ANALYTIC HYPOElliPticITY OF KELDySH OPERATORS

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Abstract. We consider Keldysh-type operators,
\[ P = x_1 D_{x_1}^2 + a(x) D_{x_2} + Q(x, D_{x'}) \]
x = (x_1, x') with analytic coefficients, and with \( Q(x, D_{x'}) \) second order, principally real and elliptic in \( D_{x'} \) for \( x \) near zero. We show that if \( Pu = f \), \( u \in C^\infty \), and \( f \) is analytic in a neighbourhood of 0 then \( u \) is analytic in a neighbourhood of 0. This is a consequence of a microlocal result valid for operators of any order with Lagrangian radial sets. That result proves a generalized version of a conjecture made in [Zw17], [LeZw19] and has applications to scattering theory.

1. Introduction

We consider analytic regularity for generalizations of the Keldysh operator [Ke51],
\[ P := x_1 D_{x_1}^2 + D_{x_2}^2. \] (1.1)
The operator \( P \) has the feature of changing from an elliptic to a hyperbolic operator at \( x_1 = 0 \). It appears in various places including the study of transsonic flows – see for instance Čanić–Keyfitz [CaKe96]. Our interest in such operators comes from the work of Vasy [Va13] where the transition at \( x_1 = 0 \) corresponds to the boundary at infinity for asymptotically hyperbolic manifolds (see [Zw16]), crossing the event horizons of Schwartzschild black holes (see [DyZw19a, §5.7]) or the cosmological horizon for de Sitter spaces. The Vasy operator in the asymptotically hyperbolic setting is given by
\[ P(\lambda) = 4(x_1 D_{x_1}^2 - (\lambda + i) D_{x_1}) - \Delta h(x_1) + i\gamma(x) \left( 2x_1 D_{x_1} - \lambda - i\frac{n-1}{2} \right), \] (1.2)
where \( h(x_1) \) is a smooth family of Riemannian metrics in \( x' \), \( x = (x_1, x') \in \mathbb{R}^n \) and \( \gamma \in C^\infty(\mathbb{R}^n) \). The resonant states at resonant frequencies \( \lambda \) (see [DyZw19a, Chapter 5]) are the smooth solutions of \( P(\lambda)u = 0 \).

For various reasons reviewed in §1.3 it is interesting to ask if in the case of analytic coefficients the resonant states are real analytic across \( x_1 = 0 \). That lead to [Zw17, Conjecture 2] which asked if \( P(\lambda)u = f \) with \( u \) smooth and \( f \) analytic near \( x_1 = 0 \) implies that \( u \) is analytic near \( x_1 = 0 \). For \( \gamma(x) \equiv 0 \) and \( h \) independent of \( x_1 \), this was shown by Lebeau–Zworski [LeZw19] under the assumption that \( \lambda \notin -\mathbb{N}^* \).

The general case was proved by Zuily [Zu17] under the same restriction on \( \lambda \). His proof was an elegant adaptation of the work of Baouendi–Goulaouic [BoGu81], Bolley–Camus [BoCa73] and Bolley–Camus–Hanouzet [BCH74].
In this paper we prove this result for generalized Keldysh operators with analytic coefficients (1.3). In particular, we do not make any assumptions on lower order terms:

**Theorem 1.** Suppose that \( U \subset \mathbb{R}^n \) is a neighbourhood of 0,

\[
P := x_1 D_{x_1}^2 + a(x) D_{x_1} + Q(x, D_{x'}) , \quad x = (x_1, x') \in U,
\]

has analytic coefficients, \( Q(x, D_{x'}) \) is a second order elliptic operator in \( D_{x'} \) with a real valued principal symbol. Then there exists \( U' \subset U, U \cap \{ x_1 = 0 \} \subset U' \), such that

\[
Pu \in C^\omega(U), \quad u \in C^\infty(U) \implies u \in C^\omega(U').
\]

We will show in §1.1 that this result follows from a more general microlocal result valid for operators of all orders satisfying a natural geometric condition.

**Remarks:**

1. In the statement of the theorem \( U' \) can be replaced by \( U \) provided we include a bicharacteristic convexity condition. That follows from propagation of analytic singularities – see [Ma02, Theorem 4.3.7] or [HiSj18, Theorem 2.9.1]: since there are no singularities near \( x_1 = 0 \) there will be no singularities on trajectories hitting \( x_1 = 0 \) – see Figure 1.

2. The result is false for the Tricomi operator

\[
P := D_{x_1}^2 + x_1 D_{x_2}^2.
\]
This can be seen using results about propagation of analytic singularities (unlike (1.3) this operator can be microlocally conjugated to \(D_{y_1} - \text{see Figure 1}\) but is also easily demonstrated by the following example:

\[
u(x) := \int_0^\infty \text{Ai}(\tau^{4/3}x_1)e^{i\tau^2x_2}e^{-\tau}d\tau, \quad Pu = 0, \quad u \in C^\infty(\mathbb{R}^2)\] (1.6)

Here, \(\text{Ai}\) is the Airy function which satisfies

\[\text{Ai}''(t) + t\text{Ai}(t) = 0, \quad |\partial^\ell \text{Ai}(t)| \leq C\ell \langle t \rangle^{\ell - \frac{1}{2}}, \quad t \in \mathbb{R}, \quad \ell \in \mathbb{N}, \quad \text{Ai}(0) > 0.\]

We then have

\[D_{x_2}^k u(0) = \text{Ai}(0) \int_0^\infty \tau^{2k}e^{-\tau}d\tau = \text{Ai}(0)(2k)!\]

and \(u\) is not analytic at 0.

3. Results similar to (1.4) have been obtained in the setting of other operators. In addition to the works [BoCa73],[BCH74] cited above, we mention the work of Baouendi–Sjöstrand [BaSj76] who considered a class of Fuchsian operators generalizing

\[P = |x|^2\Delta + \mu\langle x, D_x \rangle + \lambda\] (1.7)

In the case of (1.7), (1.4) holds for any \(\lambda, \mu \in \mathbb{C}\) and [BaSj76] established (1.4) for more general operators satisfying appropriate conditions.

4. The operators (1.3), (1.5) and (1.7) are not \(C^\infty\) hypoelliptic, that is, \(Pu \in C^\infty \not\Rightarrow u \in C^\infty\). The study of operators which are \(C^\infty\) hypoelliptic but not analytic hypoelliptic has a long tradition with a simple example [HöI, §8.6, Example 2] given by

\[P = D_{x_1}^2 + x_1^2D_{x_2}^2 + D_{x_3}^3.\]

For more complicated cases, references, and connections to several complex variables, see Christ [Ch96] and for some recent progress and additional references, Bove–Mughetti [BoMu17].

1.1. A microlocal result. We make the following general assumptions. Let \(P\) be a differential operator of order \(m\) with analytic coefficients:

\[P := \sum_{|\alpha| \leq m} a_\alpha(x)D_x^\alpha, \quad a_\alpha \in C^\omega(U), \quad p(x, \xi) := \sum_{|\alpha| = m} a_\alpha(x)\xi^\alpha,\] (1.8)

where \(U\) is an open neighbourhood of \(x_0 \in \mathbb{R}^n\). We make the following assumptions valid in a conic neighbourhood of \((x_0, \xi_0) \in T^*\mathbb{R}^n \setminus 0\): \(p\) is real valued and there exists a conic Lagrangian submanifold \(\Lambda\), such that

\[(x_0, \xi_0) \in \Lambda \subset p^{-1}(0), \quad dp|_\Lambda \neq 0, \quad H_p|_\Lambda \parallel \xi \cdot \partial_\xi|_\Lambda.\] (1.9)

Here \(\parallel\) means that the two vector fields are positively proportional, that is the Lagrangian is radial (the positivity assumptions can be achieved by multiplying \(P\) by
\[ \pm 1 \]. Except for the analyticity assumption in (1.8) these are the assumptions made in Haber [Ha14] and Haber–Vasy [HaVa15].

Theorem 1 follows from the following microlocal result. We denote by WF the \( C^\infty \)-wave front set and by WF\(_a\) the analytic wave front set – see [HöI, §8.1] and [HöI, §8.5,9.3], respectively.

**Theorem 2.** Suppose that \( P \) and \((x_0, \xi_0) \in T^* \mathbb{R}^n \setminus 0 \) satisfy the assumptions (1.8) and (1.9). Then for \( u \in \mathcal{D}'(\mathbb{R}^n) \),

\[ (x_0, \xi_0) \notin WF(u), \quad (x_0, \xi_0) \notin WF_a(Pu) \implies (x_0, \xi_0) \notin WF_a(u). \quad (1.10) \]

The proof is based on the theory of microlocal symbolic weights developed by Galkowski–Zworski [GaZw19b] and based on the work of Sjöstrand – see [Sj96, §2] (and also [HeSj86] and [Ma02, §3.5]). With this theory in place we can use escape functions, \( G, H \geq 0 \), which are logarithmically bounded in \( \xi \) (hence the \( C^\infty \) wave front set assumption on \( u \) allows the use of such weights) and which tend to \( \langle \xi \rangle \) in a neighbourhood of \((x_0, \xi_0)\). The normal form for \( p \) constructed in [Ha14] (following much earlier work of Guillemin–Schaeffer [GuSc77] which was based in turn on Sternberg’s linearization theorem [St57]) was helpful in the construction of the specific weights needed here. We indicate the method of the proof in §1.2.

**Proof of Theorem 1.** Under the assumptions of Theorem 1 the characteristic set of \( P \) over \( x_1 = 0 \) is given by (in \( T^* \mathbb{R}^n \setminus 0 \))

\[ p^{-1}(0) \cap \{ x_1 = 0 \} = \{(0, x_2, \xi_1, 0) : \xi_1 \in \mathbb{R} \setminus 0; x_2 \in \text{neigh}_{\mathbb{R}^{n-1}}(0)\} = \Lambda_+ \sqcup \Lambda_-, \]

where \( \pm \xi_1 > 0 \) on \( \Lambda_\pm \). These two components are Lagrangian and conic and \( H_p|_{\Lambda_\pm} = -\xi_1^2 \partial_{\xi_1}|_{\Lambda_\pm} \) is radial. Since \( Pu \in C^\omega(U) \) we have \( WF_a(Pu) \cap \{ x \in U : x_1 = 0 \} = \emptyset \) and hence Theorem 2 shows that \( WF_a(u) \cap \Lambda_\pm = \emptyset \). On the other hand, ([HöI, Theorem 8.6.1]), \( WF_a(u) \cap \{ x_1 = 0 \} \subset p^{-1}(0) \cap \{ x_1 = 0 \} = \Lambda_+ \sqcup \Lambda_- \). Hence \( WF_a(u) \cap \{ x_1 = 0 \} = \emptyset \) and, since \( \text{singsupp}_a u = \pi WF_a(u) \), \( u \) is analytic near \( x_1 = 0 \). \( \square \)

1.2. **A proof in a special case.** To indicate the ideas behind the proof we consider \( P \) given by

\[ P = x_1D_{x_1}^2 + D_{x_2}^2 + aD_{x_1}, \quad a \in \mathbb{C}, \]

and a very special \( u \):

\[ u = e^{ix_2}v(x_1), \quad v \in \mathcal{S}(\mathbb{R}), \quad Pu = e^{ix_2}f(x_1), \quad e^{i\xi_1}|\hat{f}| \in L^2(\mathbb{R}). \quad (1.11) \]

This assumption is a stronger version of the assumption that \( f \) is analytic. We consider a family of smooth functions \( G_\epsilon(\xi_1) \) satisfying

\[ 0 \leq G_\epsilon(\xi_1) \leq \min(\frac{1}{\epsilon} \log(1 + |\xi_1|), |\xi_1|) \quad (1.12) \]
In view of (1.11),
\[ \|v_\epsilon\|_{L^2(\mathbb{R})} \leq C_\epsilon, \quad \|f_\epsilon\|_{L^2(\mathbb{R})} \leq C_0 \quad v_\epsilon := e^{G_\epsilon(D_x)}v, \quad f_\epsilon := e^{G_\epsilon(D_x)}f. \]
where \( C_0 \) is independent of \( \epsilon \). We then consider
\[ P_\epsilon := e^{G_\epsilon(D_x)}(x_1 D_{x_1}^2 + aD_{x_1} + \tau^2)e^{-G_\epsilon(D_x)} = x_1 D_{x_1}^2 + iG'_\epsilon(D_{x_1})D_{x_1}^2 + aD_{x_1} + \tau^2. \]
We have \( P_\epsilon v_\epsilon = f_\epsilon \), and
\[ \text{Im} \langle P_\epsilon v_\epsilon, v_\epsilon \rangle_{L^2(\mathbb{R})} = \langle G'_\epsilon(D_{x_1})D_{x_1}^2 v_\epsilon, v_\epsilon \rangle_{L^2(\mathbb{R})} + \langle (\text{Im} a + 1)D_{x_1}v_\epsilon, v_\epsilon \rangle_{L^2(\mathbb{R})} \]
\[ = \langle (\xi_1^2G'_\epsilon(\xi_1) + (\text{Im} a + 1)\xi_1)\hat{v}_\epsilon, \hat{v}_\epsilon \rangle_{L^2(\mathbb{R} \xi_1)}, \]
where we took \( d\xi_1/(2\pi) \) as the measure on \( L^2(\mathbb{R} \xi_1) \). Let \( \chi \in C^\infty(\mathbb{R}; [0, 1]) \) satisfy \( \chi|_{\xi \leq 1} = 1, \chi|_{\xi \geq 2} = 0 \) and \( \chi' \leq 0 \). We define
\[ G_\epsilon(\xi_1) = (1 - \chi(\xi_1)) \int_0^{\xi_1} (\chi(et) + (1 - \chi(et))(et)^{-1})dt, \]
which satisfies (1.12) and \( G'_\epsilon \geq 0 \). Moreover, for \( \xi_1 \geq M \geq 2 \) and \( \epsilon < 1/M \),
\[ \xi_1^2G'_\epsilon(\xi_1) \geq \xi_1^2\chi(\epsilon\xi_1) + \epsilon^{-1}\xi_1(1 - \chi(\epsilon\xi_1)) \geq M\xi_1. \]
Hence, by taking \( M = \max(-\text{Im} a + 1, 2) \), and \( \epsilon < 1/M \),
\[ \|f_\epsilon\|_H \geq \text{Im} \langle P_\epsilon v_\epsilon, v_\epsilon \rangle = \langle (\xi_1^2G'_\epsilon(\xi_1) + (\text{Im} a + 1)\xi_1)\hat{v}_\epsilon, \hat{v}_\epsilon \rangle \]
\[ \geq \|\hat{v}_\epsilon\|^2 - \|(1 + |\xi_1|(|\text{Im} a| + 1))\hat{v}_\epsilon|_{\xi_1 \leq M}\|\|\hat{v}_\epsilon\| \geq \|\hat{v}_\epsilon\|^2 - C_1\|\hat{v}_\epsilon\|, \]
where \( C_1 := (|\text{Im} a| + 1)e^M\|v\|_{H^1} \) is independent of \( \epsilon \). This implies that
\[ \|\hat{v}_\epsilon\| \leq \|f_\epsilon\| + C_1 \leq C_0 + C_1. \]
Letting \( \epsilon \to 0 \) gives \( \|e^{\xi_1^2}v|_{\xi_1 \geq 0}\| \leq C. \) A similar argument applies to \( \xi_1 \leq 0 \) which shows that
\[ e^{\xi_1^2}\hat{v} \in L^2, \]
and consequently that \( u(x) = e^{ix_1^2}v(x_1) \) is analytic.

In the actual proof, the Fourier transform is replaced by the FBI transform (2.1) and its deformation (2.5) defined using a suitably chosen \( G_\epsilon \) satisfying (1.12) (see Lemma 3.1 which is the heart of the argument). One difficulty not present in the simple one dimensional case is the localization in other variables. It is here that the \( C^\infty \) normal forms of [St57],[GuSc77] and [Ha14] are particularly useful. It is essential that no analyticity is needed in the construction of \( G_\epsilon \).
1.3. Applications to scattering theory. As already indicated in [Zu17] analyticity of smooth solution to the Vasy operator (1.2) implies analyticity of resonant states and of their radiation patterns. We review this here and, in Theorem 3, present a slightly stronger result.

For a detailed presentation of scattering on asymptotically hyperbolic manifolds we refer to [DyZw19a, Chapter 5]. To state Theorem 3, let $M$ be a compact $n + 1$ dimensional manifold with boundary $\partial M \neq \emptyset$ and let $M := \overline{M} \setminus \partial M$. We assume that $\overline{M}$ is a real analytic manifold near $\partial M$. A metric $g$ on $M$ is called asymptotically hyperbolic and analytic near infinity if there exist functions $y' \in C^\infty(M; \partial M)$ and $y_1 \in C^\infty(\overline{M}; (0, 2))$, $y_1|_{\partial M} = 0$, $dy_1|_{\partial M} \neq 0$, such that

$$\overline{M} \supset y_1^{-1}([0, 1)) \ni m \mapsto (y_1(m), y'(m)) \in \{0, 1\} \times \partial M$$

(1.13)

is a real analytic diffeomorphism, and near $\partial M$ the metric has the form,

$$g|_{y_1 \leq \epsilon} = \frac{dy_1^2 + h(y_1)}{y_1^2},$$

(1.14)

where $[0, 1) \ni t \mapsto h(t)$, is an analytic family of real analytic Riemannian metrics on $\partial M$.

Let

$$R_g(\lambda) = (-\Delta_g - \lambda^2 - (n/2)^2)^{-1} : L^2(M, d\text{vol}_g) \rightarrow H^2(M, d\text{vol}_g), \text{ Im } \lambda > 0.$$ 

Mazzeo–Melrose [MM87] and Guillarmou [Gu05] proved that

$$R_g(\lambda) : C^\infty_c(M) \rightarrow C^\infty(M),$$

(1.15)

continues to a meromorphic family of operators for $\lambda \in \mathbb{C} \setminus i(-\frac{1}{2} - N)$. In addition, Guillarmou [Gu05] showed that if the metric is even, that is,

$$g|_{y_1 \leq \epsilon} = \frac{dy_1^2 + h(y_1)}{y_1^2},$$

(1.16)

(see [DyZw19a, Theorem 5.6] for an invariant formulation), then $R_g(\lambda)$ is meromorphic in $\mathbb{C}$. In particular, for $\lambda \neq 0$ we have the following Laurent expansion

$$R_g(\zeta) = \sum_{j=1}^{J(\lambda)} \frac{(-\Delta_g - \lambda^2 - (n/2)^2)j^{-1} \Pi(\lambda)}{(\zeta^2 - \lambda^2)^j} + A(\zeta, \lambda), \quad \Pi(\lambda) := \frac{1}{2\pi i} \oint_{\lambda} R_g(\zeta) 2\zeta d\zeta,$$

where $\zeta \mapsto A(\zeta, \lambda)$ is holomorphic near $\lambda$. For $\lambda = 0$ we have a Laurent expansions in powers of $\zeta^{-j}$.

The operator $\Pi(\lambda)$ has finite rank and its range consists of generalized resonant states. We then have
Theorem 3. Suppose that \((M, g)\) is an even asymptotically hyperbolic manifold (in the sense of (1.16)) analytic near conformal infinity \(\partial M\). Then for \(\lambda \in \mathbb{C} \setminus 0\),

\[
  u \in \Pi(\lambda)C^\infty_c(M) \implies u = y_1^{-i\lambda + \frac{n}{2}}F, \quad F|_{\partial M} \in C^\omega(\partial M).
\]

Moreover, in coordinates of (1.16), \(F(y) = f(y_1^2, y'), y' \in \partial M\) where \(f \in C^\omega((-\delta, \delta) \times \partial M)\).

Proof. The metric (1.14) (in the coordinates valid near the boundary) gives the following Laplace operator:

\[
  -\Delta_y = (y_1D_{y_1})^2 + i(n + y_1\gamma_0(y_1^2, y'))y_1D_{y_1} - y_1^2\Delta_{h(y_1)},
  \gamma_0(t, y') := -\frac{1}{2}\partial_t h(t)/\bar{h}(t), \quad \bar{h}(t) := \det h(t), \quad D := \frac{i}{4}\partial.
\]

(1.18)

Following Vasy [Va13] we change the variables to \(x_1 = y_1^2, x' = y'\) so that

\[
  y_1^{i\lambda - \frac{n}{2}}(-\Delta_y - \lambda^2 - \left(\frac{n}{2}\right)^2)y_1^{-i\lambda + \frac{n}{2}} = x_1P(\lambda),
\]

(1.19)

where, near \(\partial M\), \(P(\lambda)\) is given by (1.2). This operator is considered on \(X := ((-\delta, 0) \times \partial M) \sqcup M\). The key fact is that \(P(\lambda)\) is a Fredholm family operators on suitable spaces, \(P(\lambda)^{-1}\) is meromorphic and its poles can be studied using microlocal methods – see [Va13], [DyZw19a, Chapter 5] and also [Zw16, §2] for a short self-contained presentation.

From meromorphy of \(P(\lambda)^{-1}\) we obtain meromorphy of (1.15) using (1.19):

\[
  R_y(\lambda)f := y_1^{\frac{n}{2} - i\lambda}(P(\lambda)^{-1}y_1^{i\lambda - \frac{n+2}{2}}f)|_M \in C^\infty(M).
\]

(1.20)

Here we make \(y_1^{i\lambda - \frac{n+2}{2}}f\) into an element of \(C^\infty_c(X)\) by extending it by zero outside of \(M\). Near any \(\lambda\), \(P(\zeta)^{-1} = \sum_{k=1}^{\infty} Q_j(\lambda)(\zeta - \lambda)^{-j} + Q_0(\zeta, \lambda)\), with \(Q_j(\lambda)\) operators of finite rank and \(\zeta \mapsto Q_0(\zeta, \lambda)\) is analytic near \(\lambda\). We then have

\[
  \Pi(\lambda) = \frac{1}{2\pi} y_1^{\frac{n}{2} - i\lambda} Q_1(\lambda)y_1^{i\lambda - \frac{n+2}{2}}.
\]

Hence, the claim about the range of \(\Pi(\lambda)\) follows from analyticity of functions in the range of \(Q_1(\lambda)\). This follows from Theorem 1. In fact, \(P(\zeta) = P(\lambda) + (\zeta - \lambda)V, V := -4D_{x_1} + i\gamma(x)\), and hence

\[
  P(\lambda)Q_k(\lambda) = -VQ_{k+1}(\lambda), \quad Q_{K+1}(\lambda) := 0.
\]

Since we already know that the ranges of \(Q_k\)’s are in \(C^\infty\) (see [DyZw19a, (5.6.10)]) we inductively conclude that the ranges are in \(C^\infty\). \(\square\)

Remark. Vasy’s adaptation of Melrose’s radial estimates [Me94] shows that to conclude that \(u \in C^\infty\) when \(P(\lambda)u \in C^\infty\) (see (1.2)), we only need to assume that \(u \in H^{s+1}\) near \(m_0\), where \(s + \frac{1}{2} > -\text{Im} \lambda\), see [Zw16, §4, Remark 3].
2. Preliminaries on FBI transforms and their deformations

We will use the FBI transform defined in [GaZw19b] in its $\mathbb{R}^n$ (rather than $\mathbb{T}^n$) version. Since the weights we use will be compactly supported in $x$ the same theory applies. The constructions there are inspired by the works of Boutet de Monvel–Sjöstrand [BoSj76], Boutet de Monvel–Guillemin [BoGu81], Helffer–Sjöstrand [HeSj86] and Sjöstrand [Sj96]. An alternative approach to using the classes of weights we need here was developed independently and in greater generality by Guedes Bonthonneau–Jézéquel [GuJe20].

2.1. Deformed FBI transforms. We define

$$Tu(x, \xi) := h^{-\frac{3n}{4}} \int_{\mathbb{R}^n} e^{\frac{h}{2}((x-y)\cdot \xi + \frac{i}{2}\langle \xi \rangle (x-y)^2)} \langle \xi \rangle^{-\frac{n}{4}} u(y) dy, \ u \in C_\infty^\infty(\mathbb{R}^n),$$

(2.1)

recalling that the left inverse of $T$ is given by

$$Sv(y) = \frac{2^n h^{-\frac{3n}{2}}}{(2\pi)^{\frac{3n}{2}}} \int_{\mathbb{R}^{2n}} e^{-\frac{h}{2}((x-y)\cdot \xi + \frac{i}{2}\langle \xi \rangle (x-y)^2)} \langle \xi \rangle^{-\frac{n}{4}} (1 + \frac{i}{2} \langle x - y, \xi / \langle \xi \rangle \rangle) v(x, \xi) dx d\xi,$$

(2.2)

see [GaZw19b, Proposition 2.2].

The first fact we need is the characterization of Sobolev spaces and of the $C^\infty$ wave front set using the FBI transform (2.1). To formulate it we use semiclassical Sobolev spaces $H^s_h$ (see for instance [Zw12, §7.1] or [DyZw19a, Definition E.18]) but we should in general think of $h$ as being fixed.

Proposition 2.1. There exists a constant $C$ such that for $u \in \mathcal{S}'(\mathbb{R}^n),$

$$\|u\|_{H^s_h} \leq C \|\langle \xi \rangle^s Tu\|_{L^2(T^*\mathbb{R}^n)} \leq C^2 \|u\|_{H^s_h},$$

(2.3)

Moreover,

$$(x_0, \xi_0) \notin \text{WF}(u) \iff \exists \chi \in S^0(T^*\mathbb{R}^n), \chi \equiv 1 \text{ in a conic neighbourhood of } (x_0, \xi_0), \forall N \exists C_N \|\langle \xi \rangle^N T u\|_{L^2(T^*\mathbb{R}^n)} \leq C_N.$$

Proof. This follows from the characterization of the $H^s$ based wave front sets in Gérard [Gé90] as stated in [De, Theorem 1.2]. Since the arguments are similar to the more involved analytic case presented in Proposition 2.3 we omit the details. \qed

As in [Sj96, §2] and [GaZw19b, §3] we introduce a geometric deformation of $\mathbb{R}^{2n},$ $\Lambda = \Lambda_G:

$$\Lambda := \{(x - iG(x, \xi), \xi + iG(x, \xi)) \mid (x, \xi) \in \mathbb{R}^{2n}\} \subset \mathbb{C}^{2n},$$

$$\text{supp} G \subset K \times \mathbb{R}^n, \ K \in \mathbb{R}^n,$$

$$\sup_{|\alpha| + |\beta| \leq 2} \langle \xi \rangle^{-1 + |\beta|} |\partial_\xi^\alpha \partial_x^\beta G(x, \xi)| \leq \epsilon_0, \ |\partial_\xi^\alpha \partial_x^\beta G(x, \xi)| \leq C_{\alpha \beta} \langle \xi \rangle^{1 - |\beta|},$$

(2.4)
where $\epsilon_0$ is small and fixed (so that the constructions below remain valid as in [GaZw19b]). For convenience, we change here the convention from [GaZw19b]: it amounts to replacing $G$ by $-G$ everywhere.

This provides us with the following new objects: the deformed FBI transform (see [GaZw19b, §4]),

$$T_\Lambda u(x, \xi) := T u(x - iG_\xi(x, \xi), \xi + iG_x(x, \xi)), \ u \in \mathcal{B}_\delta,$$

(2.5)

the spaces $H^s_\Lambda$, defined as in [GaZw19b, §4],

$$H^s_\Lambda := \mathcal{B}_0^{\frac{1}{2} \| \cdot \|^s_{\Lambda}}, \ ||u||^2_{H^s_\Lambda} := \int_\Lambda \langle \text{Re} \alpha \xi \rangle^{2s} |T_\Lambda u(\alpha)|^2 e^{-2H(\alpha)/h} d\alpha,$$

(2.6)

and the orthogonal projector

$$\Pi_\Lambda : L^2(\Lambda, e^{-2H(\alpha)/h} d\alpha) \to T_\Lambda H_\Lambda, \ H_\Lambda := H^0_\Lambda,$$

described asymptotically (as $h \to 0$ and as $\xi \to \infty$) in [GaZw19b, §5]. The weight $H$ appears naturally in this subject and is given by [GaZw19b, (3.3),(3.4)] i.e. $H(x, \xi) = \xi \cdot G_\xi(x, \xi) - G(x, \xi)$. The deformed FBI transform $T_\Lambda$ has an exact left inverse $S_\Lambda$ obtained by deforming $S$ in (2.2).

We now prove a slightly modified version of [GaZw19b, Proposition 6.2]:

\textbf{Proposition 2.2.} Suppose that $P = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$ is a differential operator with $a_\alpha \in C^\infty_c(\mathbb{R}^n)$ satisfying,

$$a_\alpha \in C^\omega(U), \ K \subset U,$$

for an open set $U$ and $K$ as in (2.4). Then

$$\Pi_\Lambda T_\Lambda h^m PS_\Lambda = \Pi_\Lambda b_P \Pi_\Lambda + O(h^\infty)_{H^-N \to H^N},$$

where

$$b_P(x, \xi) \sim \sum_{j=0}^\infty h^j b_j(x, \xi), \ b_j \in S^{m-j}(\mathbb{R}^{2n}),$$

(2.7)

$$b_0 = p|_\Lambda := p(x - iG_\xi(x, \xi), \xi + iG_x(x, \xi)).$$

We remark that the expansion remains valid when $h$ is fixed. We can use smallness of $h$ to dominate the lower order terms and then keep it fixed.

\textbf{Proof.} The result follows from the analogue of [GaZw19b, Lemma 6.1] where the operator $T_\Lambda h^m PS_\Lambda$ is described in the case where the coefficients of $P$ are globally analytic. Here we point out that the analyticity of the coefficients is only needed in the neighbourhood $U$ of $K \subset \mathbb{R}^n$ such that in (2.4) supp $G \subset K \times \mathbb{R}^n$ and $\epsilon_0$ is small enough depending on the size of the complex neighbourhood to which the coefficients extend holomorphically.
In fact, arguing as in the proof of [GaZw19b, Proposition 6.2] all we need is that for \( a \in C_c^\infty(\mathbb{R}^n) \) and \( a \in C^\infty(U) \), the Schwartz kernel of \( T_\Lambda M_a S_\Lambda \), \( M_a f(x) := a(x) f(x) \), is given by
\[
K_a(\alpha, \beta) = c_0 h^{-n} e^{\frac{i}{h} \Psi(\alpha, \beta)} A(\alpha, \beta) + r(\alpha, \beta), \quad \alpha, \beta \in \Lambda = \Lambda_G,
\]
where the amplitude satisfies
\[
r(\alpha, \beta) \text{ is the kernel of an operator } R = O(h^\infty) : H^N_\Lambda \to H^N_\Lambda.
\]

The phase in (2.8) is given by
\[
\Psi(\alpha, \beta) = \frac{i}{2} \frac{(\alpha_\xi - \beta_\xi)^2}{\langle \alpha_\xi \rangle + \langle \beta_\xi \rangle} + \frac{i}{2} \frac{\langle \beta_\xi \rangle (\alpha_x - \beta_x)^2}{\langle \alpha_\xi \rangle + \langle \beta_\xi \rangle} + \frac{\langle \beta_\xi \rangle \alpha_\xi + \langle \alpha_\xi \rangle \beta_\xi}{\langle \alpha_\xi \rangle + \langle \beta_\xi \rangle} \cdot (\alpha_x - \beta_x), \quad (2.9)
\]
and the amplitude satisfies
\[
A \sim \sum_{j=0}^\infty h^j \langle \alpha_\xi \rangle^{-j} A_j, \quad A_0(\alpha, \alpha) = a|_\Lambda(\alpha),
\]
and \( A_j \) are supported in a small conic neighbourhood of the diagonal in \( \Lambda \times \Lambda \). We note that if \( \epsilon_0 \) is small enough, \( a \) extends to some neighbourhood of \( K \) in \( \mathbb{C}^n \) and hence \( a|_\Lambda = a(x - iG\xi(x, \xi)) \) is well defined.

To see (2.8) we use the definitions of \( T_\Lambda \) and \( S_\Lambda \) to write
\[
K_a(\alpha, \beta) = c_n \langle \beta_\xi \rangle^\frac{n}{2} \langle \alpha_\xi \rangle^\frac{n}{2} h^{-\frac{3n}{2}} \int e^{\frac{i}{h} (\varphi_G(\alpha,y) + \varphi_G^*(\beta,y))} a(y) \left(1 + \frac{\langle \beta_x - y, \beta_\xi \rangle}{\langle \beta_\xi \rangle}\right) dy,
\]
where
\[
\varphi_G(\alpha, y) = \Phi(z, \zeta, \gamma)|_{z=\alpha_x, \zeta=\alpha_\xi, \gamma = \alpha_\xi}, \quad \varphi_G^*(\alpha, y) = -\Phi(z, \zeta, \gamma)|_{z=\alpha_x, \zeta=\alpha_\xi},
\]
\[
\alpha_x = x - iG\xi(x, \xi), \quad \alpha_\xi = \xi + iG_x(x, \xi), \quad (2.11)
\]
\[
\Phi(z, \zeta, \gamma) = \langle z - y, \zeta \rangle + \frac{i}{2} \langle \gamma \rangle (z - y)^2, \quad \Phi(z, \zeta, \gamma) := \Phi(z, \zeta, \gamma).
\]

Let \( V, V_1 \) open such that \( K \subset V_1 \subset V \subset U \). We start by showing that the contribution to \( K_a \) away from the diagonal is negligible. For that let \( \chi \in C_c^\infty(\mathbb{R}) \) with \( \chi \equiv 1 \) near 0. Then for all \( \delta > 0 \) small enough, the operator \( R_1 \) with kernel
\[
R_1(\alpha, \beta) = K_a(\alpha, \beta) \tilde{\chi}_\delta(\alpha, \beta),
\]
satisfies \( R_1 = O_{H^N_\Lambda \to H^N_\Lambda}(h^\infty) \). This amounts to showing that the operator with kernel
\[
R_1(\alpha, \beta) e^{\frac{i}{h} (H(\beta) - H(\alpha))} \langle \alpha_\xi \rangle^N \langle \beta_\xi \rangle^N
\]
is bounded on \( L^2(\mathbb{R}^n) \) with \( O(h^\infty) \) norm.

To see this, we first integrate by parts \( K \) times in \( y \), using that
\[
|\partial_y \Psi| = |\beta_\xi - \alpha_\xi + i(\langle \alpha_\xi \rangle (y - \alpha_x) + \langle \beta_\xi \rangle (y - \beta_x))| \geq c(1 + |\alpha_\xi| + |\beta_\xi|)
\]
on \text{supp} \( \tilde{\chi}_\delta \). This reduces the analysis to the case of (2.10) with \( a \) is replaced by \( b(\cdot, \alpha, \beta) \in C^\infty(U) \cap C_c^\infty(\mathbb{R}^n) \) with \( |b| \leq h^K(|\alpha_\xi| + |\beta_\xi|)^{-K} \).
Next, we choose \( \psi \in C_c^\infty(\mathbb{R}^n; [0, 1]) \) with \( \psi \equiv 1 \) on \( V \) and \( \text{supp} \, \psi \subset U \), and \( \psi_1 \in C_c^\infty(\mathbb{R}^n; [0, 1]) \) with \( \psi_1 \equiv 1 \) on \( V_1 \) and \( \text{supp} \, \psi_1 \subset V \). We then deform the contour

\[
y \mapsto y + i\epsilon \psi(y) \frac{\beta_x - \alpha_x}{|\beta_x - \alpha_x|}.
\]

This contour deformation is justified since \( a \in C^\infty(U) \). The phase in the integrand of (2.10) becomes

\[
\Psi = \langle \alpha_x - y, \alpha_\xi \rangle + \langle y - \beta_x, \beta_\xi \rangle + \frac{i\langle \alpha_\xi \rangle}{2} (\alpha_x - y)^2 + \frac{i\langle \beta_\xi \rangle}{2} (\beta_x - y)^2 \\
+ i\epsilon \psi(y) \frac{|\beta_x - \alpha_\xi|^2}{|\beta_x - \alpha_x|} + \frac{i\langle \alpha_\xi \rangle}{2} \left[ 2\epsilon \psi(y) \langle \alpha_x - y, \frac{\alpha_\xi - \beta_x}{|\beta_x - \alpha_x|} \rangle - \epsilon^2 \psi^2(y) \frac{|\beta_x - \alpha_\xi|^2}{|\beta_x - \alpha_x|^2} \right] \\
+ \frac{i\langle \beta_\xi \rangle}{2} \left[ 2\epsilon \psi(y) \langle \beta_x - y, \frac{\alpha_\xi - \beta_x}{|\beta_x - \alpha_x|} \rangle - \epsilon^2 \psi^2(y) \frac{|\beta_x - \alpha_\xi|^2}{|\beta_x - \alpha_x|^2} \right].
\]

In particular, for \( y \in V \), and \( (\alpha, \beta) \in \text{supp} \, \tilde{\chi}_\delta \), the integrand is bounded by

\[
e^{-c(\langle \alpha_\xi \rangle + \langle \beta_\xi \rangle) / (\alpha_x - \beta_x)} / h
\]

which is negligible (even after multiplication by \( e^{\frac{1}{h}(H(\beta) - H(\alpha))} \langle \alpha_\xi \rangle^N \langle \beta_\xi \rangle^N \)).

For the integral over \( y \notin V \), we consider three cases. First, if both \( \text{Re} \, \alpha_x \in K \) and \( \text{Re} \, \beta_x \in K \), then it is easy to see that the integrand is bounded by

\[
e^{-c(\langle \alpha_\xi \rangle + \langle \beta_\xi \rangle) / (\alpha_x - \beta_x) + |y|} / h
\]

and hence produces a negligible contribution. Next, if \( \text{Re} \, \alpha_x \notin K \) and \( \text{Re} \, \beta_x \notin K \), then \( H(\alpha) = H(\beta) = 0 \), \( \alpha, \beta \) are real, and integration by parts in \( y \) shows that the contribution is negligible.

Finally, we consider the case \( \text{Re} \, \alpha_x \in K \), \( \text{Re} \, \beta_x \notin K \), (the case \( \text{Re} \, \beta_x \in K \) and \( \text{Re} \, \alpha_x \notin K \) being similar). In this case, we have \( H(\beta) = 0 \) and \( \beta \) real. Since \( y \notin V \), we have that the integrand is bounded by \( e^{-c(\langle \alpha_\xi \rangle \langle \alpha_x - y \rangle) / h} h^{K(\beta_x) - K} \) and hence this term is also negligible.

Since \( R \) is negligible, we may assume from now on that

\[
|\alpha_x - \beta_x| \ll 1 \quad \text{and} \quad |\alpha_\xi - \beta_\xi| \ll \langle \langle \alpha_\xi \rangle \rangle + \langle \langle \beta_\xi \rangle \rangle.
\]

In particular, there are three cases: \( \text{Re} \, \alpha_x \in K \) and \( \text{Re} \, \beta_x \in V_1 \), \( \text{Re} \, \beta_x \in K \) and \( \text{Re} \, \alpha_x \in V_1 \), or \( \text{Re} \, \alpha_x \notin K \) and \( \text{Re} \, \beta_x \notin K \).

The first two cases are similar, so we consider only one of them. Since \( \text{Re} \, \alpha_x \in K \) and \( \text{Re} \, \beta_x \in V_1 \), the contribution from \( y \notin V \) is negligible. Therefore, we may deform the contour to

\[
y \mapsto y + \psi(y) y_c(\alpha, \beta), \quad y_c(\alpha, \beta) = \frac{i(\beta_x - \alpha_x) + \langle \alpha_\xi \rangle \alpha_x + \langle \beta_\xi \rangle \beta_x}{\langle \alpha_\xi \rangle + \langle \beta_\xi \rangle}.
\]

The proof in this case then follows from the method of complex stationary phase.
When, both $\Re \alpha_x \notin K$ and $\Re \beta_x \notin K$, $\alpha = \Re \alpha$, $\beta = \Re \beta$, and $H(\alpha) = H(\beta) = 0$. In order to handle this situation, we will Taylor expand $a(y)$ around $y = \alpha_x$. For that we first consider (2.10) with $a = O(|y - \alpha_x|^{2N})$. In that case, we consider the integral
\[ K_N(\alpha, \beta) := h^{-\frac{3n}{2}} \int e^{\frac{i}{h}((\alpha_x - y, \alpha_x) + \frac{i}{2}((\alpha_x - y)^2 + (\beta_x - y)^2))} O(|y - \alpha_x|^{2N}) \langle \alpha_x \rangle^{\frac{n}{2}} \beta_x^{\frac{n}{2}} (1 - \bar{\chi}_\delta(\alpha, \beta))dy. \] (2.12)
Changing variables $y \mapsto y + \alpha_x$,
\[ |K_N(\alpha, \beta)| \leq \int \langle \alpha_x \rangle^{\frac{n}{2}} \beta_x^{\frac{n}{2}} h^{N - \frac{3n}{2}} \langle \alpha_x \rangle^{\frac{n}{2}} N e^{-\frac{(\beta_x - \alpha_x - y)^2}{h}} (1 - \bar{\chi}_\delta)dy \leq C \frac{h^{N - n}}{(\langle \alpha_x \rangle + \langle \beta_x \rangle)^N} e^{-\frac{(\alpha_x - \beta_x)^2}{h}} (1 - \bar{\chi}_\delta(\alpha, \beta)). \]
Therefore, using the Schur test for boundedness, the operator $K_N$ with kernel $K_N(\alpha, \beta)$ satisfies
\[ K_N = O(h^{-\frac{3n}{2}}) : H^{N^+ \frac{3n}{2} + 0} \rightarrow H^{N^{-} - \frac{3n}{2} - 0}. \]
Now, observe that for any $N > 0$,
\[ a(y) = a_N(y) + O(|y - \alpha_x|^{2N}) \]
where $a_N(y)$ is a polynomial of order $2N - 1$ in $(y - \alpha_x)$. In particular,
\[ K_a(\alpha, \beta) = K_{a_N}(\alpha, \beta) + K_N(\alpha, \beta) \]
Since $a_N$ is analytic and the integrand is exponentially decaying in $y$, we may deform the contour with $y \mapsto y + y_c(\alpha, \beta)$ in the integral forming the kernel of $K_{a_N}$ and apply complex stationary phase as in the case where $\Re \alpha_x \in K$ or $\Re \beta_x \in K$. This finishes the proof of the proposition after taking $N$ large enough. \(\square\)

2.2. Analytic wave front set. We now relate weighted estimates to analyticity.

**Proposition 2.3.** Let $T$ be the FBI transform defined in (2.1) for some fixed $h$, and let $\psi \in S^1(T^*\mathbb{R}^n)$ satisfy
\[ |\psi(x, \xi)| \geq |\xi|/C, \ (x, \xi) \in U \times \Gamma, \] (2.13)
where $U \subset \mathbb{T}^n$ and $\Gamma \subset \mathbb{R}^n \setminus 0$ is an open cone. Then, for $u \in H^{-N}(\mathbb{R}^n)$,
\[ e^{\psi}(\xi)^{-N}Tu \in L^2(T^*\mathbb{R}^n) \implies \WF_a(u) \cap (U \times \Gamma) = \emptyset. \] (2.14)
Conversely, suppose $u \in H^{-N}(\mathbb{R}^n)$, $\Gamma_0 \subset \mathbb{R}^n$ is a conic open set such that $\Gamma_0 \cap S^{n-1} \subset \Gamma \cap S^{n-1}$, $U_0 \subset U$. Then for any $\psi \in S^1(\mathbb{R}^n \times \mathbb{R}^n)$ with $\text{supp} \psi \subset U_0 \times \Gamma_0$,
\[ \WF_a(u) \cap (U \times \Gamma) = \emptyset \implies \exists \theta > 0 \langle \xi \rangle^{-N} e^{\theta \psi}Tu \in L^2(T^*\mathbb{R}^n). \] (2.15)
Remark: Here we do not consider uniformity in \( h \) in the \( L^2 \) bounds. If we demanded that, than we would only need \( \psi \in C_c^\infty(T^*\mathbb{R}^n) \), \( \psi > 0 \) on \( U \times (\Gamma \cap \mathbb{S}^{n-1}) \).

The proof is based on the following

Lemma 2.4. Let \( T \) and \( S \) be given by (2.1) and (2.2), respectively, with \( h \) fixed. Suppose that \( \chi, \tilde{\chi} \in \mathcal{S}^0(\mathbb{R}^n \times \mathbb{R}^n) \) and \( \text{supp} \chi, \text{supp} \chi_1 \subset K \times \mathbb{R}^n, K \subset \mathbb{R}^n \). Then for any \( a > 0 \) there exists \( b > 0 \) such that

\[
\chi e^{b(\xi)}TS\chi_1 e^{-a(\xi)} = \mathcal{O}_N(1) : L^2(\mathbb{R}^{2n}) \to H^N(\mathbb{R}^{2n}),
\]

for any \( N \).

If in addition \( \chi_1 \equiv 1 \) on a a conic neighbourhood of the support of \( \chi \), then there exists \( b > 0 \) such that

\[
\chi e^{b(\xi)}TS(1 - \chi_1)\langle \xi \rangle^M = \mathcal{O}_{N,M}(1) : L^2(\mathbb{R}^{2n}) \to H^N(\mathbb{R}^{2n}),
\]

for any \( N \).

Proof. We analyse the Schwartz kernel of the operator in (2.16), \( K(x, \xi, y, \eta) \). As in the proofs of [GaZw19b, Lemma 2.1, Proposition 4.5] (the phase of resulting operator can be computed by completion of squares and is given by [GaZw19b, (4.10)] with \( \Lambda = T^*\mathbb{R}^n \)) we see that

\[
|\langle hD_{x,\xi}^a K(x, \xi, y, \eta) \rangle| \leq C_{\alpha} e^{b(\xi) - a(\eta) - \psi(x,\xi,y,\eta)},
\]

\[
\psi := c(\langle \xi \rangle + \langle \eta \rangle)^{-1} \left( |\xi - \eta|^2 + \langle \xi \rangle \langle \eta \rangle |x - y|^2 \right).
\]

We have

\[
b < \frac{1}{8} \min(a, c) \Rightarrow b \langle \xi \rangle - a \langle \eta \rangle - c(\langle \xi \rangle + \langle \eta \rangle)^{-1}|\xi - \eta|^2 \leq -\frac{1}{2} (b \langle \xi \rangle + a \langle \eta \rangle),
\]

if \( b \) is sufficiently small. (By taking \( b < a/8 \) we can assume that \( |\eta| \leq |\xi|/2 \). But then \( |\xi - \eta| \geq \frac{1}{2} |\xi| \) and \( \langle \xi \rangle + \langle \eta \rangle \leq 2 \langle \eta \rangle \).) This proves (2.16) as we can use the Schur criterion.

To see (2.17) we note that we can now assume that \( |\xi/\langle \xi \rangle - \eta/\langle \eta \rangle| > \delta \) or \( |x - y| > \delta \). But then if the kernel of the operator in (2.17) is given by \( K_M(x, \xi, y, \eta) \) where

\[
|\langle hD_{x,\xi}^a K_M(x, \xi, y, \eta) \rangle| \leq C_{\alpha, N} e^{b(\xi) - M \log(\eta) - \psi(x,\xi,y,\eta)}.
\]

Now, fix \( 0 < \delta < 1 \) small. Then, when \( |\xi/\langle \xi \rangle - \eta/\langle \eta \rangle| > \delta \) or \( |x - y| > \delta \),

\[
|\xi - \eta|^2 + \langle \xi \rangle \langle \eta \rangle |x - y|^2 \geq \frac{\delta^2}{16} (\langle \xi \rangle + \langle \eta \rangle)^2.
\]

To see this, observe that on

\[
\left| \frac{\langle \xi \rangle - \langle \eta \rangle}{\langle \xi \rangle + \langle \eta \rangle} \right| \geq \frac{\delta}{4},
\]
we have
\[ \frac{\delta}{4} \leq \left|\frac{\langle \xi \rangle^2 - \langle \eta \rangle^2}{\langle \xi \rangle + \langle \eta \rangle}\right| \leq \frac{\|\xi - \eta\|}{\langle \xi \rangle + \langle \eta \rangle}. \]

On the other hand, when
\[ \left|\frac{\langle \xi \rangle - \langle \eta \rangle}{\langle \xi \rangle + \langle \eta \rangle}\right| \leq \frac{\delta}{4}, \]
we have
\[ \frac{2\langle \xi \rangle\langle \eta \rangle}{\langle \xi \rangle + \langle \eta \rangle} = \frac{\langle \xi \rangle + \langle \eta \rangle}{2} \left(1 - \left[\frac{\langle \xi \rangle - \langle \eta \rangle}{\langle \xi \rangle + \langle \eta \rangle}\right]^2\right) \geq \frac{1}{4} (\langle \xi \rangle + \langle \eta \rangle) \]
Therefore, if \(|x - y| \geq \delta\), (2.19) follows. If instead, \(|\xi/\langle \xi \rangle - \eta/\langle \eta \rangle| \geq \delta\), then
\[ \frac{\|\xi - \eta\|}{\langle \xi \rangle + \langle \eta \rangle} \geq \frac{\delta}{2} \left[\frac{\|\xi - \eta\|}{\langle \xi \rangle + \langle \eta \rangle}\right] \geq \frac{\delta}{4} \]
and (2.19) follows.

From (2.19), we have that there is \(C_{M,\delta} > 0\) such that if \(|\xi/\langle \xi \rangle - \eta/\langle \eta \rangle| > \delta\) or \(|x - y| > \delta\),
\[ b(\langle \xi \rangle - c(\langle \xi \rangle + \langle \eta \rangle))^{-1} (|\xi - \eta|^2 + \langle \xi \rangle \langle \eta \rangle |x - y|^2) + M \log \langle \eta \rangle \]
\[ \leq b(\langle \xi \rangle - \frac{1}{64} c\delta^2 (\langle \xi \rangle + \langle \eta \rangle) - \frac{1}{2} c(\langle \xi \rangle + \langle \eta \rangle))^{-1} (|\xi - \eta|^2 + \langle \xi \rangle \langle \eta \rangle |x - y|^2) + C_{M,\delta}, \]
and the Schur criterion and gives (2.17) for \(b \leq \frac{c\delta^2}{64}\). \(\square\)

**Proof of Proposition 2.3.** We start by recalling the characterization of the analytic wave front set using the standard FBI/Bargmann–Segal transform:
\[ \mathcal{F} u(x, \xi; h) := c_n h^{-\frac{3n}{2}} \int_{\mathbb{R}^n} e^{\frac{i}{h} r(x-y, \xi) + \frac{i}{h} (x-y)^2} u(y) dy, \quad u \in \mathcal{S}'(\mathbb{R}^n). \]

Then
\[ (x_0, \xi_0) \notin \WF_a(u) \iff \left\{ \exists \delta, U = \text{neigh}( (x_0, \xi_0) ) \right. \left. \quad |\mathcal{F} u(x, \xi, h)| \leq C e^{-\delta/h}, \quad (x, \xi) \in U, \quad 0 < h < h_0. \right. \quad (2.20) \]
see [Höl, Theorem 9.6.3] for a textbook presentation; note the somewhat different convention: \(\mathcal{F} u(x, \xi; h) = e^{-\frac{i}{h} \xi^2 T_1/h} u(x - i\xi).\)

We first prove (2.14). Hence suppose that \((x_0, \xi_0) \in U \times \Gamma\). Let \(\chi \in S^0\) be supported in a small conic neighbourhood, \(U_0 \times \Gamma_0\), of \((x_0, \xi_0)\) and choose \(\chi_1 \in S^0\) which is supported in \(U \times \Gamma\) and is equal to 1 on a conic neighbourhood of the support of \(\chi\) and \(\chi_2 \in S^0\) supported in \(U \times \Gamma\) and equal to 1 on a conic neighborhood of the support of \(\chi_1\). Our assumptions then show that \(e^{a(\xi)/h} \chi_2 T u \in L^2(\mathbb{R}^{2n})\) for some \(a > 0\). We now write
\[ \chi e^{b(\xi)} T u = \chi e^{b(\xi)} TS \left( \chi_1 e^{-a(\xi)} e^{a(\xi)} \chi_2 T u + (1 - \chi_1) \langle \xi \rangle^N \langle \xi \rangle^{-N} T u \right). \]
Since \( u \in H^{-N} \), \( \langle \xi \rangle^{-N}Tu \in L^2(\mathbb{R}^{2n}) \) and (2.16), (2.17), now show that \( e^{b\langle \xi \rangle}Tu \in H^K \) for some \( b > 0 \) and any \( K \). By taking \( K > n \) and applying [Höl, Corollary 7.9.4] we obtain a uniform bound

\[
|Tu(x, \xi)| \leq Ce^{-b\langle \xi \rangle}, \quad (x, \xi) \in U_0 \times \Gamma_0, \quad (x_0, \xi_0) \in U_0 \times \Gamma_0.
\]

Let \( h_1 \) be the fixed \( h \) in the definition of \( T \). Then,

\[
\mathcal{T}(x, \xi/\langle \xi \rangle; h_1/\langle \xi \rangle) = Tu(x, \xi) = O(e^{-b\langle \xi \rangle}), \quad (x, \xi) \in U_0 \times \Gamma_0.
\]

Putting \( \omega_0 := \xi_0/\langle \xi_0 \rangle \), it follows that \( \mathcal{T}(x, \omega, h) = O(e^{-\delta/h}) \) for \( (x, \omega) \) in a small neighbourhood of \( (x_0, \omega_0) \). But then (2.20) shows that \( (x_0, \omega_0) \notin WF_a(u) \). Since \( WF_a(u) \) is a closed conic set, we conclude that \( (x_0, \xi_0) \notin WF_a(u) \).

Now suppose that \( WF_a(u) \cap (U \times \Gamma) = \emptyset \). Then for \( (x, \omega) \) near \( U_0 \times (\Gamma_0 \cap S^{n-1}) \) (with \( U_0 \) and \( \Gamma_0 \), as in the statement of the theorem), \( \mathcal{T}(x, \omega, h) = O(e^{-\delta/h}) \). Reversing the argument in (2.21) we see that

\[
|Tu(x, \xi)| \leq Ce^{-b\langle \xi \rangle}, \quad (x, \xi) \in U_0 \times \Gamma_0.
\]

Now, since \( u \in H^{-N}(\mathbb{R}^n) \), \( \langle \xi \rangle^{-N}Tu \in L^2(\mathbb{R}^{2n}) \). In particular, since \( |\psi| \leq C\langle \xi \rangle \) and the support of \( \psi \) is contained in \( U_0 \times \Gamma_0 \), (2.15) follows.

The next proposition relates weighted estimates to deformed FBI transform:

**Proposition 2.5.** Suppose that \( H_\Lambda \), \( \Lambda = \Lambda_G \), is defined in [GaZw19b, (4.7)] with \( G \) satisfying (2.4) with \( \epsilon_0 \) chosen as in the definition of \( H_\Lambda \).

Then there exists \( \psi \in S^1(T^*\mathbb{R}^n) \) such that \( T : \mathcal{B}_\delta \to L^2(T^*\mathbb{R}^n, e^{\delta\langle \xi \rangle/Ch}dxd\xi) \) extends to

\[
T = O(1) : H_\Lambda \to L^2(T^*\mathbb{R}^n, e^{2\psi(x,\xi)/h}dxd\xi),
\]

and \( S : L^2(T^*\mathbb{R}^n, e^{-C\delta\langle \xi \rangle/h}dxd\xi) \to \mathcal{B}_\delta \), extends to

\[
S = O(1) : L^2(T^*\mathbb{R}^n, e^{2\psi(x,\xi)/h}dxd\xi) \to H_\Lambda.
\]

In addition,

\[
\psi(x, \xi) = G(x, \xi) + O(\epsilon_0^2)_{S^1(T^*\mathbb{R}^n)}.
\]

For a simpler version of this result in the case of compactly supported weights see [GaZw19a, §8].

**Proof.** The statement (2.22) is equivalent to

\[
TS_\Lambda = O(1) : L^2(\Lambda, e^{-2H(\alpha)/h}d\alpha) \to L^2(T^*\mathbb{R}^n, e^{2\psi(\beta)/h}d\beta)
\]

and hence we analyse the kernel of the operator \( TS_\Lambda \) which is given by

\[
K(\alpha, \beta) = C_n h^{-\frac{3n}{2}} \int_{\mathbb{R}^n} e^{\tilde{h}(\varphi_0(\alpha, y) + \varphi_0(\beta, y))} (\beta_\xi^2 + \frac{2}{h} (\alpha_\xi - \beta_\xi))(\alpha_\xi + \frac{1}{2} (\alpha_\xi - \beta_\xi)) dy.
\]
where the notation (and also notation for \( \Phi \) below) comes from (2.11). The integral in \( y \) converges and can be evaluated by a completion of squares as in [GaZw19b, Proposition 4.4]. That gives the phase (2.9) with \( \alpha \in T^*\mathbb{R}^n \) and \( \beta \in \Lambda \). The critical point in \( y \) is given by

\[
y_c(\alpha, \beta) = \frac{1}{\langle \alpha \xi \rangle + \langle \beta \xi \rangle} (\langle \alpha \xi \rangle \alpha_x + \langle \beta \xi \rangle \beta_x + i(\beta_\xi - \alpha_\xi)).
\]

(2.25)

We then have (2.22) with

\[
\psi(\alpha) := \max_{\beta \in \Lambda} (-\text{Im} \Psi(\alpha, \beta) + H(\beta)).
\]

(2.26)

We have (see [GaZw19b, (3.3),(3.4)])

\[
d_{\beta}(-\text{Im} \Psi(\alpha, \beta) + H(\beta)) = \text{Im}(-\partial_{z,\xi} \Psi(\alpha, (z, \zeta)) - \zeta dz|_{\Lambda})|_{(z, \zeta) = \beta \in \Lambda}.
\]

Now, if \( y_c(\alpha, (z, \zeta)) \) is the critical point in \( y \), then

\[
\partial_{z,\xi} \Psi(\alpha, z) = \partial_{z,\xi} \Phi(\alpha, y_c(\alpha, (z, \zeta))) - \Phi((z, \zeta), y_c(\alpha, (z, \zeta))) = -\partial_{z,\xi} \Phi|_{y=y_c(z,\zeta)}(z, \zeta)
\]

\[
= -\zeta \cdot dz + (y_c - z) \cdot d\zeta + i(z - y_c) \cdot dz + \frac{1}{2}(z - y_c)^2 \zeta \cdot d\zeta/\langle \zeta \rangle.
\]

For \( G = 0 \) the critical point (see (2.25)) is given by \( \alpha = \beta \). Hence

\[
\beta_c = \beta_c(\alpha) = (\alpha_x + O(\epsilon_0)_{S^0}, \alpha_\xi + O(\epsilon_0)_{S^1}),
\]

(2.27)

with \( \epsilon_0 \) as in (2.4).

Hence we obtain \( \psi \) by inserting the critical point \( \beta_c \) into the right hand side of (2.26)

\[
\psi(\alpha) = -\text{Im} \Psi(\alpha, \beta_c(\alpha)) + H(\beta_c(\alpha)) \in S^1(T^*\mathbb{R}^n).
\]

(2.28)

(We note that for \( G = 0 \) the maximum in (2.26) is non-degenerate and unique and it remains such under small symbolic perturbations.) From (2.9) we see that

\[
\text{Im} \Psi(\alpha, \beta_c(\alpha)) = \text{Im} \Psi(\alpha, \alpha + O(\epsilon_0)_{S^0} \times S^1) = \alpha_\xi \cdot G_\xi(\alpha) + O(\epsilon_0^2)_{S^1}.
\]

Inserting this into (2.28) and recalling that \( H = \xi G_\xi - G \) we obtain (2.24).

To obtain (2.23) we apply the same analysis to \( T_\Lambda S \) and we need to show that two weights coincide. That is done as in [GaZw19a, §8].

\[\square\]

3. Proof of Theorem 2

As already indicated in §1.2, to prove the theorem we construct a family of weights \( G_\epsilon \in S^1 \), uniformly bounded in \( S^1 \), supported in a conic neighbourhood of \( \Gamma = \{(0, 0, \xi_1, 0) : \xi_1 > M \}, M \gg 1 \), and satisfying \( 0 \leq G_\epsilon \leq C_\epsilon \log(\xi) \). In addition,

\[
H_p G_\epsilon \geq 0, \quad G_\epsilon \rightarrow \xi_1 \text{ on } \Gamma \text{ (in } S^1+),
\]

(3.1)

with \( H_p G_\epsilon \gg \xi_1^{m-1} \) in a suitable sense (see (3.4)) for \( \epsilon \ll 1 \).
We will then put \( \Lambda_\epsilon := \Lambda_{G_\epsilon} \) so that the assumption \( u \in C^\infty \) will give \( u \in H_{\Lambda_\epsilon} \). On the other hand the assumption that \( \Gamma \cap \text{WF}_a(Pu) \) shows that \( \|Pu\|_{H_{\Lambda_\epsilon}} \leq C \) with the constant \( C \) independent of \( \epsilon \). But then [GaZw19b, Proposition 6.2] and the properties of \( G_\epsilon \) show that \( \|u\|_{H_{\Lambda_\epsilon}} \) is bounded independently of \( \epsilon \). Propositions 2.3 and 2.5 then show that \( \text{WF}_a(u) \cap \Gamma_0 = \emptyset \).

### 3.1. Construction of the weight.

We now construct a family of weights, \( G_\epsilon \), satisfying (3.1). In fact, we need more precise conditions on \( G_\epsilon \) given in the following

**Lemma 3.1.** Suppose that \( p \) satisfies (1.9) at \( \rho_0 = (x_0, \xi_0) \in T^*\mathbb{R}^n \setminus 0 \) and \( \Gamma \) is an open conic neighbourhood of \( \rho_0 \). Then, there exists \( G_\epsilon \in S^1(T^*\mathbb{R}^n) \), supp \( G_\epsilon \subset \Gamma \), such that

\[
|\partial_x^\alpha \partial_\xi^\beta G_\epsilon| \leq C_{\alpha\beta} \langle \xi \rangle^{-1-|\beta|}, \quad 0 \leq G_\epsilon \leq C\epsilon^{-1} \log \langle \xi \rangle,
\]

\[
G_\epsilon(x, \xi)|_{1 \leq |\xi| \leq 1/\epsilon} = \Phi(x, \xi)|x|, \quad \Phi \in S_0^0(T^*\mathbb{R}^n), \quad \Phi(x_0, t\xi_0) = 1, \quad t \gg 1,
\]

\[
H_p G_\epsilon(x, \xi) \geq c_0 \left( \langle \xi \rangle^m |\partial_\xi G_\epsilon(x, \xi)|^2 + \langle \xi \rangle^{m-2} |\partial_x G_\epsilon(x, \xi)|^2 \right),
\]

\[
\forall M_1, \gamma \geq 0 \ \exists M_2, \ K, \epsilon_0 < \epsilon < \epsilon_0, \quad H_p G_\epsilon e^{\gamma G_\epsilon} + M_2 \langle \xi \rangle^K \geq M_1 \langle \xi \rangle^{m-1} e^{\gamma G_\epsilon}. \tag{3.4}
\]

We stress that the constants \( C_{\alpha\beta} \) and \( c_0 \) are independent of \( \epsilon \) and \( M_1 \).

**Proof.** We use the normal form for \( p \) constructed in [Ha14, §3]. That means that we take \( x_0 = 0 \) and \( \xi_0 = e_1 := (1,0,\cdots,0) \) and can assume that \( p(x, \xi) = -\xi_1^m x_1 \) in a conic neighbourhood of \( \rho = (0, e_1) \). For simplicity we can assume that \( m = 1 \) as the argument is the same otherwise.

Let \( \chi \in C_c^{\infty}(\mathbb{R}; [0, 1]) \) satisfy

\[
\text{supp } \chi \subset [-2, 2], \quad \chi|_{|t| \leq 1} = 1, \quad t\chi'(t) \leq 0. \tag{3.5}
\]

and put \( \varphi(t) := \chi(t/\delta) \). Here \( \delta \) will be fixed depending on \( \Gamma \). Using this function we define \( \Phi = \Phi(x, \xi) := \varphi_1 \varphi_2 \varphi_3 \psi \) where

\[
\varphi_1 := \varphi(x_1), \quad \varphi_2 := \varphi(|x'|/\xi_1), \quad \varphi_3 = \varphi(|x'|), \quad \psi := (1 - \varphi((\xi_1)_+)). \tag{3.6}
\]

We choose \( \delta \) small enough so that \( \text{supp } \Phi \subset \Gamma \).

We define \( G_\epsilon \) as follows

\[
G_\epsilon(x, \xi) = \Phi(x, \xi) q_\epsilon(\xi_1), \quad q_\epsilon(t) := \int_0^t \left( (1 - \chi(\epsilon s)) + (1 - \chi(\epsilon s))(se)^{-1} \right) ds. \tag{3.7}
\]

We check that

\[
\xi_1 \partial_{\xi_1} q_\epsilon \geq \min(\xi_1, \epsilon^{-1}),
\]

\[
\xi_1 \mathbb{1}_{|\xi_1| \leq 1/\epsilon} + \epsilon^{-1}(1 + \log(\epsilon \xi_1)) \mathbb{1}_{|\xi_1| \leq 1/\epsilon} \leq q_\epsilon \leq \xi_1 \mathbb{1}_{|\xi_1| \leq 1/\epsilon} + \epsilon^{-1}(2 + \log(\epsilon \xi_1)) \mathbb{1}_{|\xi_1| \leq 1/\epsilon}. \tag{3.8}
\]
Uniform boundedness of $G_\epsilon$ in $S^1$ means that $q_\epsilon$ in (3.7) satisfies $|\partial_{\xi_1}^k q_\epsilon| \leq C_k \xi_1^{1-k}$ with $C_k$’s independent of $\epsilon$. But this is immediate from the definition. We also easily see that $G_\epsilon$ converges to $G := \Phi(x, \xi)\xi_1$ in $S^{1+}$ as $\epsilon \to 0$. This proves (3.2).

To see (3.3), we first note that, since $\Phi \geq 0$, $\Phi \in S^0$, the standard estimate $f(z) \geq 0 \implies |df(z)|^2 \leq C f(z)$ gives,

$$\Phi(x, \xi) \geq c_1 \left( \xi_1^2 |\partial_\xi \Phi(x, \xi)|^2 + |\partial_x \Phi(x, \xi)|^2 \right).$$

(3.9)

Note also that we have $H_p = \xi_1 \partial_{\xi_1} - x_1 \partial_{x_1}$ and therefore

$$H_p \Phi = -x_1 \varphi'(x_1) \varphi_2 \varphi_3 \psi - (|\xi'|/\xi_1) \varphi'(\xi'/\xi_1) \varphi_1 \varphi_3 \psi - \varphi_1 \varphi_2 \varphi_3 \xi_1 \varphi'((\xi_1)+) \geq 0.$$  

(3.10)

Since $q_\epsilon \in S^1$, $\xi_1 \partial_{\xi_1} q_\epsilon(\xi_1) \geq c_2 \xi_1 (\partial_{\xi_1} q_\epsilon(\xi_1))^2$. We also claim that

$$\xi_1 \partial_{\xi_1} q_\epsilon(\xi_1) \geq c_2 \xi_1^{-1} q_\epsilon(\xi_1)^2.$$  

(3.11)

In fact, using (3.8) we see that to prove (3.11) it is enough to have

$$\min(t, \epsilon^{-1}) \geq c_2 t^{-1} \left( t \mathbb{I}_{\xi_1 \leq 1/\epsilon}(t) + \epsilon^{-1}(2 + \log(t\epsilon)) \mathbb{I}_{\xi_1 \geq 1/\epsilon}(t) \right)^2.$$  

This clearly holds (with $c_2 = 1$) for $t \leq 1/\epsilon$ and for $t \geq \epsilon$ is equivalent to $c_2 (2 + \log(s)) \leq s$, $s = t\epsilon \geq 1$, which holds with $c_2 = \frac{1}{4}$. It follows that

$$\xi_1 \partial_{\xi_1} q_\epsilon(\xi_1) \geq c_2 \left( \xi_1^{-1} q_\epsilon(\xi_1)^2 + \xi_1 (\partial_{\xi_1} q_\epsilon(\xi_1))^2 \right),$$  

which combined with (3.9) and (3.10) gives

$$H_p G_\epsilon = \Phi(\xi_1 \partial_{\xi_1} q_\epsilon) + (H_p \Phi) q_\epsilon \\
\geq \Phi(\xi_1 \partial_{\xi_1} q_\epsilon) \geq c_2 \xi_1 \Phi(\partial_{\xi_1} q_\epsilon)^2 + c_3 \left( \xi_1^2 |\partial_\xi \Phi|^2 + |\partial_x \Phi|^2 \right) \xi_1^{-1} q_\epsilon^2 \\
\geq c_0 \left( \xi_1^2 |\partial_{\xi_1} G_\epsilon|^2 + \xi_1^{-1} |\partial_x G_\epsilon|^2 \right).$$

Since $(\xi) \sim \xi_1$ on the support of $G_\epsilon$, we obtain (3.3).

Finally we prove (3.4). Since by (3.10) we have $H_p G_\epsilon \geq \Phi H_p q_\epsilon$, we see that (3.4) follows from proving that for any $M_1$ we can find $K$, $M_2$ and $\epsilon_0$ such that for $\xi_1 \geq 1$,

$$\Phi H_p q_\epsilon e^{\gamma \Phi q_\epsilon} + M_2 \xi_1^K \geq M_1 e^{\gamma \Phi q_\epsilon}.$$  

(3.12)

Using (3.8), we see that for $\xi_1 \leq 1/\epsilon$ we need $G_\epsilon e^{\gamma G_\epsilon} + M_2 \xi_1^K \geq M_1 e^{\gamma G_\epsilon}$. This holds for

$$K = 0, \quad M_2 = 2 \gamma^{-1} e^{\gamma M_1^{-1}}$$

since for $\gamma > 0$ and $a \geq 0$, $ae^{\gamma a} - M_1 e^{\gamma a} \geq -2 \gamma^{-1} e^{\gamma M_1^{-1}}$.

For $\xi_1 \geq 1/\epsilon$, we need to find $K$ and $M_2$ for which

$$e^{-1} \Phi e^{\gamma \Phi q_\epsilon} + M_2 \xi_1^K \geq M_1 e^{\gamma \Phi q_\epsilon}.$$  

(3.13)

Using $ae^{ab} + M_1 e^{M_1 b} \geq M_1 e^{ab}$ with $a := -1/\Phi$ and

$$b := \gamma \epsilon q_\epsilon \leq \gamma (2 + \log(\epsilon \xi_1)) \leq \gamma (2 + \log \xi_1),$$

we obtain

$$e^{-1} \Phi e^{\gamma \Phi q_\epsilon} + M_2 \xi_1^K \geq M_1 e^{\gamma \Phi q_\epsilon}.$$
we obtain (3.13) with $M_2 = M_1 e^{2\gamma M_1}$ and $K = \gamma M_1$. Hence we obtain (3.12) proving (3.4).

3.2. Microlocal analytic hypoelliticity. We will have bounds which are uniform in $\epsilon$ but not in $h$. We start with the following

**Lemma 3.2.** Suppose that $P$ is of the form (1.8) with real valued principal symbol $p$ and suppose that $\Gamma \subset U \times \mathbb{R}^n \setminus \{0\}$ is an open cone, $\Gamma \cap S^{n-1} \Subset U \times S^{n-1}$ and $G \in S^1(\Gamma; \mathbb{R}), \ |G| \leq C \log(\xi), \ H_p G(x, \xi) \geq c_0 (\langle \xi \rangle^m |\partial_\xi G(x, \xi)|^2 + \langle \xi \rangle^{m-2}|\partial_x G(x, \xi)|^2)$. From (2.7) and (3.14) we obtain

$$\text{Im}(\langle \xi \rangle^{-N} T_\Lambda h^m \Psi_\Lambda T_\Lambda u, \langle \xi \rangle^{-N} T_\Lambda u \rangle_{L_\Lambda^2} = \langle \langle \text{Im} b_{p,N} \rangle T_\Lambda u, T_\Lambda u \rangle_{L_\Lambda^2} + O(h^\infty)\|u\|_{H^{-N}}^2 \geq \langle \langle \text{Im} p_{\Lambda} \rangle \langle \xi \rangle^{-N} T_\Lambda u, \langle \xi \rangle^{-N} T_\Lambda u \rangle_{L_\Lambda^2} - M \|u\|_{H^{-N}}^2. \tag{3.15}$$

Then for $T_\Lambda, H_\Lambda$, $\Lambda = \Lambda_{BG}$ defined in (2.4) and (2.6), $h$ and $\theta$ sufficiently small, and $u \in H^{-N}_\Lambda$,

$$\text{Im}(h^m P u, u)_{H^{-N}_\Lambda} \geq \frac{1}{2} \theta\langle H_p G \langle \xi \rangle^{-N} T_\Lambda u, \langle \xi \rangle^{-N} T_\Lambda u \rangle_{L_\Lambda^2} - M h\|u\|_{H^{-N}}^2. \tag{3.16}$$

Proof. We use Proposition 2.2 and [GaZw19b, Proposition 6.3] to see that for any $K > 0$, $\text{Im}(h^m P u, u)_{H^{-N}_\Lambda} = \text{Im}(\langle \xi \rangle^{-2N} T_\Lambda h^m \Psi_\Lambda T_\Lambda u, T_\Lambda u \rangle_{L_\Lambda^2} = \text{Im}(\langle \Pi_\Lambda \langle \xi \rangle^{-2N} \Pi_\Lambda h^m \Psi_\Lambda \Pi_\Lambda T_\Lambda u, T_\Lambda u \rangle_{L_\Lambda^2} = \langle \langle \text{Im} b_{p,N} \rangle T_\Lambda u, T_\Lambda u \rangle_{L_\Lambda^2} + O(h^\infty)\|u\|_{H^{-N}}^2 \geq \langle \langle \text{Im} p_{\Lambda} \rangle \langle \xi \rangle^{-N} T_\Lambda u, \langle \xi \rangle^{-N} T_\Lambda u \rangle_{L_\Lambda^2} - M \|u\|_{H^{-N}}^2. \tag{3.16}$$

From (2.7) and (3.14) we obtain

$$\text{Im} p_{\Lambda} = \text{Im} p(x - i\theta \partial_\xi G(x, \xi), \xi + i\theta \partial_x G(x, \xi)) = \theta H_p G(x, \xi) + \theta^2 \mathcal{O}(\langle \xi \rangle^m |\partial_\xi G(x, \xi)|^2 + \langle \xi \rangle^{m-2}|\partial_x G(x, \xi)|^2) \geq \frac{1}{2} \theta H_p G(x, \xi),$$

if $\theta$ is small enough. \qed

The next lemma allows us to use smoothness of $u$ to obtain weaker weighted estimates:

**Lemma 3.3.** Suppose $U \subset \mathbb{R}^n$ is an open set,

$$G \in S^1(T^*\mathbb{R}^n), \ G \geq 0, \ \text{supp} G \subset K \times \mathbb{R}^n, \ K \Subset U,$$

and $T_\Lambda, H_\Lambda$, $\Lambda = \Lambda_{BG}$ are defined in (2.4) and (2.6). Then, there exists $a > 0$ such that for every $\chi, \bar{\chi} \in S^1$ with $\bar{\chi} \equiv 1$ in a conic neighborhood of $\text{supp} \chi$ and every $K, N > 0$, there exists $c, C > 0$ such that for all $u \in H^{-N}(\mathbb{R}^n)$,

$$\|\langle \xi \rangle^{-N} T_\Lambda u\|_{L_\Lambda^2} \leq C(\|\langle \xi \rangle^{K} \bar{\chi} T_\Lambda u\|_{L^2(T^*\mathbb{R}^n)} + e^{-c/h}\|\langle \xi \rangle^{-N} T_\Lambda u\|_{L^2(T^*\mathbb{R}^n)}). \tag{3.17}$$
In particular, if $\chi \equiv 1$ on $\text{supp} \, G$, then
\[
\| (\langle \xi \rangle^K e^{-a/h} \chi + \langle \xi \rangle^{-N} (1 - \chi)) T_\Lambda u \|_{L^2_{\chi}} 
\leq C(\| \langle \xi \rangle^N \chi T u \|_{L^2(T^*\mathbb{R}^n)} + e^{-C/h} \| \langle \xi \rangle^{-N} T u \|_{L^2(T^*\mathbb{R}^n)}).
\] (3.18)

Proof. First, observe that by [GaZw19b, Lemma 4.5], for any $\delta > 0$,
\[ T_\Lambda S = K_\delta + O_{N, \delta}(e^{-c(h)}) \langle \xi \rangle^N L^2(T^*\mathbb{R}^n) \rightarrow \langle \xi \rangle^{-N} L^2_{\chi}, \]
and $K_\delta$ has kernel, $K_\delta(\alpha, \beta)$, given by
\[ h^{-n} e^{\frac{i}{h} \Psi(\alpha, \beta)} k(\alpha, \beta) \psi(\delta^{-1} | \text{Re} \alpha - \beta |) \psi(\delta^{-1} \min(|\text{Re} \alpha |, |\beta |)) \]
where $(\alpha, \beta) \in \Lambda \times T^*\mathbb{R}^n$ and $\Psi$ is as in (2.9), and $\psi \in C^\infty_c(\mathbb{R})$ is identically 1 near 0. Therefore, we need only consider $K_\delta(\alpha, \beta)$.

To do this, let $\tilde{\chi} \in S^0$ be identically 1 on a conic neighborhood of $\text{supp} \, \chi$. Then, for $\delta > 0$ small enough,
\[ \chi(\text{Re} \alpha) K_\delta(\alpha, \beta)(1 - \tilde{\chi})(\beta) \equiv 0. \]
Therefore,
\[ \chi e^{-aG/h} \langle \xi \rangle^K T_\Lambda S (1 - \tilde{\chi}) = O_N(e^{-c(h)}) \langle \xi \rangle^N L^2(T^*\mathbb{R}^n) \rightarrow \langle \xi \rangle^{-N} L^2_{\chi}. \]

For the mapping properties
\[ \chi e^{-aG/h} T_\Lambda S \tilde{\chi} : \langle \xi \rangle^{-K} L^2(T^*\mathbb{R}^n) \rightarrow \langle \xi \rangle^{-K} L^2_{\chi}, \]
we consider the operator
\[ \chi e^{-aG/h} e^{-H/h} \langle \xi \rangle^K T_\Lambda S \tilde{\chi} \langle \xi \rangle^{-K} : L^2(T^*\mathbb{R}^n) \rightarrow L^2(\Lambda; dx d\xi). \]
Modulo negligible terms, the kernel of this operator is given by
\[ h^{-n} e^{\frac{i}{h} \psi((x, \xi), (y, \eta))} \tilde{k}((x, \xi), (y, \eta)) \]
where $\tilde{k} \in S^0$ has
\[ \text{supp} \, \tilde{k} \subset \{ |\xi - \eta| \leq C \delta |\xi| \} \cap \{ |x - y| \leq C \delta \}. \] (3.19)
and
\[ \psi = i H(x, \xi) + ia \theta G(x, \xi) + \Psi((x - i \theta G \xi, x \xi + i \theta G x(x, \xi)), (y, \eta)), \]
with $H(x, \xi) = \theta \langle \xi, G \xi(x, \xi) \rangle - \theta G(x, \xi)$. Using (3.19), we have
\[ \text{Im} \, \varphi = a G + \theta \xi \cdot G \xi - \theta G + \frac{\langle \eta \rangle \langle \xi \rangle}{2(\langle \eta \rangle + \langle \xi \rangle)} ((x - y)^2 - (\theta G \xi)^2) + \frac{(\xi - \eta)^2 - (\theta G \xi)^2}{2(\langle \eta \rangle + \langle \xi \rangle)} \]
\[ + \theta \xi \cdot G \xi + O(\theta |x - y||G_x| + \langle \xi \rangle^{-1}|x - \eta||G_\xi|) \]
\[ + O(\theta^2(|\xi|^{-1} G_x^2 + |\xi| |G_\xi|^2)) \]
\[ \geq (a - \theta) G - C \theta^2 (|\langle \xi \rangle^{-1}(G_x)^2 + \langle \xi \rangle |G_\xi|^2) + c \langle \xi \rangle (x - y)^2 + c \langle \xi \rangle^{-1}(\xi - \eta)^2. \]
In particular, taking \( a \) large enough and using that \( G \geq 0, G \in S^1 \), (see the argument for (3.9)), we have
\[
\text{Im } \varphi \geq \frac{a}{2} G(x, \xi) + c\langle \xi \rangle (x - y)^2 + c\langle \xi \rangle^{-1}(\xi - \eta)^2.
\]
Therefore, applying the Schur test for \( L^2 \) boundedness completes the proof that
\[
\chi\langle \xi \rangle^K e^{-aG/h} T_\Lambda S\langle \xi \rangle^{-K} = O(1) : L^2(T^*\mathbb{R}^n) \to L^2_\Lambda
\]
and the lemma follows. \( \square \)

With these two lemmas in place we can prove the main result:

Proof of Theorem 2. By multiplying \( u \) by a \( C^\infty_c \)-function which is 1 in a neighbourhood of \( x_0 \), we can assume that \( u \in H^{-N+m} \), for some \( N \), is compactly supported in \( U \) and \( \rho_0 := (x_0, \xi_0) \notin \text{WF}(u) \). By Proposition 2.1, there exists \( \tilde{\chi} \in S^0 \) with \( \tilde{\chi} \equiv 1 \) in an open conic neighborhood, \( \Gamma \), of \( \rho_0 \) such that for any \( K > 0 \),
\[
\|\langle \xi \rangle^K \tilde{\chi} T_\Lambda u\|_{L^2} \leq C_K. \tag{3.20}
\]
Also, since \( u \in H^{-N+m} \),
\[
\|\langle \xi \rangle^{-N+m} T_\Lambda u\|_{L^2} \leq C. \tag{3.21}
\]
Let \( \Gamma_1 \Subset \Gamma \) be an open conic neighborhood of \( \rho_0 \) and \( \chi \in S^1 \) with \( \chi \equiv 1 \) on \( \Gamma_1 \) and \( \text{supp } \chi \subset \Gamma \).

We choose \( \theta \) small enough so that (2.4) and (3.16) hold. We then fix \( 0 < h \leq 1 \) small enough so that (3.16) holds. From now we neglect the dependence on \( h \) which is considered to be a fixed parameter. We choose for \( G = G_\epsilon \) constructed in Lemma 3.1 and supported in \( \Gamma_1 \). We recall that the estimates depend only on the \( S^1 \)-seminorms of \( G \) and these are uniform in \( \epsilon \). We now claim that
\[
u \in H_{\Lambda_\epsilon}^{-N+m}, \quad \Lambda_\epsilon := \Lambda_{\theta G_\epsilon}.
\]
In fact, we can use (3.18) together with (3.20) and (3.21), observing that \( \exp(aG_\epsilon/h) = O_\epsilon(\langle \xi \rangle^{Ca/(he)}) \) and taking \( K = Ca/(he) \).

Next, note that \( Pu \in H^{-N} \) is supported in \( U \) and \( \rho_0 \notin \text{WF}_a(Pu) \). Propositions 2.3 and 2.5 (see (2.15) and (2.23) respectively) then show that for \( G = G_\epsilon \) satisfying the assumptions of Lemma 3.2 and \( \theta \) sufficiently small \( \| Pu \|_{H_{\Lambda_\epsilon}^{-N}} \leq C_0 \), where \( C_0 \) depends only on \( Pu \) and \( S^1 \)-seminorms of \( \theta G_\epsilon \).

We now apply (3.15) to obtain with \( \Lambda_\epsilon \) as above,
\[
\frac{1}{2}\| u \|_{H_{\Lambda_\epsilon}^{-N}}^2 + 2C_0^2 \geq \langle \theta H_{\rho G_\epsilon} - M \langle \xi \rangle^{m-1} \rangle \langle \xi \rangle^{-N-m} T_{\Lambda_\epsilon} u, \langle \xi \rangle^{-N} T_{\Lambda_\epsilon} u \rangle_{L^2_{\Lambda_\epsilon}}, \tag{3.22}
\]
Let \( a \) be given by Lemma 3.3 (so that (3.17) holds). Then by (3.4), there exist \( M_2 \) and \( K \) such that
\[
\theta H_{\rho G_\epsilon} + M_2 \langle \xi \rangle^{2K} e^{-2aG_\epsilon/h} \geq (M + 1) \langle \xi \rangle^{m-1}.
\]
From (3.17) we have
\[ \| M_2 \chi \langle \xi \rangle^K e^{-aG_\epsilon/h} \langle \xi \rangle^{-N} T_{\Lambda_\epsilon} u \|_{L^2_{\Lambda_\epsilon}}^2 \]
\[ \leq C(\| \langle \xi \rangle^K T u \|_{L^2(T^*R^n)}^2 + \| \langle \xi \rangle^{-N} T u \|_{L^2(T^*R^n)}^2) \leq C_1^2 \tag{3.23} \]
Therefore, adding (3.23) to (3.22), and using that supp \( G_\epsilon \subset \chi \equiv 1 \), we have
\[ \frac{1}{2} \| u \|_{H^{-N}_{H\epsilon}}^2 + C_1^2 + 2C_0^2 \]
\[ \geq \langle \chi^2 \langle \xi \rangle^{m-1} \langle \xi \rangle^{-N} T_{\Lambda_\epsilon} u, \langle \xi \rangle^{-N} T_{\Lambda_\epsilon} u \rangle_{L^2_{\Lambda_\epsilon}} \]
\[ - \langle M(1-\chi^2) \langle \xi \rangle^{m-1} \langle \xi \rangle^{-N} T_{\Lambda_\epsilon} u, \langle \xi \rangle^{-N} T_{\Lambda_\epsilon} u \rangle_{L^2_{\Lambda_\epsilon}} \]
\[ \geq \langle \langle \xi \rangle^{m-1} \langle \xi \rangle^{-N} T_{\Lambda_\epsilon} u, \langle \xi \rangle^{-N} T_{\Lambda_\epsilon} u \rangle_{L^2_{\Lambda_\epsilon}} - (M + 1) \| u \|_{H^{-N_+ \frac{m-1}{2}}}^2, \tag{3.24} \]
where in the last line we use that \( \chi \equiv 1 \) on supp \( G_\epsilon \).

Using \( m \geq 1 \) and rearranging, this yields
\[ \| u \|_{H^{-N}_{H\epsilon}}^2 \leq 2C_1^2 + 4C_0^2 + 2(M + 1) \| u \|_{H^{-N_+ \frac{m-1}{2}}}^2. \]
where \( C_1, C_0 \) and \( M \) are constants independent of \( \epsilon \).

Since \( \Lambda_\epsilon \cap \{ |\xi| < 1/\epsilon \} = \Lambda_0 \cap \{ |\xi| < 1/\epsilon \} \) where \( G_0 := \Phi|\xi| \), we have that \( H_\epsilon |_{|\xi|<1/\epsilon} = H_0 |_{|\xi|<1/\epsilon} \), where \( H_\epsilon = \theta \xi \partial_\xi G_\epsilon + \theta G \) is the corresponding weight. Therefore, the monotone convergence theorem implies that \( u \in H_{\Lambda_0} \). Since \( \Phi(x_0, t\xi_0) = 1, t \gg 1 \), Proposition 2.3 shows that \( (x_0, \xi_0) \notin \text{WF}_a(u) \). \( \square \)

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