Internal waves in fluids and spectral theory of 0th order operators

Seminarium algebry operatorów
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Motivation
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Pillet et al '18

Bouzet '16
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Boussinesq approximation:

\[
\begin{align*}
\partial_t \eta + \mathbf{u} \cdot \nabla \rho_0 &= 0, \\
\rho_0 \partial_t \mathbf{u} &= -\eta \mathbf{e}_3 \cdot \nabla P + \mathbf{F} e^{-i\omega_0 t}, \\
\mathbf{n} \cdot \mathbf{u} &= 0.
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Formal diagonalization gives

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\mathbf{u} = \mathbf{u}_+ e_+ + \mathbf{u}_- e_-
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i \frac{\partial t}{\mathbf{u}_\pm} - P \mathbf{u}_\pm = e^{-i\omega_0 t} f_\pm
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i \partial_t u_\pm - P u_\pm = e^{-i\omega_0 t} f_\pm
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\[
P = H_\pm(x, D), \quad H_\pm(x, \xi) = \pm(-g \rho'_0(x)/\rho_0(x))^{\frac{1}{2}} \xi_1/|\xi|
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Other related models: rotating fluids Ralston '73

\[
\begin{align*}
\partial_t^2 \Delta_x \mathbf{u} &= \partial_{x_1}^2 \mathbf{u}, \\
\mathbf{u} |_{\partial \Omega} &= 0 \\
i \partial_t \mathbf{u} - P \mathbf{u} &= 0, \\
P &= \pm \Delta^{-\frac{1}{2}} \partial_{x_1}
\end{align*}
\]
Mathematical Model

\[ H \pm (x, D) \rightarrow P \in \Psi_0(T^2), P^* = P_{\sigma} = \text{homogeneous of degree 0}, \]
\[ dp|_{p-1}(\omega_0) \neq 0, \]
the flow of \( \langle \xi \rangle \)
\[ H_p \rightarrow \partial \xi_p \cdot \partial x - \partial x_p \cdot \partial \xi, \]
\( (x, \xi) \sim (y, \eta) \iff x = y, \xi = t \eta, t > 0 \)
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\[ H_\xi \big|_{p - 1} (\omega_0) / \sim \text{is Morse–Smale with no fixed points} \]
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\[ H_p = \partial_\xi p \cdot \partial_x - \partial_x p \cdot \partial_\xi, \quad (x, \xi) \sim (y, \eta) \Leftrightarrow x = y, \quad \xi = t\eta, \quad t > 0 \]
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\[ i \partial_t u - Pu = e^{-i \omega_0 t} f, \quad P \in \Psi^0(\mathbb{T}^2), \quad P^* = P, \quad u|_{t=0} = 0, \quad f \in C^\infty(\mathbb{T}^2) \]
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The surface \( \Sigma := p^{-1}(\omega_0) / \sim \) lies on the boundary of \( \overline{T^*\mathbb{T}^2} \setminus 0 \)

\( \langle \xi \rangle H_p \) is tangent to \( \Sigma \).
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**Morse–Smale on \( \Sigma \):**

(i) \( \langle \xi \rangle H_p \) has a finite number of fixed points all of which are hyperbolic;

(ii) \( \langle \xi \rangle H_p \) has a finite number of hyperbolic limit cycles;

(iii) there are no separatrix connections between saddle fixed points

(iv) every trajectory different from (i) and (ii) has a unique trajectory (i) or (ii) as its \( \alpha, \omega \)-limit set.

If there are no fixed points \( \Sigma \) is a finite union of tori. This is why we do not consider more general manifolds in this case.

(Some comments about fixed points at the end.)
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Main result

Let $\tilde{\Lambda}_+$ be the attractor of the flow of $\langle \xi \rangle H_p$ on $\Sigma = p^{-1}(\omega_0)/\sim$. 
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$\Lambda_+ := \{(x, \xi) : [(x, \xi)]_\sim \in \tilde{\Lambda}_+\} \subset T^*\mathbb{T}^2 \setminus 0$ is a conic Lagrangian

$I^m(\Lambda_+) \subset H^{-m-\frac{1}{2}}$

is the space of Lagrangian distributions of order $m$. 
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$w \in I^m(\Lambda_+) \iff w(x) = \int_{\mathbb{R}} a(x_2, \xi_1) e^{i x_1 \xi_1} d\xi_1$

$|\partial_{x_2}^k \partial_{\xi_1}^\ell a(x_2, \xi_1)| = \begin{cases} O(\xi_1^{m-\ell}) & \xi_1 \rightarrow +\infty \\ O(|\xi_1|^{-\infty}) & \xi_1 \rightarrow -\infty. \end{cases}$
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For instance, $w(x) = (x_1 - i0)^{-1} \varphi(x_1, x_2), \varphi \in C^\infty(\mathbb{T}^2)$. 
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$$I^m(\Lambda_+) \subset H^{-m - \frac{1}{2}}$$

is the space of Lagrangian distributions of order $> m$.

**Theorem** Suppose that $\omega_0 \notin \text{Spec}_{pp}(P)$ and that $u$ solves

$$i\partial_t u - Pu = e^{-i\omega_0 t} f, \quad u|_{t=0} = 0, \quad f \in C^\infty(\mathbb{T}^2).$$

Then,

$$u(t) = e^{-i\omega_0 t} u_\infty + b(t) + \epsilon(t), \quad u_\infty \in I^0(\Lambda_+)$$

$$\|b(t)\|_{L^2} \leq C, \quad \|\epsilon(t)\|_{-\frac{1}{2}} \to 0, \quad t \to \infty$$
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\[ P := \langle D \rangle^{-1} D_{x_2} - 2 \cos x_1 \]

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Main Tool: spectral theory

\[ i \partial_t u - Pu = f, \quad u|_{t=0} = 0, \quad f \in C^\infty(\mathbb{T}^2). \]

\[ u(t) = \int_0^t e^{-isP} f \]
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\[ u(t) = \int_0^t e^{-isP} f = iP^{-1}(1 - e^{-itP})f \]

The operator \( P^{-1}(1 - e^{-itP}) \) is well defined for all \( t \) using the spectral theorem (recall that \( P = P^* \)). We need to show that

\[ (P - \omega - i0)^{-1} f \]

exists for \( \omega \) near 0

\[ P^{-1}(1 - e^{-itP}) \chi(\Lambda + \cdots) \]

\( \rightarrow \) \( (P - i0)^{-1} \chi(\Lambda + \cdots) \).
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\( P^{-1}(1 - e^{-itP})\chi(P)f \overset{\text{in } H^{-\frac{1}{2}}}{\longrightarrow} (P - i0)^{-1}\chi(P)f. \)
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- \( P^{-1}(1 - e^{-itP}) \chi(P)f \xrightarrow{\text{in } H^{-\frac{1}{2}}_{-}} (P - i0)^{-1} \chi(P)f \).

- \( (P - i0)^{-1}f \in l^0(\Lambda_+) \)
Main Tool: radial estimates

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Some relevant ones:

- scattering by 0th order potentials Hassell–Melrose–Vasy '04
- hyperbolic scattering Vasy '13, Datchev–Dyatlov '13
- general relativity Vasy, Hintz–Vasy '13..., Dyatlov '11–'14
- Lagrangian regularity Haber–Vasy '15
- Anosov flows Dyatlov–Zworski '16, '17
- Axiom A flows Dyatlov–Guillarmou '16, '18
Main Tool: radial estimates

Radial sources and sinks: definition by (a very special) example
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\[ \|A_- u\|_s \lesssim \|\tilde{B}_-(P - i\epsilon) u\|_{s+1} + \|u\|_{-N}, \quad s > -\frac{1}{2}, \text{ source} \]
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\[ \| A_+ u \|_s \lesssim \| \widetilde{B}_+(P - i \epsilon) u \|_{s+1} + \| B_+ u \|_s + \| u \|_{-N}, \quad s < -\frac{1}{2}, \quad \text{sink} \]
Main Tool: radial estimates

Radial sources and sinks: definition by (a very special) example

\[ p = \xi_1^{-1}(\xi_2 + \lambda \xi_1 x_1), \quad \xi_1 > |\xi_2|, \quad \langle \xi \rangle \sim \xi_1, \quad P = P^*, \quad p = \sigma(P) \]

\[ \xi_1 H_p|_{p^{-1}(0)} = \partial x_2 + \lambda(x_1 \partial x_1 - \xi_1 \partial \xi_1), \quad \Lambda := \{x_1 = 0, \xi_2 = 0\}, \quad L := \Lambda/\sim \]

\[ \lambda > 0, \text{ source/repeller} \quad \lambda < 0, \text{ sink/attractor} \]

\[ \|A_- u\|_s \lesssim \|\widetilde{B}_-(P - i\epsilon)u\|_{s+1} + \|u\|_{-N}, \quad s > -\frac{1}{2}, \quad \text{source} \]

\[ \|A_+ u\|_s \lesssim \|\widetilde{B}_+(P - i\epsilon)u\|_{s+1} + \|B_+ u\|_s + \|u\|_{-N}, \quad s < -\frac{1}{2}, \quad \text{sink} \]

Uniform for \( \epsilon > 0 \).
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Standard arguments in scattering theory (cf. Melrose ’94) show that the limit

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exists except at finitely many eigenvalues.
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**Lemma** (Dyatlov–Zworski '18; related to Haber–Vasy '15)

*Suppose that*

\[ (P - \omega)u \in C^\infty, \ \text{WF}(u) \subset \Lambda_+(\omega), \ u \in H^{-\frac{1}{2}-}. \]

*Then* \( u \in I^0(\Lambda_+(\omega)). \)

*Moreover, if* \( u(\omega) = (P - \omega - i0)^{-1} f, \ f \in C^\infty, \) *then*

\[ u(\omega) \in C^\infty((-\delta, \delta)_\omega; I^0(\Lambda_+(\omega))) \]
Geometry of attracting Lagrangians

The general set up:

1. $M$ is a compact surface without boundary;
2. $p(x,\xi) : T^*M \setminus 0 \to \mathbb{R}$ is smooth and homogeneous of order 0;
3. $\Lambda_\omega \subset p^{-1}(\omega) \subset T^*M \setminus 0$ is a family of conic embedded Lagrangian submanifolds depending smoothly on $\omega \in I$;
4. $H_p$ is tangent to each $\Lambda_\omega$.

Lemma (Dyatlov–Zworski '18)

Suppose that for all $\omega \in I$ and all $(x,\xi) \in \Lambda_\omega$, $\exp(tH_p)(x,\xi)$ converges to infinity of the fibers at linear rate as $t \to \infty$.

Suppose that, locally, $\Lambda_\omega = \{(x,\xi) : x = \partial_\xi F(\omega,\xi)\}$ where $\xi \mapsto F(\omega,\xi)$ is a family of homogeneous functions of order one. Then for some $c > 0$, $\partial_\omega F(\omega,\xi) < -c|\xi|$, $\xi \in \Gamma_0$. 
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Theorem Suppose that $0 \notin \text{Spec}_{pp}(P)$ and that $u$ solves

$$i\partial_t u - Pu = f, \quad u|_{t=0} = 0, \quad f \in C^\infty(\mathbb{T}^2).$$

Then, $u(t) = u_\infty + b(t) + \epsilon(t)$, where $u_\infty \in L^0(\Lambda_+)$, $\|b(t)\|_{L^2} \leq C$ and $\|\epsilon(t)\|_{-\frac{1}{2}} \to 0$, as $t \to \infty$. 
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Proof: From spectral theorem and Stone's formula

$$u(t) = \frac{1}{2\pi} \int_0^t e^{-is\omega} \left( (P - \omega - i0)^{-1} - (P - \omega + i0)^{-1} \right) f d\omega$$

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Proof continued...

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\[ w(\omega) \in C_0^\infty (\mathbb{R}_\omega; L^0(\Lambda_\omega)), \quad \Lambda_\omega = \{ (\partial_\xi F(\omega, \xi), \xi) \} \]

*Suppose that* \( \epsilon \partial_\omega F(0, \xi) < 0 \). *Then for* \( w(\omega) \) *supported near 0,*

\[ \int_0^t e^{is\omega} w(\omega) d\omega = w_\infty + b(t) + \epsilon(t), \quad w_\infty = \begin{cases} 2\pi w(0) & \epsilon = +, \\ 0 & \epsilon = - \end{cases} \]
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The geometric lemma provides the sign condition! QED
More general geometries

\[ P = A + A^*, A := \langle D \rangle^{-1} \left( D_x^1 \cos x^1 - 2 D_x^2 \cos x^2 \right) i u - P u = f, \quad f = \chi(x^1 - \pi/2, x^2 - \pi/2) \]
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In the Morse–Smale case, Colin de Verdière ’18 used a hybrid of Mourre and radial estimates to show that \( \|\epsilon(t)\|_{H^{-\frac{1}{2}}} \to 0. \)
More general geometries

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\[ \pi(\Lambda_+) = \{x_1 = x_2 = -\frac{1}{2}\pi\} \cup \{x_1 = -\frac{\pi}{2}, x_2 = \frac{\pi}{2}\} \cup \{x_1 = \frac{\pi}{2}, x_2 = -\frac{\pi}{2}\} \]

\[ \pi(\Gamma_+) = \{x_1 = \frac{\pi}{2}\} \cup \{x_2 = \frac{\pi}{2}\} \]
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