\ R_+(\lambda) & T(\lambda) \end{pmatrix}, \tag{3.2}$$

is well defined. It is called the *scattering matrix*, $T(\lambda)$ is the transmission coefficient and $R_\pm(\lambda)$ are the reflection coefficients. They extend to *meromorphic* functions in $\mathbb{C}$ satisfying

$$\begin{gathered} T(\lambda)\overline{T(\bar{\lambda})} + R_\pm(\lambda)\overline{R_\pm(\bar{\lambda})} = 1, \quad T(\lambda)\overline{R_\pm(\bar{\lambda})} + R_\pm(\lambda)\overline{T(\bar{\lambda})} = 0, \\ T(-\lambda) = \overline{T(\bar{\lambda})}, \quad R_\pm(-\lambda) = \overline{R_\pm(\bar{\lambda})}, \end{gathered} \tag{3.3}$$

and

$$\prod_{j=1}^N \frac{|\lambda - i\mu_j|}{|\lambda + i\mu_j|} |T(\lambda)| \leq e^{2R \operatorname{Im} \lambda}, \quad \prod_{j=1}^N \frac{|\lambda - i\mu_j|}{|\lambda + i\mu_j|} |R_\pm(\lambda)| \leq e^{2R \operatorname{Im} \lambda}, \quad \operatorname{Im} \lambda \geq 0, \tag{3.4}$$

where $\mu_j > 0$ and $-\mu_j^2$, $j = 1, \cdots, N$, are the eigenvalues of $D_x^2 + V(x)$ – see [DyZw19, §2.4, Exercise 3.14.10]. The normalised eigenfunctions $w_j \in L^2(\mathbb{R}; \mathbb{R})$ satisfy

$$(D_x^2 + V(x))w_j = -\mu_j^2 w_j, \quad \|w_j\|_{L^2} = 1, \quad w_j(x)|_{\pm x > R} = a_j^\pm e^{\mp \mu_j x}, \quad \mu_j > 0. \tag{3.5}$$

When $V \geq 0$ there are no eigenvalues and in that case we can consider multiplication by T and $R_\pm$ on Hardy spaces:

$$\begin{gathered} H_\alpha := \{ f \in \mathscr{O}(\{\operatorname{Im} \lambda > 0\}) : \sup_{\sigma > 0} e^{-2a\sigma} \int_{\mathbb{R}} |f(\lambda + i\sigma)|^2 d\lambda < +\infty \}, \\ H_\alpha \ni f(\lambda) \mapsto T(\lambda) f(\lambda) \in H_{a+2R}, \quad H_\alpha \ni f(\lambda) \mapsto R_\pm(\lambda) f(\lambda) \in H_{a+2R}. \end{gathered} \tag{3.6}$$

We remark that the spaces are isometric as $f(\omega) \mapsto e^{ia\omega} f(\omega)$ provides a unitary map between H_α and H.

The relation to the wave equation is given in the following

Proposition 4. *Suppose that* $g \in \mathscr{S}'(\mathbb{R})$ *and that* $\operatorname{supp} g \subset (R, \infty)$. *If*

$$(D_t^2 - D_x^2 - V(x))u(t, x) = 0, \quad t \in \mathbb{R}, \quad u(t, x) = g(t - x), \quad t \leq 0, \tag{3.7}$$

then, in the notation of (3.2) *and* (3.5),

$$u(t, x) = \begin{cases} g(t - x) + M_- g(x + t), & x < -R, \\ M_+ g(t - x), & x > R, \end{cases} \tag{3.8}$$

where

$$M_-g(x) = R_+(D)g(x) + \sum_{j=1}^{N} \langle g, w_j \rangle a_j^- e^{\mu_j x},$$

$$M_+g(x) = T(D)g(x) + \sum_{j=1}^{N} \langle g, w_j \rangle a_j^+ e^{\mu_j x},$$

and where T and R_+ are defined in (3.2).

Remark. At first it is not clear that the expression in (3.8) has the desired support properties. To see this when there are no eigenvalues, we first note that by the Paley–Wiener Theorem $\widehat{g}(\lambda)$ (with the convention of (2.10)) is holomorphic for $\operatorname{Im}\lambda > 0$ and satisfies the bound $\langle \lambda \rangle^N e^{-(R+\delta)\operatorname{Im}\lambda}$, $\delta > 0$, there. In view of (3.4) we then have

$$T(\lambda)\widehat{g}(\lambda),\ R_+(\lambda)\widehat{g}(\lambda) = \mathcal{O}(\langle \lambda \rangle^N) e^{(R-\delta)\operatorname{Im}\lambda}, \quad \operatorname{Im}\lambda > 0.$$

The Paley–Wiener Theorem then shows that $\operatorname{supp} T(D)g, \operatorname{supp} R_+(D)g \subset (-R, \infty)$ and

$$\operatorname{supp} \mathbb{1}_{-\infty,-R)}(\bullet) R_+(D) g(\bullet + t) \cap (-\infty, R) \subset \{x : -t - R < x < -R\},$$
$$\operatorname{supp} \mathbb{1}_{(R,\infty)} T(D) g(t - \bullet) \cap (R, \infty) \subset \{R < x < t + R\}.$$

In particular for $t \leq 0$ the scattering terms both vanish. When there are eigenvalues, $-\mu_j^2$, then the entries of $S(\lambda)$ have poles $i\mu_j$, $\mu_j > 0$. A contour deformation then provides the needed cancellation.

Proof. We use the spectral representation of the wave propagator constructed using distorted plane waves, that is solutions to $(D_x^2 + V - \lambda^2)e_\pm = 0$ satisfying

$$e_\pm(x, \lambda) = \begin{cases} T(\lambda) e^{\pm i\lambda x} & \text{for} \quad \pm x > R, \\ e^{\pm i\lambda x} + R_\pm(\lambda) e^{\mp i\lambda x} & \text{for} \quad \pm x < -R. \end{cases} \tag{3.9}$$

Then, for $g \in C_c^\infty(\mathbb{R})$, $\operatorname{supp} g \subset (R, \infty)$,

$$u(t, x) = \sum_{j=1}^{N} (\langle g, w_j \rangle \cosh \mu_j t - \langle g', w_j \rangle \mu_j^{-1} \sinh \mu_j t) w_j(x)$$
$$+ \frac{1}{2\pi} \int_0^\infty \int_{\mathbb{R}} (e_+(x, \lambda)\overline{e_+(y, \lambda)} + e_-(x, \lambda)\overline{e_-(y, \lambda)})(\cos t\lambda - \lambda^{-1} \sin t\lambda \partial_y) g(-y) dy d\lambda.$$

In view of the support properties of g and the form of w_j for $x < -R$ we have

$$\langle g', w_j \rangle = \int g'(x) a_j^- e^{x\mu_j} dx = -\mu_j \int g(x) a_j^- e^{x\mu_j} dx = -\mu_j \langle g, w_j \rangle.$$

Hence the contribution of w_j's to $u(t,x)$ is given by

$$\sum_{j=1}^N e^{t\mu_j}\langle g, w_j\rangle w_j(x) = \sum_{j=1}^N a_j^\pm \langle g, w_j\rangle e^{\mu_j(t\mp x)}, \quad \pm x > R.$$

To see the contribution of the continuous spectrum, we now use the expressions for $\overline{e_\pm(y,\lambda)}$ for $y < -R$:

$$\begin{aligned} E_+g(\lambda,t) &:= \int_{\mathbb{R}} \overline{e_+(y,\lambda)}(\cos t\lambda - \lambda^{-1}\sin t\lambda \partial_y) g(-y)dy = \widehat{g}(\lambda)e^{-i\lambda t} + \overline{R_+(\lambda)}\widehat{g}(-\lambda)e^{i\lambda t}, \\ E_-g(\lambda,t) &:= \int_{\mathbb{R}} \overline{e_-(y,\lambda)}(\cos t\lambda - \lambda^{-1}\sin t\lambda \partial_y) g(-y)dy = \overline{T(\lambda)}\widehat{g}(-\lambda)e^{i\lambda t}. \end{aligned} \tag{3.10}$$

Then for $x < -R$,

$$\begin{aligned} e_+(x,\lambda)E_+g(\lambda,t) &= (e^{i\lambda x} + R_+(\lambda)e^{-i\lambda x})(\widehat{g}(\lambda)e^{-i\lambda t} + \overline{R_+(\lambda)}\widehat{g}(-\lambda)e^{i\lambda t}) \\ &= \widehat{g}(\lambda)e^{i\lambda(x-t)} + |R_+(\lambda)|^2\widehat{g}(-\lambda)e^{i\lambda(t-x)} \\ &\quad + R_+(\lambda)e^{-i\lambda(x+t)}\widehat{g}(\lambda) + R_+(-\lambda)e^{i\lambda(x+t)}\widehat{g}(-\lambda)e^{i\lambda(t-x)}, \\ e_-(x,\lambda)E_-g(\lambda,t) &= |T(\lambda)|^2\widehat{g}(-\lambda)e^{-i\lambda(x-t)}. \end{aligned}$$

Using (3.3) in the expression for $u(t,x)$ gives (3.8) for $x < -R$. Similar arguments give the expression for $x > R$. ☐

3.2. Construction of $u(x,\omega)$ in Theorem 1.

To motivate the construction of $u(x,\omega)$ we recall one way to do it in the case of (3.1). For $V \in L^\infty_{\rm comp}(\mathbb{R};[0,\infty))$, $\operatorname{supp} V \subset (-R,R)$, we consider the resolvent $R_V(\omega) := (D_x^2 + V - \omega^2)^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ which is holomorphic for $\operatorname{Im}\omega > 0$. When $V \not\equiv 0$ (note we assumed that $V \geq 0$), it is defined $L^2_{\rm comp}(\mathbb{R}) \to L^2_{\rm loc}(\mathbb{R})$ for $\operatorname{Im}\omega = 0$. (See [DyZw19, Theorem 2.7].)

For F with $\operatorname{supp} F \subset (-R_1,R_1)$, $R < R_1$,

$$R_V(\omega)F(x) = \begin{cases} A_-(\omega)e^{-i\omega x}, & x < -R_1, \\ A_+(\omega)e^{i\omega x}, & x > R_1, \end{cases} \tag{3.11}$$

which is the meaning of being outgoing for 1D for compactly supported perturbations. (Here $A_\pm$ depend on F and ω.) To construct a solution of $(D_x^2 + V - \omega^2)u = 0$ satisfying (1.6) we choose $\rho \in C^\infty_{\rm c}(\mathbb{R};[0,1])$ supported in $(-R_1,R_1)$, $R_1 > R$, and equal to 1 near the support of V. We then put

$$u(x,\omega) = (1-\rho(x))e^{i\omega x}f(\omega) + R_V(\omega)[D_x^2,\rho](e^{i\bullet\omega}f(\omega))(x),$$

Then (3.11) shows that (1.6) holds. We will mimic this strategy and construct $R_V(\omega)$ now acting on spaces of functions of both x and ω. The key is the existence of R_V in $\mathscr{H}_\alpha$ which will be established in §3.3.

We start by recording some simple facts:

Lemma 5. *For* $R_0(\omega) = (D_x^2 - \omega^2)^{-1}$, $\operatorname{Im} \omega > 0$, *the outgoing resolvent, and* $0 \leq s \leq 2$, *define*

$$\mathscr{R}_s : f(x,\omega) \mapsto \frac{\omega}{(\omega+i)^s} R_0(\omega) f(\bullet, \omega)(x). \tag{3.12}$$

Then, in the notation of (1.5), *for* $\rho \in C_c^\infty(\mathbb{R})$, *and any* $\alpha \in \mathbb{R}$,

$$\begin{gathered} \|\rho \mathscr{R}_s \rho\|_{\mathscr{H}_\alpha \to \mathscr{H}_\alpha^s} \leq C, \\ \rho \mathscr{R}_s \rho f(x,\omega) := \rho(x)\omega(\omega+i)^{-s} R_0(\omega)(\rho(\bullet) f(\bullet,\omega))(x). \end{gathered} \tag{3.13}$$

Proof. This follows immediately from he definition (1.5) and the explicit formula for $R_0(\omega)$ as an operator $L^2_{\mathrm{comp}}(\mathbb{R}) \to H^2_{\mathrm{loc}}(\mathbb{R})$:

$$\frac{\omega}{(\omega+i)^s} R_0(\omega) g(x) = \frac{i}{2(\omega+i)^s} \int_{\mathbb{R}} e^{i\omega|x-y|} f(y) dy.$$

(See [DyZw19, Theorem 2.1] for estimates on $\|\rho R_0 \rho\|_{L^2 \to H^s}$.) ☐

The next lemma gives a compactness result:

Lemma 6. *Suppose that* $A_0 := a(x, D_\omega)(\omega + i\gamma)^{-1}$, $\gamma > 0$, *and the assumptions in* (1.1) *hold. Then, for any* $\alpha \in \mathbb{R}$, $A_0 : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ *defines a bounded operator*

$$A_0 : \mathscr{H}_\alpha \to \mathscr{H}_\alpha, \tag{3.14}$$

and, in the notation of Lemma 5,

$$A_0 \mathscr{R}_0 \rho : \mathscr{H}_\alpha \to \mathscr{H}_\alpha \quad \text{is a compact operator.} \tag{3.15}$$

Proof. To establish (3.14) we need to show that $a(x, D_\omega)$ extends to an operator on $\mathscr{H}_\alpha$. The holomorphy can be seen by applying the Cauchy Riemann operator and integration by parts justified by the rapid decay of $\tau \mapsto \widehat{a(x,\bullet)}(\tau)$ as $\operatorname{Re} \tau \to \infty$.

Since $x \mapsto a(x,t)$ is compactly supported and bounded with values in $C_c^\infty(\mathbb{R})$ (and the support in t is uniformly bounded) it is enough to show that for $b \in C_c^\infty(\mathbb{R})$, $b(D_\omega)$ extends to an operator on $\mathscr{H}_\alpha$, defined in (3.6).

We will need slightly more for (3.15) and with this in mind we show that for any $s \in \mathbb{R}$, $(\omega+i)^s b(D_\omega)(\omega+i)^{-s}$ extends to a bounded operator on $\mathscr{H}_\alpha$.

In fact, recalling (1.4) we have,

$$\begin{aligned} &[(\omega+i)^s b(D_\omega)(\omega+i)^{-s} f](\lambda + i\sigma) \\ &\quad = \frac{1}{2\pi} \int (\lambda + i\sigma + i)^s \hat{b}(\lambda - \omega')(\omega' + i\sigma + i)^{-s} f(\omega' + i\sigma) d\omega', \end{aligned}$$

and, since $\hat{b}$ is rapidly decaying,

$$\sup_{\lambda} \int |\lambda + i\sigma + i|^s |\hat{b}(\lambda - \omega)| |\omega + i\sigma + i|^{-s} d\omega \leq C$$
$$\sup_{\omega} \int |\lambda + i\sigma + i|^s |\hat{b}(\lambda - \omega)| |\omega + i\sigma + i|^{-s} d\lambda \leq C.$$

Thus, by the Schur test for boundedness,

$$\sup_{\sigma>0} e^{-2\sigma\alpha} \int \left| \frac{1}{2\pi} \int (\lambda + i\sigma + i)^s \hat{b}(\lambda - \omega')(\omega' + i\sigma + i)^{-s} f(\omega' + i\sigma) d\omega' \right|^2 d\lambda$$
$$\leq C \sup_{\sigma>0} e^{-2\sigma\alpha} \int |f(\omega' + i\sigma)|^2 d\omega',$$

that is, $(\omega + i)^s b(D_\omega)(\omega + i)^{-s}$ is bounded on $\mathscr{H}_\alpha$. In particular, (3.14) follows.

For compactness, suppose that $\{f_n\}_{n=1}^\infty$ is bounded in $\mathscr{H}_\alpha$. Then, using Lemma 5, $\{\rho \mathscr{R}_s \rho f_n\}_{n=1}^\infty$ is bounded in $\mathscr{H}_\alpha^s$ for for $0 \leq s \leq 2$. Moreover, using the equality

$$(\omega + i)^{\frac{1}{2}} A_0 \mathscr{R}_0 \rho f_n = (\omega + i)^{\frac{1}{2}} a_0(x, D_\omega)(\omega + i)^{-\frac{1}{2}} (\omega + i)(\omega + i\gamma)^{-1} \rho \mathscr{R}_{\frac{1}{2}} \rho f_n,$$

together with the facts that $(\omega + i)^{\frac{1}{2}} a_0(x, D_\omega)(\omega + i)^{-\frac{1}{2}} : \mathscr{H}_\alpha \to \mathscr{H}_\alpha$ (proved above) and $(\omega + i + i\sigma)/(\omega + i\gamma + i\sigma)| \leq 1 + 1/\gamma$, $\omega \in \mathbb{R}$, $\sigma > 0$, we have

$$\sup_n \|(\omega + i)^{\frac{1}{2}} A_0 \mathscr{R}_0 \rho f_n\|_{\mathscr{H}_\alpha} < \infty \tag{3.16}$$

Fixing σ we want to extract a convergent subsequence of $A_0 \mathscr{R}_0 \rho f_{n_{\sigma,k}}(\bullet, \bullet + i\sigma)$ in $L^2_x L^2_\omega$. To apply Rellich's theorem we need improved regularity in x and ω and decay in both. In x the decay comes from compact support property of the cut-off function ρ, in ω from the factor $(\omega + i)^{\frac{1}{2}}$ in (3.16). The regularity improvement in x is a consequence of the boundedness of $\{\rho \mathscr{R}_{1/2} \rho f_n, n \in \mathbb{N}\}$ in $\mathscr{H}_\alpha^{1/2}$ given in Lemma 5. Compact support in D_ω provides smoothness in ω

We conclude that for all $\sigma \geq 0$ (recall that the singularity at $\omega = 0$ is removed in (3.12)), there exists a subsequence $n_{\sigma,k}$ such that

$$e^{-\sigma\alpha} A_0 \mathscr{R}_0 \rho f_{n_{\sigma,k}}(\bullet, \bullet + i\sigma) \overset{L^2_x L^2_\omega}{\to} e^{-\sigma\alpha} g_\sigma.$$

Now, by a diagonal argument, we may find n_k such that

$$e^{-\ell\alpha} A_0 \mathscr{R}_0 \rho f_{n_k}(\bullet, \bullet + i\ell) \overset{L^2_x L^2_\omega}{\to} e^{-\ell\alpha} g_\ell$$

for all $\ell = 0, 1, \ldots$. Since $\zeta \mapsto A_0 \mathscr{R}_0 \rho f_{n_k}(\bullet, \zeta)$ are holomorphic in $\operatorname{Im} \zeta \geq 0$, we have by the Phragmén–Lindelöf principle,

$$\sup_{0<\sigma<j} \|e^{i\alpha(\omega+i\sigma)}[A_0 \mathscr{R}_0 \rho(f_{n_k} - f_{n_{k'}})](x, \omega + i\sigma)\|_{L^2_\omega L^2_x}$$
$$\leq \max_{\sigma \in \{0,j\}} \|e^{i\alpha(\omega+i\sigma)}[A_0 \mathscr{R}_0 \rho(f_{n_k} - f_{n_{k'}})](x, \omega + i\sigma)\|_{L^2_x L^2_x}.$$

We now recall (3.16) to see that $\omega \mapsto (\omega+i\sigma+i)^{\frac{1}{2}}e^{i\alpha(\omega+i\sigma)}A_0\mathscr{R}_0\rho f_n(\omega+i\sigma)$ is uniformly bounded in $L^2_\omega L^2_x$ and hence

$$\sup_{j<\sigma<\infty} \|e^{i(\alpha(\omega+i\sigma)}[A_0\mathscr{R}_0\rho(f_{n_k}-f_{n_j})](x,\omega+i\sigma)\|_{L^2_\omega L^2_x} \leq Cj^{-\frac{1}{2}}$$

Hence, we obtain a Cauchy sequence in $\mathscr{H}_\alpha$, $A_0\mathscr{R}_0\rho f_{n_k} \overset{\mathscr{H}_\alpha}{\to} g$, which concludes the proof. ☐

We now solve $Pv = f$ where P is given in (1.2):

Proposition 7. *For $\alpha \in \mathbb{R}$ and $R_1 > R$, let $f \in \mathscr{H}_\alpha$ and* $\operatorname{supp} f(\bullet, \omega) \subset (-R_1, R_1)$ *for all $\omega \in \mathbb{C}_+$. Then there is a unique $v \in \omega^{-1}\mathscr{H}_\alpha$ such that*

$$(D_x^2 - \omega^2 + A(x))v(x,\omega) = f(x,\omega).$$

Moreover, for any $\rho, \rho_1 \in C_c^\infty(\mathbb{R};[0,1])$ such that $\rho \equiv 1$ near the support of ρ_1 and $\rho_1 \equiv 1$ on a neighbourhood of $(-R_1, R_1)$,

$$v = R_0(\omega)\rho_1(I + A(x)R_0(\omega)\rho)^{-1}\rho_1 f.$$

Proof. We will solve solve

$$(D_x^2 - \omega^2 + A(x))v(x,\omega) = f(x,\omega),$$

so that (3.11) holds for $v(x,\omega)$ and

$$\rho \in C_c^\infty((-R,R);[0,1]) \implies \rho v \in \omega^{-1}\mathscr{H}_a. \tag{3.17}$$

To start we note that

$$(1-\rho)A = 0 \quad \text{and} \quad (I + A(x)R_0(\omega)(1-\rho))^{-1} = (I - A(x)R_0(\omega)(1-\rho)).$$

Hence,

$$\begin{aligned} P = D_x^2 - \omega^2 + A(x) &= (I + A(x)R_0(\omega))(D_x^2 - \omega^2) \\ &= (I + A(x)R_0(\omega)(1-\rho))(I + A(x)R_0(\omega)\rho)(D_x^2 - \omega^2), \end{aligned}$$

and solving $Pu = f = \rho_1 f$ is equivalent to solving

$$(I + A(x)R_0(\omega)\rho)(D_x^2 - \omega^2)v = (I + A(x)R_0(\omega)(1-\rho))\rho_1 f = \rho_1 f. \tag{3.18}$$

To continue, we use the following crucial proposition which will be proved in Section 3.3.

Proposition 8. *For A given in (1.2),*

$$\ker_{\mathscr{H}_\alpha}(I + A(x)R_0(\omega)\rho) = \{0\}. \tag{3.19}$$

Lemma 6 shows that at $A(x)R_0(\omega)\rho : \mathscr{H}_\alpha \to \mathscr{H}_\alpha$ is compact and hence $(I + A(x)R_0(\omega)\rho) : \mathscr{H}_\alpha \to \mathscr{H}_\alpha$ is a Fredholm operator of index 0. Proposition 8 then shows that we have an inverse

$$(I + A(x)R_0(\omega)\rho)^{-1} : \mathscr{H}_\alpha \to \mathscr{H}_\alpha. \tag{3.20}$$

Going back to (3.18) and recalling that $\rho_1 f \in \mathscr{H}_a$ we define

$$v = R_0(\omega)(I + A(x)R_0(\omega)\rho)^{-1}\rho_1 f, \tag{3.21}$$

and notice that

$$(I + A(x)R_0(\omega)\rho)^{-1}\rho_1 f = -A(x)R_0(\omega)\rho(I + A(x)R_0(\omega)\rho)^{-1}\rho_1 f + \rho_1 f.$$

Hence, the support properties of f and A imply that

$$(I + A(x)R_0(\omega)\rho)^{-1}\rho_1 f = \rho_1(I + A(x)R_0(\omega)\rho)^{-1}\rho_1 f,$$

which together with (3.21) completes the proof of the proposition. □

Proof of Theorem 1. **Existence:** Let $f \in (\omega + i)^{-1}H_\alpha$, $R_1 > R$, $\rho, \rho_1 \in C_c^\infty(\mathbb{R})$ with $\rho \equiv 1$ near $[-R, R]$, $\operatorname{supp}\rho_1 \subset (-R_1, R_1)$ and $\rho_1 \equiv 1$ near $\operatorname{supp}\rho$.

Since $f \in (\omega + i)^{-1}H_\alpha$ and $\rho \in C_c^\infty(-R_1, R_1)$,

$$F := f(\omega)[D_x^2, \rho](e^{i\omega x}) \in \mathscr{H}_{a+R_1}.$$

By Proposition 7 there is a unique $v \in \omega^{-1}\mathscr{H}_{a+R_1}$ such that

$$(D_x^2 - \omega^2 + A(x))v = F,$$

and

$$v = R_0(\omega)\rho_1(I + A(x)R_0(\omega)\rho)^{-1}\rho_1 F.$$

Setting $u := (1 - \rho(x))e^{i\omega x}f(\omega) + v$, we have

$$(D_x^2 - \omega^2 + A(x))u = 0.$$

Define

$$Tf(\omega) := f(\omega) + \frac{i}{2\omega}\int e^{-iy\omega}\left[(I + AR_0\rho)^{-1}\left(\rho_1 f(\omega)[D_\bullet^2, \rho](e^{i\omega\bullet})\right)\right](y, \omega)dy, \tag{3.22}$$

$$R_+f(\omega) := \frac{i}{2\omega}\int e^{iy\omega}\left[(I + AR_0\rho)^{-1}\left(\rho_1 f(\omega)[D_\bullet^2, \rho](e^{i\omega\bullet})\right)\right](y, \omega)dy. \tag{3.23}$$

Then Theorem 1 follows from the form of $R_0(\omega)$, the estimates (3.20) and (3.13), the fact that

$$\operatorname{supp}\{x : \exists\, \omega \in \mathbb{C}_+, (x, \omega) \in \operatorname{supp}(I + AR_0\rho)^{-1}\rho_1 F\} \subset (-R_1, R_1),$$

and that $R_1 > R$ is arbitrary.

Uniqueness: It is enough to show that for any $R > 0$, $\alpha > 0$, any solution, $v \in \omega^{-1}\mathscr{H}_\alpha$ to $Pv = 0$ with

$$v(x,\omega) = \begin{cases} g_-(\omega)e^{-i\omega x}, & x < -R \\ g_+(\omega)e^{i\omega x}, & x > R \end{cases}$$

and $g_\pm \in \omega^{-1}\mathscr{H}_\alpha$, satisfies $v \equiv 0$.

To see this we recall that $Pv = 0$ means that

$$(D_x^2 - \omega^2)v = -A(x)v \in (\omega + i)^{-1}\mathscr{H}_\alpha \subset \mathscr{H}_\alpha \tag{3.24}$$

Therefore, since $v(\omega, \bullet)$ is outgoing and $\operatorname{supp} A(\bullet) \subset (-R, R)$,

$$v = -R_0(\omega)A(x)v \tag{3.25}$$

Hence

$$A(x)v = -A(x)R_0(\omega)A(x)v,$$

so that for $\rho \in C_c^\infty$ with $\operatorname{supp}(1-\rho) \cap [-R, R] = \emptyset$,

$$0 = A(x)v + A(x)R_0(\omega)A(x)v = (I + A(x)R_0(\omega)\rho)A(x)v.$$

In view of the inclusion in (3.24), we can now use Proposition 8 sto see that $A(x)v = 0$. Together with (3.25), this implies $v = 0$, completing the proof of uniqueness. ☐

3.3. **Non-existence of purely outgoing solutions.** We will now prove Proposition 8. Suppose that, in the notation of Proposition 8,

$$(I + A(x)R_0(\omega)\rho)w = 0, \quad w \in \mathscr{H}_\alpha. \tag{3.26}$$

Since $\rho w = w$,

$$u := R_0(\omega)w = -R_0(\omega)A(x)R_0(\omega)\rho w, \quad (D_x^2 - \omega^2 + A(x))u = 0. \tag{3.27}$$

Moreover, since

$$R_0(\omega) : \mathscr{H}_\alpha \to \omega^{-1}(\mathscr{H}_\alpha \cap \langle\omega\rangle\mathscr{H}_\alpha^1 \cap \langle\omega\rangle^2\mathscr{H}_\alpha^2) \tag{3.28}$$

by Lemma 5, $u(x, \bullet) \in \mathscr{O}(\mathbb{C}_+)$, and for $\sigma > 0$,

$$\int_{\mathbb{R}} |u(x, \omega + i\sigma)|^2 d\omega < \infty.$$

We want to show that $u(x, \omega + i\sigma) \equiv 0$, $\sigma > 0$ and for that we use a variant of a positive commutator argument. Define

$$\langle u, v\rangle_\sigma := e^{-2\sigma\alpha} \int u(x, \omega + i\sigma)\overline{v(x, \omega + i\sigma)} dx d\omega,$$

$$P_\sigma := D_x^2 - (\omega + i\sigma)^2 + a(x, D_\omega)\frac{\omega + i\sigma}{\omega + i\sigma + i\gamma}. \tag{3.29}$$

Notice that for $u \in \mathscr{H}^2_\alpha$, $P_\sigma(u|_{\operatorname{Im}\omega=\sigma}) = (Pu)|_{\operatorname{Im}\omega=\sigma}$ where the action of P on u is explained in (1.4). For $u \in \omega^{-3/2}\mathscr{H}^2_\alpha$, we compute

$$\begin{aligned}\langle P_\sigma u, (\omega+i\sigma)u\rangle_\sigma &= \langle(\omega-i\sigma)D_x u, D_x u\rangle_\sigma - \langle(\omega^2+\sigma^2)(\omega+i\sigma)u,u\rangle_\sigma \\ &\quad + \langle a(x,D_\omega)\frac{\omega+i\sigma}{\omega+i\sigma+i\gamma}u, (\omega+i\sigma)u\rangle_\sigma.\end{aligned} \tag{3.30}$$

We now take the imaginary part to obtain

$$\begin{aligned}\operatorname{Im}\langle P_\sigma u, (\omega+i\sigma)u\rangle_\sigma = &-\sigma\big(\|D_x u\|_\sigma^2 + \||\omega+i\sigma|u\|_\sigma^2\big) \\ &+ \operatorname{Im}\langle a(x,D_\omega)\frac{\omega+i\sigma}{\omega+i\sigma+i\gamma}u, (\omega+i\sigma)u\rangle_\sigma \\ \leq &-\sigma\big(\|D_x u\|_\sigma^2 + \||\omega+i\sigma|u\|_\sigma^2\big) \\ &+ \|a(x,D_\omega)\|_{\mathscr{H}_\alpha\to\mathscr{H}_\alpha}\Big\|\frac{\omega+i\sigma}{\omega+i\sigma+i\gamma}u\Big\|_\sigma \||\omega+i\sigma|u\|_\sigma \\ \leq &-\sigma(\|D_x u\|_\sigma^2 + \||\omega+i\sigma|u\|_\sigma^2) + C\|u\|_\sigma\||\omega+i\sigma|u\|_\sigma \\ \leq &-\tfrac{1}{2}\sigma^3\|u\|_\sigma^2,.\end{aligned}$$

if σ is large enough.

Since $u = R_0(\omega)w$ with $w \in \mathscr{H}_\alpha$, we use (3.28) to obtain $u \in \omega^{-1}(\mathscr{H}_\alpha \cap \langle\omega\rangle\mathscr{H}^1_\alpha \cap \langle\omega\rangle^2\mathscr{H}^2_\alpha)$. Therefore, to justify multiplication by ω and integration as above, we consider a modified version of (3.30):

$$\begin{aligned}&\langle P_\sigma u, (\omega+i\sigma)\,\mathbb{1}_{|\omega|\leq T}\,u\rangle_\sigma = \langle(\omega-i\sigma)\,\mathbb{1}_{|\omega|\leq T}\,D_x u, D_x u\rangle_\sigma \\ &\quad - \langle\mathbb{1}_{|\omega|\leq T}((\omega^2+\sigma^2)(\omega+i\sigma)u,u\rangle_\sigma + \langle a(x,D_\omega)\frac{\omega+i\sigma}{\omega+i\sigma+i\gamma}u, (\omega+i\sigma)\,\mathbb{1}_{|\omega|\leq T}\,u\rangle_\sigma,\end{aligned}$$

estimate the imaginary part as above to obtain (recall that $P_\sigma u = 0$ from (3.27))

$$0 = \langle P_\sigma u, (\omega+i\sigma)\,\mathbb{1}_{|\omega|\leq T}\,u\rangle_\sigma \leq -\tfrac{1}{2}\sigma^3\|1_{|\omega|\leq T}u\|_\sigma^2.$$

For each σ large enough, sending $T \to \infty$ then implies $u|_{\operatorname{Im}\omega=\sigma} \equiv 0$. Since u is holomorphic in $\operatorname{Im}\omega > 0$, this implies that $u \equiv 0$.

4. Proof of Theorem 2

Let $R_1 > R$ and $\rho \in C_c^\infty((-R_1,R_1))$ with $\rho \equiv 1$ near $[-R,R]$, $\rho_1 \in C_c^\infty((-R_1,R_1))$ with

$$\operatorname{supp}\rho \cap \operatorname{supp}(1-\rho_1) = \emptyset.$$

We then define

$$u_1(t,x) := (1-\rho(x))g(t-x), \quad u_2 := u - u_1.$$

It follows that $u_2 \equiv 0$ for $t \ll -1$ and that

$$\begin{aligned}(D_t^2 - D_x^2 - A)u_2 &= -(D_t^2 - D_x^2 - A)(1-\rho)g(t-x) \\ &= -[D_x^2,\rho]g(t-x) =: f(t,x),\end{aligned}$$

Proposition 2 and (2.11) imply that there is $C > 0$ such that $\|u_2\|_{L^2(\mathbb{R}_x)} \leq Ce^{Ct}$ and hence for $\sigma > 0$ large enough,

$$\|\widehat{u}_2(\omega + i\sigma, x)\|_{L^2_{\omega,x}} < \infty,$$

and there is $\sigma_0 > 0$ such that $\widehat{u}_2$ is holomorphic in $\{\operatorname{Im}\omega > \sigma_0\}$. Moreover, (recall the convention (2.10)) since f is smooth and compactly supported in time, there is $a > 0$ such that $\widehat{f} \in \mathscr{H}_\alpha$. Thus, for $\sigma > 0$ large enough,

$$P_\sigma \widehat{u}_2(\omega + i\sigma, x) = -\widehat{f}(\omega + i\sigma, x),$$

where P_σ is defined in (3.29). Hence, for $\sigma > 0$ large enough,

$$\widehat{u}_2(\omega + i\sigma, x) = -\Big(R_0(\omega + i\sigma)\Big[(I + AR_0\rho)^{-1}\rho_1\widehat{f}\Big](\omega + i\sigma, \bullet)\Big)(x). \tag{4.1}$$

(See (1.2) to (1.4) for a description of the action of A.) By Lemma 5, this implies that for $0 \leq s \leq 2$,

$$\frac{\omega}{(\omega + i)^s}\widehat{u}_2(\omega, x) \in \mathscr{H}^s_\alpha. \tag{4.2}$$

Now, taking the inverse Fourier transform and using (4.1)

$$u_2(t, x) = -\int_{\operatorname{Im} z = \sigma} e^{izt}\Big(R_0(I + AR_0\rho)^{-1}\rho_1\widehat{f}\Big)(z, x)dz.$$

Define

$$\Gamma_{r,\pm,\varepsilon} := \pm r + i[\varepsilon, \sigma], \qquad \Gamma_{\sigma,r} := [-r, r] + i\sigma.$$

First, using (4.2) to see that $(\omega + i\sigma)\widehat{u}_2(\omega + i\sigma, x) \in L^2_{x,\omega}$, we obtain

$$\begin{aligned}&\Big\|\int_{|\omega|>R} e^{-i\omega t + \sigma t}\widehat{u}_2(\omega + i\sigma, x)d\omega\Big\|_{L^2_x}\\ &\qquad \leq e^{\sigma t}\Big(\int_{|\omega|>r}|\omega + i\sigma|^{-2}d\omega\Big)^{1/2}\|(\omega + i\sigma)\widehat{u}_2(\omega + i\sigma, x)\|_{L^2_{x,\omega}} \leq Cr^{-1/2}.\end{aligned}$$

Next, since $(I + AR_0\rho)^{-1} : \mathscr{H}_\alpha \to \mathscr{H}_\alpha$ is bounded and

$$\|R_0(z)\|_{L^2 \to L^2} \leq \frac{1}{|z|\operatorname{Im} z}, \qquad \operatorname{Im} z > 0,$$

we have

$$\sup_{\operatorname{Im} z > 0}\|e^{-a\operatorname{Im} z}|z||\operatorname{Im} z|\big(R_0(I + AR_0\rho)^{-1}\rho_1\widehat{f}\big)(z, \bullet)\|_{L^2(\mathbb{R}_x)} < \infty,$$

and hence

$$\Big\|\int_{\Gamma_{R,\pm,\varepsilon}}\widehat{u}_2(z, \bullet)dz\Big\|_{L^2_x} \leq Cr^{-1}\log\varepsilon^{-1}.$$

Deforming the contour and letting $r \to \infty$, we obtain that for $\varepsilon > 0$

$$u_2(t, x) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(\omega + i\varepsilon)t}R_0(\omega + i\varepsilon)[(I + AR_0\rho)^{-1}\rho_1\widehat{f}](\omega + i\varepsilon)d\omega.$$

Now, sending $\varepsilon \to 0^+$, we obtain

$$u_2(t,x) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\frac{1}{\omega+i0}\omega R_0(\omega)[(I+AR_0\rho)^{-1}\rho_1\widehat{f}\,](\omega,x)d\omega.$$

The integral makes sense as a distributional pairing with $(\omega+i0)^{-1}$ since, by Lemma 5, $\omega R_0(\omega) : \mathscr{H}_{a,\mathrm{comp}} \to \mathscr{H}_{a,\mathrm{loc}}$.

Recalling the definition of f, we have

$$\begin{aligned}\widehat{f}(\omega,x) &= \int_{-\infty}^{\infty} e^{i\omega s}[D^2,\rho]g(s-x)ds = [D^2,\rho]\int_{-\infty}^{\infty} e^{-i\omega(x-s)+i\omega x}g(s-x)ds\\ &= \widehat{g}(\omega)([D^2,\rho]e^{i\omega\bullet})(x).\end{aligned}$$

Thus,

$$u_2(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\frac{1}{\omega+i0}\omega R_0(\omega)(I+AR_0\rho)^{-1}\rho_1\widehat{g}(\omega)([D^2,\rho]e^{i\omega\bullet})d\omega.$$

Therefore, using the definition (3.23) for $x < -R_1$,

$$u_2(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega(t+x)}[R_+\widehat{g}(\bullet)](\omega)d\omega = [\mathscr{R}_+ g](t+x)$$

and, using the definition (3.22) for $x > R_1$,

$$u_2(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega(t-x)}[T\widehat{g}(\bullet) - \widehat{g}(\bullet)](\omega)d\omega = [\mathscr{T} g](t-x) - g(t-x)$$

Hence, $u = u_1 + u_2$ satisfies

$$u(t,x) = \begin{cases} [\mathscr{T} g](t-x) & x > R_1,\\ g(t-x) + [\mathscr{R}_+ g](t+x) & x < -R_1.\end{cases}$$

Since $R_1 > R$ is arbitrary the Theorem follows. □

Appendix by Zhen Huang and Maciej Zworski

A.1. **Mathematical setup.** We consider the following special case of operators defined in §2.2:

$$D_t^2 u - a(x,t)\mathcal{B}D_t u - D_x^2 u = 0, \quad D_t := \tfrac{1}{i}\partial_t, \quad D_x = \tfrac{1}{i}\partial_x. \tag{A.1}$$

where

$$a(x,t) \in L^\infty(\mathbb{R}_x, \mathscr{S}(\mathbb{R}_t)), \quad \operatorname{supp} a(\bullet,t) \subset [-R,R] \tag{A.2}$$

and $\mathcal{B}$ is a memory term:

$$\mathcal{B}v(t,x) := \int_{-\infty}^{t} e^{-\gamma(t-t')}v(t',x)dt', \quad \gamma > 0. \tag{A.3}$$

An example in (A.2) in the spirit of [HGW23] would be

$$a(x,t) = V(x)\exp(-\alpha(t-t_0)^2), \quad \alpha > 0, \quad V(x) = A\,\mathbb{1}_{[-1,1]}.$$

We also allow any step potential:

$$V(x) = \sum_{j=1}^{J} V_j \, \mathbb{1}_{[x_j, x_{j+1}]}, \quad x_1 < x_2 < \cdots < x_{J+1}.$$

As the initial condition we take

$$u(t,x)|_{t\le 0} = g(x-t), \quad g \in C_{\mathrm{c}}^{\infty}((-\infty, -R), \tag{A.4}$$

and in practice we could take

$$g(x) = \exp(-(x-x_0)^2/\sigma)\exp(i\lambda(x-x_0)), \quad 0 < \sigma \ll 1, \quad x_0 \ll -R.$$

We can rewrite (A.1) as a system

$$D_t \mathbf{u} = \begin{pmatrix} 0 & 1 \\ D_x^2 & a(x,t)\mathcal{B} \end{pmatrix} \mathbf{u}, \quad \mathbf{u}(0,x) = \begin{pmatrix} g(x) \\ ig'(x) \end{pmatrix}, \quad \mathbf{u}(t,x) = \begin{pmatrix} u(t,x) \\ D_t u(t,x) \end{pmatrix}. \tag{A.5}$$

The code can be tested in the special case of no memory effect by setting $\gamma = 0$, $\alpha = 0$, and $A = -iV_0$. Then

$$\mathcal{B} D_t u(t,x) = \frac{1}{i} \int_{-\infty}^{t} \partial_t u(t',x) dt' = -iu(t,x).$$

Hence, for our choice of parameters,

$$a(x,t)\mathcal{B} D_t u = -ia(x,t)u(t,x) = V_0 \, \mathbb{1}_{[-1,1]}(x) u(t,x),$$

that is, our equation becomes

$$D_t^2 u - (D_x^2 + V(x))u = 0, \quad V(x) = V_0 \, \mathbb{1}_{[-1,1]}.$$

For this there is a standard code for comparing solutions.

Algorithm 1 Leapfrog scheme for (A.5) with memory term (A.3)

Input: Initial condition $\mathbf{u}_0 = (u_0, v_0)$, time step Δt, number of time steps N, uniform spatial grid points x.
Output: Solution $\mathbf{u}_n$ at times $t_n = n\Delta t$ for $n = 1, \dots, N$
Initialize memory term $\boldsymbol{\mathcal{M}}_0$ using trapezoidal rule from $-\infty$ to 0.
Startup: Compute $\mathbf{u}_2$ from $\mathbf{u}_1$ using forward Euler (one step).
for $n = 2$ to $N-1$ **do**
    Update memory term:

$$\boldsymbol{\mathcal{M}}_n = e^{-\gamma \Delta t} \boldsymbol{\mathcal{M}}_{n-1} + \Delta t v_n.$$

    Leapfrog update:

$$u_{n+1} = u_{n-1} + 2\mathrm{i}\Delta t \, v_n, \quad v_{n+1} = v_{n-1} + 2\mathrm{i}\Delta t \, (D_x^2 u_n + a(x,t_n)\boldsymbol{\mathcal{M}}_n).$$

end for

A.2. **Numerical schemes.** Now we aim to write down a numerical scheme for (A.5), which we rewrite as

$$D_t \mathbf{u} = L\mathbf{u} + a(x,t) \int_{-\infty}^{t} C(t-s)\mathbf{u}(s)ds,$$

where

$$L = \begin{pmatrix} 0 & 1 \\ D_x^2 & 0 \end{pmatrix}, \quad C(t-s) = \begin{pmatrix} 0 & 0 \\ 0 & e^{-\gamma(t-s)} \end{pmatrix}.$$

Let $\mathbf{u}_n(x) = \mathbf{u}(t_n, x)$, where $t_n = n\Delta t$ and Δt is the time step. Due to the simple exponential form of $C(t)$, we use recurrence to update the memory term:

$$\boldsymbol{\mathcal{M}}_n := \int_{-\infty}^{t_n} C(t_n - s)\mathbf{u}(s)ds = e^{-\gamma\Delta t} \int_{-\infty}^{t_{n-1}} C(t_{n-1}-s)\mathbf{u}(s)ds + \int_{t_{n-1}}^{t_n} C(t_n - s)\mathbf{u}(s)ds.$$

For $n = 0$, we evaluate $\boldsymbol{\mathcal{M}}_0 = \int_{-\infty}^{0} C(t_n - s)\mathbf{u}(s)ds$ using the trapezoidal rule. For $n \geq 1$, we have the approximate recurrence relation:

$$\boldsymbol{\mathcal{M}}_n \approx e^{-\gamma\Delta t}\boldsymbol{\mathcal{M}}_{n-1} + \Delta t C(0)\,\mathbf{u}_n,$$

where $\mathbf{u}_n$ is updated using the following leapfrog scheme: for $\mathbf{u}_n = (u_n, v_n)^T$,

$$u_{n+1} = u_{n-1} + 2\mathrm{i}\Delta t\, v_n, \tag{A.6}$$

$$v_{n+1} = v_{n-1} + 2\mathrm{i}\Delta t\,(D_x^2 u_n + a(x,t_n)\boldsymbol{\mathcal{M}}_n), \tag{A.7}$$

where the memory term $\boldsymbol{\mathcal{M}}_n$ is updated via the recurrence relation at each step. A one-step startup procedure using the forward Euler scheme is required to compute $\mathbf{u}_1$ from $\mathbf{u}_0$:

$$u_1 = u_0 + \mathrm{i}\Delta t\, v_0, \tag{A.8}$$

$$v_1 = v_0 + \mathrm{i}\Delta t\,(D_x^2 u_0 + a(x,t_0)\boldsymbol{\mathcal{M}}_0). \tag{A.9}$$

The leapfrog scheme is explicit and has second-order accuracy in time. The overall algorithm is in Algorithm 1.

A.3. **Matlab codes.** This code produces the movie and the graphs shown in Figure 1.

```
function wave_memory(gamma,alpha,V0,x0,lambda)
% Computes a toy wave-like PDE with memory
% Usage: wave_memory(gamma,alpha,V0,x0,lambda)
% If arguments are omitted, reasonable defaults are used.
%
% The code evolves three parameter cases in parallel (three gammas/alphas).
% Subfunction Vnew(x,V0,x0) returns piecewise-constant potential.
% --- Defaults and input sanitizing ---
if (nargin < 1 ) gamma = [0,3,4]; end
if (nargin < 2 ) alpha = [0,2,10]; end
```

```
if (nargin < 3 ) V0 = 10*[40,-5,30]; end
if (nargin < 4 ) x0 = [-0.5,-0.25,0.25,0.5]; end
if (nargin < 5 ) lambda = 10; end
% Ensure alpha and gamma have length at least 3 (pad or truncate)
if numel(alpha) < 3, alpha = [alpha(:).' zeros(1,3-numel(alpha))]; end
if numel(gamma) < 3, gamma = [gamma(:).' zeros(1,3-numel(gamma))]; end
% --- Spatial / temporal grid ---
L = pi;         % domain [-L, L]
T = 3.65;        % final time
dx = 0.01;
dt = 0.005;     % dt should be smaller than dx for stability here
x = -L:dx:L;
t = 0:dt:T;
Nx = length(x);
Nt = length(t);
% Memory parameter and potential
t0 = 2;
V = Vnew(x,V0,x0);
% --- Initial condition (rename centre to avoid colliding with x0 input) ---
y0 = -2.0;
sigma = 0.05;
g_func = @(x1) exp(-((x1 - y0).^2) / sigma) .* exp(1i * lambda * (x1 - y0));
% derivative: ( -2*(x-y0)/sigma + i*lambda ) * g
g_prime = @(x1) ( -2*(x1 - y0)/sigma + 1i*lambda ) .* g_func(x1);
% Preallocate arrays for three cases
u1 = zeros(Nt, Nx); v1 = zeros(Nt, Nx);
u2 = zeros(Nt, Nx); v2 = zeros(Nt, Nx);
u3 = zeros(Nt, Nx); v3 = zeros(Nt, Nx);
u1(1,:) = g_func(x);   v1(1,:) = 1i*g_prime(x);
u2(1,:) = g_func(x);   v2(1,:) = 1i*g_prime(x);
u3(1,:) = g_func(x);   v3(1,:) = 1i*g_prime(x);
% Laplacian (second-order central difference), Dirichlet at boundaries
e = ones(Nx,1);
Dxx = -spdiags([e -2*e e], -1:1, Nx, Nx) / dx^2;
Dxx(1,:) = 0; Dxx(end,:) = 0;
% --- Initialize memory B (discrete convolution with g') ---
s = 0:dt:10;  % integration variable s = x - t' in your approximation
ds = dt;
ws1 = exp(-gamma(1)* s);
ws2 = exp(-gamma(2)* s);
```

```
ws3 = exp(-gamma(3)* s);
B1 = zeros(1, Nx); B2 = B1; B3 = B1;
for j = 1:Nx
    xj = x(j);
    gprime_vals = g_prime( xj + s );  % vector
    B1(j) = sum(ws1 .* gprime_vals) * ds;
    B2(j) = sum(ws2 .* gprime_vals) * ds;
    B3(j) = sum(ws3 .* gprime_vals) * ds;
end
% --- One-step startup (forward Euler) ---
axt1 = 1i * V .* exp(-alpha(1)*(t(1)-t0)^2);
axt2 = 1i * V .* exp(-alpha(2)*(t(1)-t0)^2);
axt3 = 1i * V .* exp(-alpha(3)*(t(1)-t0)^2);
% Step from n=1 to n=2
u1(2,:) = u1(1,:) + 1i * dt * v1(1,:);
v1(2,:) = v1(1,:) + 1i * dt * ((Dxx * (u1(1,:).')).' + axt1 .* B1);
u2(2,:) = u2(1,:) + 1i * dt * v2(1,:);
v2(2,:) = v2(1,:) + 1i * dt * ((Dxx * (u2(1,:).')).' + axt2 .* B2);
u3(2,:) = u3(1,:) + 1i * dt * v3(1,:);
v3(2,:) = v3(1,:) + 1i * dt * ((Dxx * (u3(1,:).')).' + axt3 .* B3);
% --- Video writer and figure setup ---
filename = 'wave_multi_one.mp4';
writerObj = VideoWriter(filename,'MPEG-4');
writerObj.FrameRate = 40;
open(writerObj);
fig = figure;
fig.Position = [100 100 800 600];
tl = tiledlayout(fig, 1, 1);
ax1 = nexttile(tl, 1);
% --- Time stepping (leapfrog) ---
for n = 2:Nt-1
    tn = t(n);
    % update memory B[v] (exponential recurrence)
    B1 = exp(-gamma(1)*dt)*B1  + dt * v1(n,:);
    B2 = exp(-gamma(2)*dt)*B2  + dt * v2(n,:);
    B3 = exp(-gamma(3)*dt)*B3  + dt * v3(n,:);
    axt1 = 1i * V .* exp(-alpha(1)*(tn - t0)^2);
    axt2 = 1i * V .* exp(-alpha(2)*(tn - t0)^2);
    axt3 = 1i * V .* exp(-alpha(3)*(tn - t0)^2);
    % leapfrog updates
```

```
    u1(n+1,:) = u1(n-1,:) + 2i * dt * v1(n,:);
    v1(n+1,:) = v1(n-1,:) + 2i * dt * ((Dxx * (u1(n,:).')).' + axt1 .* B1);
    u2(n+1,:) = u2(n-1,:) + 2i * dt * v2(n,:);
    v2(n+1,:) = v2(n-1,:) + 2i * dt * ((Dxx * (u2(n,:).')).' + axt2 .* B2);
    u3(n+1,:) = u3(n-1,:) + 2i * dt * v3(n,:);
    v3(n+1,:) = v3(n-1,:) + 2i * dt * ((Dxx * (u3(n,:).')).' + axt3 .* B3);
    % normalize potential for plotting (avoid overwriting input V0)
    Vmax = max(abs(V));
    if Vmax == 0, W = V; else W = V / Vmax; end
    % Plot case 1
    axes(ax1); cla(ax1);
plot(ax1, x, real(u1(n,:)), 'c',x,real(u2(n,:)), ...
'r',x,real(u3(n,:)),'b', x, W, 'k', 'LineWidth', 2);
    xlabel(ax1, '$x$','Interpreter','latex','FontSize',20);
    axis(ax1,[min(x) max(x) -1.5 2]);
legend(ax1, ...
    sprintf('$\\alpha=%.1f,\\ \\gamma=%.1f$', alpha(1), gamma(1)), ...
    sprintf('$\\alpha=%.1f,\\ \\gamma=%.1f$', alpha(2), gamma(2)), ...
    sprintf('$\\alpha=%.1f,\\ \\gamma=%.1f$', alpha(3), gamma(3)), ...
    'Interpreter','latex','FontSize',20,'Location','southeast');
title(ax1, ...
      sprintf('${\\rm{Re}} u(x,t),\\ t=%.2f\\quad V(x)/\\max|V|,\\ \\max|V|=%.2f$', ...
            t(n), Vmax), ...
    'Interpreter','latex', ...
    'FontSize', 20);
    drawnow
    F = getframe(gcf);
    writeVideo(writerObj, F);
end
close(writerObj);
end
function V = Vnew(x, V0, x0)
% V(x) = V0(m) for x0(m) < x <= x0(m+1)
if nargin < 2
    V0 = [10 2 20];
end
if nargin < 3
    x0 = [-2 -1 1 2]/5;
end
% number of intervals should satisfy length(x0) = length(V0)+1
```

```
if length(x0) ~= length(V0)+1
    error('x0 must have length length(V0)+1');
end
V = zeros(size(x));
for j = 1:length(V0)
    idx = (x > x0(j)) & (x <= x0(j+1));
    V(idx) = V0(j);
end
end
```

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