A QUANTITATIVE VERSION OF CATLIN-D'ANGELO-QUILLEN THEOREM

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Abstract. Let $f(z,\bar{z})$ be a positive bi-homogeneous hermitian form on $\mathbb{C}^n$, of degree m. A theorem proved by Quillen and rediscovered by Catlin and D'Angelo states that for N large enough, $\langle z,\bar{z}\rangle^N f(z,\bar{z})$ can be written as the sum of squares of homogeneous polynomials of degree $m+N$. We show this works for $N \geq C_f((n+m)\log n)^3$ where C_f has a natural expression in terms of coefficients of f. The proof uses a semi-classical point of view on which $1/N$ plays a role of the small parameter h.

1. Introduction and main result

Let $f = f(z,\bar{z})$ be a bi-homogeneous form of degree $m \geq 1$ on $\mathbb{C}^n$:

$$f(z,\bar{z}) := \sum_{|\alpha|=|\beta|=m} c_{\alpha\beta} z^\alpha \bar{z}^\beta, \quad z \in \mathbb{C}^n, \quad c_{\alpha\beta} \in \mathbb{C}. \tag{1.1}$$

Here $n \geq 2$, $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$, $|\alpha| := \alpha_1 + ... + \alpha_n$, $z^\alpha := z_1^{\alpha_1} ... z_n^{\alpha_n}$. The following theorem was proved by Quillen in 1968 [9], and rediscovered by Catlin and D'Angelo in 1996 [2]:

Theorem 1. *Suppose f is given by (1.1) and that*

$$f(z,\bar{z}) > 0, \quad z \neq 0.$$

Then there exists N_0 such that for $N > N_0$

$$\|z\|^{2N} f(z,\bar{z}) = \sum_{j=1}^{d_N} |P_j^N(z)|^2, \quad \|z\|^2 := \sum_{j=1}^{n} |z_j|^2, \tag{1.2}$$

where $P_j^N(z)$ are homogeneous polynomials of degree $m+N$, and $d_N = \binom{n+m+N}{N}$ is the dimension of the space of homogeneous polynomials of degree $m+N$.

This result can be considered as the complex variables analogue of Hilbert's 17th problem: given a multivariate polynomial that takes only non-negative values over the reals, can it be represented as a sum of squares of rational functions? The positive answer to this original question was given by Artin in 1926 [1]. For a survey of recent work on the hermitian case see the review paper by D'Angelo [3].

In this paper we give the following quantitative version of Theorem 1:

Theorem 2. *Let f satisfy the assumptions of Theorem 1 and define*

$$\lambda(f) := \min_{\|z\|=1} f(z,\bar{z}), \quad \Lambda(f) := \left(\sum_{|\alpha|=|\beta|=m} \left(\frac{\alpha!\beta!}{m!^2} \right) |c_{\alpha\beta}|^2 \right)^{1/2}. \tag{1.3}$$

Then there exists a universal constant C such that (1.2) holds for

$$N \geq C \frac{\Lambda(f)}{\lambda(f)} (m+n)^3 \log^3 n. \tag{1.4}$$

The proofs of Quillen [9] and Catlin-D'Angelo [2] are based on functional analytic methods related to the study of Toeplitz operators. The existence of N_0 such that (1.2) is satisfied is obtained by a non-constructive Fredholm compactness argument – see [7, Section 10] for outlines and comparisons of the two proofs, and also [4] for an elementary introduction to the subject.

Here we take a point of view based on the semiclassical study of Toeplitz operators – see [11, Chapter 13] and references given there. Our proof of Theorem 2 is a quantitative version of the proof of Theorem 1 given in [11, Section 13.5.4]: the compactness argument is replaced by an asymptotic argument with $N = 1/h$, where h is the semiclassical parameter. The symbol calculus for Toeplitz operators allows estimates in terms of h which then translate into a bound on N.

Better bounds on N obtained using purely algebraic methods already exist and it is an interesting question if such bounds can be obtained using semiclassical methods.

In the diagonal (real) case, $c_{\alpha\beta} = 0$ if $\alpha \neq \beta$, Theorem 1 is equivalent to a classical theorem of Pólya – see [7, Section 10.1]. In that case a sharp bound on N was given by Powers and Resnick [6]:

$$N > \frac{m(m-1)}{2} \frac{\widetilde{\Lambda}(f)}{\lambda(f)} - m, \quad \widetilde{\Lambda}(f) := \max_{|\alpha|=m} \left\{ \frac{\alpha!}{m!} |c_{\alpha\alpha}| \right\}. \tag{1.5}$$

It is remarkable that the bound does not depend on the dimension n. To compare this bound to the bound obtained using semiclassical methods, we note that in the diagonal case, the spectral radius used in Lemma (3.1) is given by $\widetilde{\Lambda}(f)$. Hence an easy modification of that lemma leads to the bound

$$N \gtrsim \frac{\tilde{\Lambda}(f)}{\lambda(f)} (n+m)^3 \log^3 n, \tag{1.6}$$

which is weaker than the bound (1.5) from [6], roughly by a factor of $m(1+n/m)^3$.

In the complex case, To-Yeung [10, Theorem 1] give an algebraic proof of a better bound than the one provided by our method in Theorem 2. They show that

$$N \geq nm(2m-1) \frac{\Lambda^\sharp(f)}{\log 2\, \lambda(f)} - n - m, \quad \Lambda^\sharp(f) := \sup_{|z|=1} |f(z,\bar{z})|.$$

The common feature of all these bounds is the denominator $\lambda(f)$ and the standard example of $|z_1|^4 + |z_2|^4 - c|z_1|^2|z_2|^2$, $0 < c < 2$, $z \in \mathbb{C}^2$ (see for instance [11, Section 13.5.4]) shows that the $1/\lambda(f)$ behaviour is optimal.

In Putinar's generalization of Pólya's theorem [8], a much larger bound was given by Nie and Schweighofer [5]:

$$N > c \exp\left(m^2 n^m \frac{\tilde{\Lambda(f)}}{\lambda(f)} \right)^c , \tag{1.7}$$

for some $c > 0$.

The paper is organized as follows. In Section 2 we recall various basic facts about the Bargmann-Fock space and Toeplitz quantization. Section 3 presents the basic inequality which leads to a bound on N. Section 4 provides quantitative estimates on the localization of homogeneous polynomials in the Bargmann-Fock space, with a stationary phase argument given in the appendix. The proof of Theorem 2 is then given in Section 5.

Notation. We denote $\langle x, y\rangle$ for $x, y \in \mathbb{C}^n$ the euclidean quadratic form on $\mathbb{C}^n$ (not the hermitian scalar product): $\langle x, y\rangle := \sum_{i=1}^n x_i y_i$. For $z = (z_1, ..., z_n) \in \mathbb{C}^n$ we define $\|z\|$ as the standard hermitian norm: $\|z\|^2 := \sum_{i=1}^n z_i \overline{z_i} = \langle z, \bar{z}\rangle$. The measure $dm(z)$ denotes the $2n$-dimensional Lebesgue measure on $\mathbb{C}^n$. The space of homogeneous polynomials of degree M is denoted $\mathcal{P}_M$. Finally, for two quantities A, B, we write $A \gtrsim B$, if there exist a (large, universal) constant C, such that $A \geq CB$.

Acknowledgments. We would like to thank Mihai Putinar for encouraging us to pursue this project and John D'Angelo for providing useful comments and and references, especially reference [10]. The partial support by the National Science Foundation for partial support under the grant DMS-1201417 is also gratefully acknowledged.

2. Preliminaries: Bargmann-Fock space and Toeplitz quantization

Quillen's original proof of Theorem 1 used the Bargmann-Fock space – see [7, Section 10],[9] and [11, Section 13.5.4]. We modify it slightly by introducing a semiclassical parameter h and considering the subpace of homogeneous polynomials of degree M, $\mathcal{P}_M$.

A Hilbert space *Bargmann-Fock* norm on $\mathcal{P}_M$ is given by

$$\|u\|^2_{\mathcal{P}_M} = \int_{\mathbb{C}^n} |u(z)|^2 e^{-\|z\|^2/h} dm\,(z)$$

and we can extend this norm to any function u such that

$$\int_{\mathbb{C}^n} |u\,(z, \bar{z})\,|^2 e^{-\|z\|^2/h} dm\,(z) < \infty.$$

We denote the resulting space by L^2_Φ. The closed subspace of holomorphic functions is denoted by H_Φ. The measure $\exp(-\|z\|^2/h)dm(z)$ will sometimes be written as $dG(z)$.

The Bergman projector Π_Φ, is the orthogonal projector $L^2_\Phi \to H_\Phi$ and to compute it we identify an orthonormal basis of H_Φ. The following standard lemma is a rephrasing of [11, Theorem 13.16]:

Lemma 2.1. *Let us define*

$$f_\alpha(z) := \frac{1}{(\pi h)^{n/2}} \left(\frac{1}{h^{|\alpha|}\alpha!} \right)^{1/2} z^\alpha. \tag{2.1}$$

Then

(i) *The set of f_α's is an othonormal basis on H_Φ.*
(ii) *The Bergman projector Π_Φ can be written*

$$\Pi_\Phi u(z) = \int_{\mathbb{C}^n} \Pi(z,w)\, u(w)\, dm(w)$$

where

$$\Pi(z,w) := \frac{1}{(\pi h)^n} \exp\left(\frac{1}{h} \left(\langle z, \overline{w} \rangle - |w|^2 \right) \right).$$

To connect the study of positive bi-homogeneous forms to Bargmann-Fock space, we recall the standard result (see [11, Lemma 13.17]):

Lemma 2.2. *A bi-homogeneous form of degree m can be written as a sum of squares of homogeneous polynomials,*

$$f(z,\bar{z}) = \sum_{j=1}^{k} |P_j(z)|^2, \quad P_j(z) = \sum_{|\alpha|=m} p^j_\alpha z^\alpha,$$

if and only if the matrix $(c_{\alpha\beta})_{|\alpha|=|\beta|=m}$ is positive semidefinite.

Thus to prove Theorem 1 we need to show that the matrix of the hermitian form $\langle z, \bar{z} \rangle^N f(z,\bar{z})$ is positive for N large enough. Let us compute this matrix. Since

$$\frac{\langle z, \bar{z} \rangle^N}{N!} = \sum_{|\mu|=N} \frac{z^\mu \bar{z}^\mu}{\mu!},$$

$$\langle z, \bar{z} \rangle^N f(z,\bar{z}) = \sum_{\substack{|\alpha|=|\beta|=m \\ |\mu|=N}} \frac{c_{\alpha\beta}}{\mu!} z^{\alpha+\mu} \bar{z}^{\beta+\mu} = \sum_{|\gamma|=|\rho|=m+N} c^N_{\gamma\rho} z^\gamma \bar{z}^\rho,$$

where

$$c^N_{\rho\gamma} = \sum_{\substack{\alpha+\mu=\rho \\ \beta+\mu=\gamma, |\mu|=N}} \frac{c_{\alpha\beta}}{\mu!}, \quad |\rho| = |\gamma| = N + m. \tag{2.2}$$

The following essential idea comes from the work of Quillen in [9]. It relates the positivity of the matrix (2.2) to the positivity of a differential operator.

Let P_f be the following differential operator

$$P_f = \sum_{|\alpha|=|\beta|=m} c_{\alpha\beta} z^{\alpha} \left(h\partial_z\right)^{\beta} : H_\Phi \longrightarrow H_\Phi. \tag{2.3}$$

Since f is real, $\overline{c_{\alpha\beta}} = c_{\beta\alpha}$. Thus the formula (2.5) shows that P_f is self adjoint. Let us explain now how the positivity condition and the operator P_f are related.

A simple calculation (see [11, Section 13.5.5]) based on the definition and (2.4) shows that for all $u, v \in \mathcal{P}_{m+N}$,

$$\langle P_f u, v\rangle_{\mathcal{P}_{m+N}} = \pi^n h^{n+N+2m} \sum_{|\gamma|=|\rho|=m+N} \rho!\gamma! c^N_{\rho\gamma} u_\gamma \overline{v}_\rho$$

where $u_\rho \in \mathbb{C}$, $v_\gamma \in \mathbb{C}$, are given in

$$u = \sum_{|\rho|=m+N} u_\rho z^\rho, \quad v = \sum_{|\gamma|=m+N} v_\gamma z^\gamma.$$

Thus proving that the matrix (2.2) is positive definite is equivalent to proving that P_f is a positive operator on $\mathcal{P}_{m+N}$. To make this quantitative we use the following lemma which is an application of a more general formula given in [11, Theorem 13.10]:

Lemma 2.3. *Let Π_Φ be the orthogonal projector from L^2_Φ to H_Φ. Then*

$$P_f|_{P_{m+N}} = \sum_{|\alpha|=|\beta|=m} c_{\alpha\beta} z^\alpha \Pi_\Phi \left(\bar{z}^\beta \cdot\right), \tag{2.4}$$

and

$$P_f|_{P_{m+N}} = \Pi_\Phi q\left(z, \bar{z}\right) \Pi_\Phi \tag{2.5}$$

where

$$q\left(z, \bar{z}\right) = \sum_{j=0}^{m} \frac{h^j}{j!} \left(-\tfrac{1}{4}\Delta\right)^j f\left(z, \bar{z}\right). \tag{2.6}$$

Using (2.5), positivity of P_f on $\mathcal{P}_{N+m}$ follows from inequality

$$\langle \Pi_\Phi q \Pi_\Phi u, u\rangle_{\mathcal{P}_{m+N}} \geq c\|u\|^2_{L^2_\Phi}, \quad u \in \mathcal{P}_{N+m},$$

for some constant $c > 0$. But since $\Pi_\Phi u = u$ and $\Pi_\Phi^* = \Pi_\Phi$, it suffices to prove that for all $u \in \mathcal{P}_{N+m}$, with L^2_Φ-norm equal to 1,

$$\langle q(z, \bar{z})u, u\rangle_{L^2_\Phi} \geq c, \tag{2.7}$$

and (2.7) is the starting point of our work.

3. The basic estimate

We define the ring Ω_ε as

$$\Omega_\varepsilon := \{z \in \mathbb{C}^n, 1-\varepsilon \le \|z\|^2 \le 1+\varepsilon\}.$$

For $u \in \mathcal{P}_{N+m}$ with L^2_Φ-norm equal to 1 we have

$$\begin{aligned}\langle qu, u\rangle_{L^2_\Phi} &= \int_{\mathbb{C}^n} q(z,\bar z)\,|u(z)|^2 e^{-\|z\|^2/h} dm(z)\\ &= \int_{\mathbb{C}^n\setminus\Omega_\varepsilon} q(z,\bar z)\,|u(z)|^2 e^{-\|z\|^2/h} dm(z) + \int_{\Omega_\varepsilon} q(z,\bar z)\,|u(z)|^2 e^{-\|z\|^2/h} dm(z)\\ &\ge \min_{\mathbb{C}^n\setminus\Omega_\varepsilon} q\left(\|u\|_{L^2_\Phi} - \|u\|^2_{L^2_\Phi(\Omega_\varepsilon)}\right) + \int_{\Omega_\varepsilon} q(z,\bar z)\,|u(z)|^2 e^{-\|z\|^2/h} dm(z)\\ &= \min_{\mathbb{C}^n\setminus\Omega_\varepsilon} q\left(1 - \|u\|^2_{L^2_\Phi(\Omega_\varepsilon)}\right) + \langle qu, u\rangle_{L^2_\Phi(\Omega_\varepsilon)}.\end{aligned}$$

Recalling (2.6) we see that

$$\begin{aligned}\langle qu, u\rangle_{L^2_\Phi(\Omega_\varepsilon)} &= \sum_{j=0}^m \frac{h^j}{j!}\int_{\Omega_\varepsilon} \frac{\left(-\frac14\Delta\right)^j f(z,\bar z)}{\|z\|^{2(m-j)}} |\|z\|^{m-j} u(z)|^2 e^{-\|z\|^2/h} dm(z)\\ &\ge -\sum_{j=0}^m \frac{h^j}{j!} \max_{\Omega_\varepsilon}\left(\frac{1}{\|z\|^{2(m-j)}}\left|\left(\tfrac14\Delta\right)^j f(z,\bar z)\right|\right) \|\|z\|^{m-j}u\|^2_{L^2_\Phi(\Omega_\varepsilon)} \qquad (3.1)\\ &\ge -\sum_{j=0}^m \frac{h^j}{j!} E_\varepsilon(h, m+N, m-j) \max_{\|z\|=1}\left|\left(\tfrac14\Delta\right)^j f(z,\bar z)\right|,\end{aligned}$$

where the quantity $E_\varepsilon(h,M,k)$ is defined as

$$E_\varepsilon(h,M,k) := \sup_{u\in\mathcal{P}_M, \|u\|_{\mathcal{P}_M}=1} \|\|z\|^k u\|_{L^2_\Phi(\Omega_\varepsilon)}, \qquad (3.2)$$

and where we used the homogeneity of $\Delta^j f$ of degree $2(m-j)$.

Rearranging the terms we obtain

$$\begin{aligned}\langle qu, u\rangle_{L^2_\Phi} \ge &\left(\left(1 - E_\varepsilon(h, m+N, 0)\right)\min_{\mathbb{C}^n\setminus\Omega_\varepsilon} q\right)\\ &- \sum_{j=0}^m \frac{h^j}{j!} E_\varepsilon(h, m+N, m-j)\max_{\|z\|=1}\left|\left(\tfrac14\Delta\right)^j f(z,\bar z)\right|.\end{aligned} \qquad (3.3)$$

Moreover,

$$\begin{aligned}\min_{1-2\varepsilon\le\|z\|^2\le 1+2\varepsilon} q(z,\bar z) &= \min_{1-2\varepsilon\le\|z\|^2\le 1+2\varepsilon} \sum_{j=0}^m \frac{h^j}{j!}\left(-\tfrac14\Delta\right)^j f(z,\bar z)\\ &\ge (1-2\varepsilon)^m \lambda(f) - \sum_{j=1}^m \frac{h^j}{j!}(1+2\varepsilon)^{m-j}\max_{\|z\|=1}\left|\left(\tfrac14\Delta\right)^j f(z,\bar z)\right|.\end{aligned}$$

We see that we need an upper bound for $\max_{\|z\|=1}\left|\left(\frac{1}{4}\Delta\right)^j f(z,\bar{z})\right|$ and that is given in the following

Lemma 3.1. *We have the estimate:*

$$\max_{\|z\|=1}\left|\left(\tfrac{1}{4}\Delta\right)^j f(z,\bar{z})\right| \leq \left(nm^2\right)^j \Lambda(f), \tag{3.4}$$

where $\Lambda(f)$ *is defined in* (1.3).

To explain the proof we note that since f is a bihomogeneous form of degree m, $\Delta^k f$ is a bihomogeneous form of degree $m-k$. If we have estimates on f, and if we find an explicit relation between estimates on f and Δf, related to the bound on $\max_{\|z\|=1}|f(z,\bar{z})|$, a recursion procedure will give (3.4)

Proof. For $z\in\mathbb{C}^n$ satisfying $\|z\|=1$, put $z=\left(r_1e^{i\theta_1},...,r_ne^{i\theta_n}\right)$, with $\sum r_i^2=1$. Then

$$\begin{aligned}
|f(z,\bar{z})| &\leq \sum_{|\alpha|=|\beta|=m}|c_{\alpha\beta}|r^\alpha r^\beta \leq \sum_{|\alpha|=|\beta|=m}\frac{\sqrt{\alpha!\beta!}}{m!}|c_{\alpha\beta}|\sqrt{\frac{m!}{\alpha!}}r^\alpha\sqrt{\frac{m!}{\beta!}}r^\beta \\
&= \langle\widetilde{C}R,R\rangle, \qquad \widetilde{C}:=\left(\frac{\sqrt{\alpha!\beta!}}{m!}|c_{\alpha\beta}|\right)_{|\alpha|=|\beta|=m}, \quad R:=\left(\sqrt{\frac{m!}{\alpha!}}r^\alpha\right)_{|\alpha|=m}.
\end{aligned}$$

Since

$$\|R\|^2=\sum_{|\alpha|=m}\frac{m!}{\alpha!}r^{2\alpha}=\|r\|^{2m}=1$$

we have

$$|f(z,\bar{z})|\leq\langle\widetilde{C}R,R\rangle\leq\rho(\widetilde{C}),$$

where $\rho(\widetilde{C})$ is the spectral radius of $\widetilde{C}$. The spectral radius can be estimated by $\Lambda(f)$ given in (1.3): we write $\widetilde{C}=UDU^{-1}$, where D and U are diagonal and orthogonal matrix, respectively. Then

$$\Lambda(f)^2=\operatorname{tr}\left(\widetilde{C}\widetilde{C}^*\right)=\operatorname{tr}\left(UD^2U^{-1}\right)=\operatorname{tr}\left(D^2\right)\geq\rho(\widetilde{C})^2$$

and hence

$$\max_{\|z\|=1}|f(z,\bar{z})|\leq\Lambda(f). \tag{3.5}$$

We now need to find a relation between $\Lambda(f)$ and $\Lambda\left(\frac{1}{4}\Delta f\right)$. Let $D:=(d_{\gamma\rho})$ be the matrix of the bi-homogeneous form $\frac{1}{4}\Delta f$, and let us chose γ,ρ with $|\gamma|=|\rho|=m-1$.

Denoting by $\tilde{c}_{\alpha\beta}$ the entries of $\widetilde{C}$ we obtain

$$d_{\gamma\rho} = \frac{1}{\gamma!\rho!}\frac{\partial^\gamma}{\partial z^\gamma}\frac{\partial^\rho}{\partial \bar{z}^\rho}\sum_{i=1}^n \frac{\partial}{\partial z^i}\frac{\partial}{\partial \bar{z}^i} f(0,0) = \sum_{i=0}^n (\gamma_i+1)(\rho_i+1)\, c_{\gamma+e_i,\rho+e_i} \tag{3.6}$$

$$= \sum_{i=1}^n (\gamma_i+1)(\rho_i+1)\frac{m!}{\sqrt{(\gamma+e_i)!(\rho+e_i)!}}\tilde{c}_{\gamma+e_i,\rho+e_i} \tag{3.7}$$

$$= \frac{m!}{\sqrt{\gamma!\rho!}}\sum_{i=1}^n \sqrt{(\gamma_i+1)(\rho_i+1)}\tilde{c}_{\gamma+e_i,\rho+e_i} \tag{3.8}$$

$$= \frac{(m-1)!}{\sqrt{\gamma!\rho!}} m^2 \sum_{i=1}^n \tilde{c}_{\gamma+e_i,\rho+e_i}. \tag{3.9}$$

If we put $\tilde{d}_{\gamma\rho} := \sqrt{\gamma!\rho!} d_{\gamma\rho}/(m-1)!$, and denote the correspoding matrix by $\widetilde{D}$, then

$$\tilde{d}_{\gamma\rho} \le m^2 \sum_{i=1}^n \tilde{c}_{\gamma+e_i,\rho+e_i}, \tag{3.10}$$

and

$$\Lambda\left(\tfrac{1}{4}\Delta f\right)^2 = \sum_{|\gamma|=|\rho|=m-1} d_{\gamma\rho}^2 \le \sum_{|\gamma|=|\rho|=m-1} m^4 \left(\sum_{i=1}^n \tilde{c}_{\gamma+e_i,\rho+e_i}\right)^2 \tag{3.11}$$

$$\le m^4 n \sum_{|\gamma|=|\rho|=m-1}\sum_{i=1}^n \tilde{c}^2_{\gamma+e_i,\rho+e_i} \tag{3.12}$$

$$\le nm^4 \cdot n\Lambda(f)^2. \tag{3.13}$$

An easy recursion then gives

$$\Lambda\left(\left(\tfrac{1}{4}\Delta\right)^j f\right) \le \left(nm^2\right)^j \Lambda(f).$$

and inequality (3.5) applied to $\left(\frac{1}{4}\Delta\right)^j f$ instead of f proves the lemma. □

The lemma and the lower bound stated after the inequality (3.3) imply

$$\min_{1-2\varepsilon\le\|z\|^2\le 1+2\varepsilon} q(z,\bar{z}) \ge \lambda(f)(1-2\varepsilon)^m - \Lambda(f)\sum_{j=1}^m \frac{1}{j!}\left(nm^2h\right)^j (1+2\varepsilon)^{m-j}. \tag{3.14}$$

This combined with (3.3) leads to the *basic inequality*:

$$\begin{aligned}
&\langle qu, u\rangle_{L^2_\Phi} \geq \\
&\quad (1 - E_\varepsilon(h, m+N, 0))\left(\lambda(f)(1-2\varepsilon)^m - \Lambda(f)\sum_{j=1}^{m}\frac{1}{j!}\left(nm^2h\right)^j(1+2\varepsilon)^{m-j}\right) \\
&\qquad - \Lambda(f)\sum_{j=0}^{m}\frac{1}{j!}\left(nm^2h\right)^j E_\varepsilon(h, m+N, m-j).
\end{aligned} \tag{3.15}$$

All the work that follows is aimed at finding h_0 such that for $h < h_0$ the right hand side of (3.15) is positive.

4. Estimates on E_ε

Our goal in this section is to prove that the quantity $E_\varepsilon(h, M, m)$ roughly decreases like $\exp(-M\varepsilon^2)$, under some assumptions relating ε, h, M, m, n. It is essentially due to the fact that the homogeneous polynomials are localised in L^2_Φ-norm around the sphere $S^{2n-1} \subset \mathbb{C}^n$, with $1/h \sim M$ – see [11, Theorem 13.16, (ii)] for an explanation of this using the harmonic oscillator. Here we prove

Lemma 4.1. *Let $\varepsilon, h, m, n, M > 0$ and let us call*

$$\sigma := h(M + m + n - 1). \tag{4.1}$$

Assume that

$$\frac{3}{2} > \sigma > 1, \quad 1 \geq \varepsilon \geq 4(\sigma - 1). \tag{4.2}$$

Then for E_ϵ defined in (3.2) we have

$$E_\varepsilon(h, M, m) \lesssim h^m(M + m + n)^{2n+m}\frac{1}{\varepsilon^2}\exp\left(-\frac{M\varepsilon^2}{16}\right). \tag{4.3}$$

Proof. Let Π_Φ^M be the projection from L^2_Φ to $\mathcal{P}_M$. For $u \in \mathcal{P}_M$. To estimate the right hand side in (3.2) we note that

$$\begin{aligned}
\|\|z\|^m u\|^2_{L^2_\Phi(\Omega_\varepsilon)} &= \langle u, \Pi_\Phi^M \|z\|^{2m}\mathbb{1}_{\Omega_\varepsilon}\Pi_\Phi^M u\rangle_{L^2_\Phi} \\
&\leq \|\Pi_\Phi^M\|z\|^{2m}\mathbb{1}_{\Omega_\varepsilon}\Pi_\Phi^M\|_{L^2_\Phi \to L^2_\Phi} \cdot \|u\|^2_{L^2_\Phi}
\end{aligned}$$

Hence it suffices to estimate the norm operator $\|\Pi_\Phi^M\|z\|^{2m}\mathbb{1}_{\Omega_\varepsilon}\Pi_\Phi^M\|$, and for that we will use the following standard variant of Schur's Lemma:

Lemma 4.2. *Let (X, μ) be a measure space, $K : L^2(X) \to L^2(X)$ a selfadjoint operator with kernel k, that is*

$$Ku(x) = \int_X k(x, y)u(y)d\mu(y).$$

Assume that there exists an almost everywhere positive function p on X and $\lambda > 0$ such that

$$\int_X |k(x, y)|p(y)d\mu(y) \leq \lambda p(x). \tag{4.4}$$

Then $\|K\| \le \lambda$.

To apply the lemma we first construct the kernel of the projector $\Pi_\Phi^M = \sum_{|\alpha|=M} f_\alpha f_\alpha^*$, where f_α was defined in (2.1), and f_α^* is the linear form $\langle f_\alpha, \cdot\rangle_{L^2_\Phi}$. Writing

$$\Pi_\Phi^M u(z) := \int_{\mathbb{C}^n} \Pi^M(z,w) u(w) e^{-\|w^2\|/h} dm(w),$$

we have

$$\begin{aligned}\Pi^M(z,w) &= \sum_{|\alpha|=M} f_\alpha(z)\overline{f_\alpha(w)} \\ &= \sum_{|\alpha|=M} \frac{1}{(\pi h)^n}\left(\frac{1}{h^M \alpha!}\right) z^\alpha \overline{w}^\alpha = \frac{1}{\pi^n h^{n+M}} \sum_{|\alpha|=M} \frac{1}{\alpha!} z^\alpha \overline{w}^\alpha = \frac{\langle z, \overline{w}\rangle^M}{M!\pi^n h^{n+M}}.\end{aligned}$$

It follows that the integral kernel of $K = \Pi_\Phi^M \|z\|^{2m} \mathbb{1}_{\Omega_\varepsilon} \Pi_\Phi^M$ with respect to the Gaussian measure $dG(z) := \exp(-\|z\|^2/h) dm(z)$, k, is given by

$$k(z,w) = \int_{\Omega_\varepsilon} \frac{\langle z, \overline{\zeta}\rangle^M}{M!\pi^n h^{n+M}} \frac{\langle \zeta, \overline{w}\rangle^M}{M!\pi^n h^{n+M}} \|\zeta\|^{2m} dG(\zeta).$$

This suggests natural choice of the weight $p = \|z\|^M$ in lemma 4.2, and we need to estimate the corresponding parameter λ in (4.4). For that, we need an upper bound on the integral

$$\int_{C^n} |k(z,w)| \|w\|^M dG(w).$$

An application of the Cauchy-Schwarz inequality inequality gives

$$\begin{aligned}\int_{C^n} |k(z,w)| \|w\|^M dG(w) &\le \int_{\mathbb{C}^n} \int_{\Omega_\varepsilon} \|w\|^M \frac{\|z\|^M \|\zeta\|^M}{M!\pi^n h^{n+M}} \frac{\|\zeta\|^M \|w\|^M}{M!\pi^n h^{n+M}} \|\zeta\|^{2m} dG(\zeta) dG(w) \\ &\le \|z\|^M \left(\int_{\mathbb{C}^n} \frac{\|w\|^{2M}}{M!\pi^n h^{n+M}} dG(w)\right) \left(\int_{\Omega_\varepsilon} \frac{\|\zeta\|^{2M+2m}}{M!\pi^n h^{n+M}} dG(\zeta)\right).\end{aligned}$$

Thus it is sufficient to estimate the following integrals:

$$I_1 = \int_{\mathbb{C}^n} \frac{\|w\|^{2M}}{M!\pi^n h^{n+M}} dG(w), \qquad I_2 = \int_{\Omega_\varepsilon} \frac{\|\zeta\|^{2M+2m}}{M!\pi^n h^{n+M}} dG(\zeta). \tag{4.5}$$

A polar coordinates change of variables, followed by a substitution $t = r^2/h$, gives

$$\begin{aligned}I_1 &= \frac{|S^{2n-1}|}{M!\pi^n h^{n+M}} \int_0^\infty r^{2M+2n-1} e^{-r^2/h} dr = \frac{|S^{2n-1}|}{2M!\pi^n h^{n+M}} h^{n+M} \int_0^\infty t^{M+n-1} e^{-t} dt \\ &= \frac{(M+n-1)!}{M!(n-1)!} \le \binom{M+n}{n} \le (M+n)^n,\end{aligned} \tag{4.6}$$

where $|S^{2n-1}| = 2\pi^n/(n-1)!$ denotes the volume of the $2n-1$ dimensional sphere.

Turning to I_2 in (4.5) we make two changes of variables, $z = r\theta$, then $r^2 = t$, so that

$$\begin{aligned} I_2 &= |S^{2n-1}| \int_{r^2 \notin [1\pm\varepsilon]} \frac{r^{2M+2m+2n-1}}{M!\pi^n h^{n+M}} \exp\left(-\frac{r^2}{h}\right) dr \\ &= \frac{|S^{2n-1}|}{2M!\pi^n h^{n+M}} \int_{t\notin[1\pm\varepsilon]} t^{M+m+n-1} e^{-t/h} dt. \end{aligned} \tag{4.7}$$

The last integral is very close to the integral appearing in the following lemma which will be proved in the appendix:

Lemma 4.3. *Let $\rho > 0, \delta < 1$. We define*

$$J(\rho,\delta) := \int_{t\notin[1-\delta,1+\delta]} t^\rho e^{-\rho t} dt.$$

Then

$$J(\rho,\delta) \lesssim \frac{1}{\rho\delta^2} \exp\left(-\rho\left(1+\frac{\delta^2}{4}\right)\right). \tag{4.8}$$

To apply this lemma to the last integral in (4.7) we make the change of variable $t/h = (M+m+n-1)s$. To assure that the interval of integration does not change much, we claim that under assumptions of Lemma 4.1 we have,

$$[1\pm\varepsilon/2] \subset \frac{1}{h(M+m+n-1)}[1\pm\varepsilon] = \frac{1}{\sigma}[1\pm\varepsilon]. \tag{4.9}$$

Indeed, (4.2) implies the following inequalities:

$$1-\frac{\varepsilon}{2} \geq \frac{1}{\sigma}(1-\varepsilon), \quad 1+\frac{\varepsilon}{2} \leq \frac{1}{\sigma}(1+\varepsilon). \tag{4.10}$$

The first one is straightforward, since it is equivalent to $2\sigma-2 \geq (\sigma-2)\varepsilon$, and $\sigma-2<0$. The second inequality in (4.10) is equivalent to $(2\sigma-2)/(2-\sigma) \leq \varepsilon$, so that in view of (4.2) we need to check that $(2\sigma-2)/(2-\sigma) \leq 4(\sigma-1)$ which follows from the assumption $\sigma < 3/2$.

Returning to (4.7) we have

$$\int_{t\notin[1\pm\varepsilon]} t^{n+m+M-1} e^{-t/h} dt \leq [h(M+m+n-1)]^{M+m+n} \int_{s\notin[1\pm\varepsilon/2]} (te^{-t})^{M+m+n-1} dt.$$

Applying Lemma 4.3 gives

$$\begin{aligned} \int_{t\notin[1\pm\varepsilon/2]} t^{n+m+M-1} e^{-t/h} dt &\lesssim \frac{[h(M+m+n-1)]^{M+m+n}}{(M+m+n-1)\varepsilon^2} e^{-(M+n+m-1)(1+\varepsilon^2/16)} \\ &\lesssim \frac{[h(M+m+n)]^{M+m+n}}{\varepsilon^2} e^{-(M+n+m)(1+\varepsilon^2/16)}. \end{aligned}$$

Hence

$$\begin{aligned} I_2 &\lesssim [h(M+m+n)]^{M+m+n} \frac{|S^{2n-1}|}{2\pi^n} \frac{e^{-M-m-n}}{h^M M!} \frac{1}{h^n\varepsilon^2} e^{-M\varepsilon^2/16} \\ &\lesssim h^m (M+m+n)^{M+m+n} \frac{e^{-M-n-m}}{M!(n-1)!} \frac{1}{\varepsilon^2} e^{-M\varepsilon^2/16} \end{aligned} \tag{4.11}$$

To simplify the upper bound in (4.11) we first use Stirling's formula to obtain (with a small irrelevant loss since $k^k \lesssim k! e^k/\sqrt{k}$)

$$(M+m+n)^{M+m+n} \lesssim (M+m+n)! e^{M+m+n}.$$

Thus the bound in (4.11) can be replaced by

$$I_2 \lesssim h^m \frac{(M+m+n)!}{M!} \frac{1}{\varepsilon^2} e^{-M\epsilon^2/16} \lesssim h^m (M+m+n)^{m+n} \frac{1}{\varepsilon^2} e^{-M\epsilon^2/16}.$$

Combining this with the bound (4.6), and applying Lemma 4.2 gives

$$\begin{aligned}\|K\| &\lesssim h^m (M+n)^n (M+m+n)^{m+n} \frac{1}{\varepsilon^2} e^{-h^{-1/3}/16} \\ &\lesssim h^m (M+n+m)^{2n+m} \frac{1}{\varepsilon^2} e^{-h^{-1/3}/16}.\end{aligned}$$

This completes the proof of Lemma 4.1. □

5. Proof of Theorem 2

We now combine the basic inequality (3.15) with the estimate on E_ε given in Lemma 4.1. We split (3.15) into four terms:

(*i*) $A_0 = \lambda(f)(1-2\varepsilon)^m$ which is the leading term;
(*ii*) $A_1 = \lambda(f) E_\varepsilon(h, m+N, 0)(1-2\varepsilon)^m$ decreases exponentially to 0 as $h \to 0$;
(*iii*) $A_2 = \Lambda(f) \sum_{j=1}^{m} \frac{1}{j!} \left(nm^2 h\right)^j (1+2\varepsilon)^{m-j}$ will be estimated by noting that it is dominated by its first term;
(*iv*) $A_3 = \Lambda(f) \sum_{j=0}^{m} \frac{1}{j!} \left(nm^2 h\right)^j E_\varepsilon(h, m+N, m-j)$ will require more care but decreases exponentially to 0 as $h \to 0$.

We want to optimize the parameters h, M, ε as functions of the order of f, m, and the dimension n. We aim to show that $A_0 \gg A_1, A_2, A_3$, using Lemma 4.1. For this we need to check that the assumption (4.2) is satisfied.

The basic strategy is outlined as follows

- (4.2) is satisfied if for all $0 \le j \le m$, $h^{-1} \sim N + 2m + n - j$ and $h(N + 2m + n - j) \le 1$. Thus we need $h^{-1} \sim N \gg m, n$.
- $A_0 \gtrsim A_1$: we want to apply Lemma 4.1 and thus we need $\varepsilon^2/h \ge -n \log(h)$;
- $A_0 \gtrsim A_2$: for this to hold A_2 has to be greater than the first term of the sum in A_2, $nm^2(1+2\varepsilon)^{m-1}h$; thus the term $(1+2\varepsilon)^m$ has to remain bounded as $m \to \infty$: we need $\varepsilon \lesssim 1/m$.
- $A_0 \gtrsim A_3$: the term A_0 has – at least – to be greater than the first term of the sum in A_3 ; thus we need to have $A_0 \gtrsim E_\varepsilon(h, N+m, m)$; using Lemma 4.1, this holds when $\varepsilon^2/h \ge -(n+m)\log(h)$.

We define ε as $\varepsilon = h^a$, where a will be determined. From the considerations above we need

$$h^{2a-1} \gtrsim (n+m)\log\frac{1}{h} \quad \text{and} \quad h^a \lesssim 1/m.$$

To express this as one condition, we demand $a = 1-2a$, that is, $a = 1/3$. This leads to the necessary relations:

$$\varepsilon = h^{-1/3}/16, \quad h \lesssim (n+m)^{-3}, \quad N = h^{-1}. \tag{5.1}$$

Application of the estimates on E_ε. To use estimates on $E_\varepsilon(h, m+N, m-j)$ for $0 \le j \le m$ we need the assumption (4.2) to hold. That means that

$$1 < h(N+2m+n-j-1) < \frac{3}{2}, \quad 1 \ge \varepsilon \ge 4(h(N+2m-j+n)-1). \tag{5.2}$$

Since $N = 1/h$, both inequalities are satisfied for all $0 \le j \le m$ if they are satisfied for $j = 0$. Recallingl that $\varepsilon = h^{-1/3}/16 \le 1$, this in turn follows from

$$4h(2m+n) \le \varepsilon, \qquad h(2m+n) \le \frac{1}{2}. \tag{5.3}$$

If $h \le \frac{1}{64}(m+n)^{-3}$, then

$$\frac{8\delta}{(m+n)^2} \le \frac{\delta^{1/3}}{(m+n)},$$

which implies (5.3). We conclude that (5.2) holds, hence also (4.2), and hence we can apply Lemma 4.1 to $E_\varepsilon(h, m+N, m-j)$, $0 \le j \le m$.

Final estimate on h. We first start by simplifying A_1. Lemma 4.1 shows that

$$E_\varepsilon\left(h, m+N, 0\right) \lesssim (n+m+N)^{2n}\varepsilon^{-2}e^{-N\varepsilon^2/16} \lesssim t(3/h)^{2n+1}e^{-h^{-1/3}/16}.$$

Thus

$$A_1 \lesssim \lambda(f)(1-2\varepsilon)^m(3/h)^{2n+1}e^{-h^{-1/3}/16}. \tag{5.4}$$

To treat A_2 we note that

$$A_2 = \Lambda\left(f\right)\sum_{j=1}^{m}\frac{\left(nm^2h\right)^j}{j!}\left(1+2\varepsilon\right)^{m-j} \le \Lambda\left(f\right)\left(1+2\varepsilon\right)^m\left(e^{nm^2h}-1\right).$$

But since $h \le (n+m)^{-3}$, $nm^2h \le 1$, and thus $\exp\left(nm^2h\right) - 1 \lesssim nm^2h$, and

$$A_2 \lesssim \Lambda\left(f\right)\left(1+2\varepsilon\right)^m nm^2h. \tag{5.5}$$

We finally treat A_3. For that, we need the estimate on $E_\varepsilon\left(h, m+N, m-j\right)$ proved in Lemma 4.1:

$$\begin{aligned} E_\varepsilon\left(h, m+N, m-j\right) &\lesssim h^{m-j}(N+n+2m-j)^{2n+m-j}\varepsilon^{-2}e^{-(m+N)\varepsilon^2/16} \\ &\lesssim (3N)^{2n}\varepsilon^{-2}(3hN)^{m-j}e^{-(m+N)\varepsilon^2/16} \\ &\lesssim (3/h)^{2n+1}\,3^m e^{-h^{-1/3}/16}. \end{aligned}$$

Inserting this in the definition of A_3,

$$A_3 = \Lambda(f) \sum_{j=0}^{m} \frac{(nm^2h)^j}{j!} E_\varepsilon(h, m+N, m-j).$$

this gives

$$\begin{aligned} A_3 &\lesssim \Lambda(f) \sum_{j=0}^{m} \frac{(nm^2h)^j}{j!} (3/h)^{2n+1} 3^m e^{-h^{-1/3}/16} \\ &\lesssim \Lambda(f) (3/h)^{2n+1} 3^m e^{-h^{-1/3}/16}. \end{aligned}$$

Here we used again $nm^2h \le 1$. Thus we get:

$$A_3 \lesssim 3^m (3/h)^{2n+1} e^{-h^{-1/3}/16}. \tag{5.6}$$

We recall that we are looking for h_0 such that for $h < h_0$,

$$\lambda(f)(1-2\varepsilon)^m \ge A_1 + A_2 + A_3 \tag{5.7}$$

is satisfied. In view of (5.4), (5.5), (5.6), to obtain (5.7) it is sufficient to have

$$\lambda(f)(1-2\varepsilon)^m \ge 3\lambda(f)(1-2\varepsilon)^m (3/h)^{2n+1} e^{-h^{-1/3}/16}, \tag{5.8}$$

$$\lambda(f)(1-2\varepsilon)^m \ge 3\Lambda(f)(1+2\varepsilon)^m nm^2h, \tag{5.9}$$

$$\lambda(f)(1-2\varepsilon)^m \ge 3\Lambda(f)3^m (3/h)^{2n+1} e^{-h^{-1/3}/16}. \tag{5.10}$$

Since $h \le \delta = 1/64$, $\varepsilon \le 1/4$ and then $(1-2\varepsilon)^m \ge 10^{-m}$; moreover

$$\left(\frac{1+2\varepsilon}{1-2\varepsilon}\right)^m \le (1+8\varepsilon)^m \le \left(1+\frac{8}{m}\right)^m \lesssim 1.$$

Thus (5.9), (5.10) can be changed in

$$\lambda(f) \ge 3\Lambda(f) nm^2h, \tag{5.11}$$

$$\lambda(f) \ge 3\Lambda(f)30^m (3/h)^{2n+1} e^{-h^{-1/3}/16}. \tag{5.12}$$

Since $\lambda(f) \le \Lambda(f)$, (5.8) and (5.12) are both implied by

$$\frac{\lambda(f)}{\Lambda(f)} \ge 3 \cdot 30^m (30/h)^{2n+1} e^{-h^{-1/3}/16}. \tag{5.13}$$

The logarithmic version of this inequality is

$$\log\left(\frac{\lambda(f)}{\Lambda(f)}\right) \ge \log(3) + (m+2n+1)\log(30) - (2n+1)\log(h) - h^{-1/3}/16,$$

and thus taking $h \lesssim \log(\Lambda(f)/\lambda(f))^{-3} (m+n)^{-3} \log(n)^{-3}$ assures its validity. Indeed,

$$h \lesssim \log\left(\frac{\Lambda(f)}{\lambda(f)}\right)^{-3} \implies \log\left(\frac{\lambda(f)}{\Lambda(f)}\right) \gtrsim -h^{-1/3} \tag{5.14}$$

and

$$h \lesssim (n\log(n))^{-3} \Rightarrow n\log(h) \gtrsim -h^{-1/3}. \tag{5.15}$$

The estimate (5.11) is straightforward: we need

$$h \lesssim \frac{\lambda(f)}{\Lambda(f)} n^{-1} m^{-2}. \tag{5.16}$$

Let us chose

$$h \lesssim \min\left(\frac{\lambda(f)}{\Lambda(f)}, \log\left(\frac{\Lambda(f)}{\lambda(f)}\right)^{-3}\right)(m+n)^{-3}\log(n)^{-3}.$$

Then h satisfies the three necessary conditions for Theorem 2 to hold: (5.14), (5.15), and (5.16). The bound on $N = 1/h$ is then given by

$$N \gtrsim \max\left(\log\left(\frac{\Lambda(f)}{\lambda(f)}\right)^3, \frac{\Lambda(f)}{\lambda(f)}\right)(m+n)^3\log^3 n$$

which is the same as

$$N \gtrsim \frac{\Lambda(f)}{\lambda(f)}(m+n)^3\log^3 n.$$

Appendix: A non-stationary phase lemma

We prove Lemma 4.3. Let $\varphi(t) = -\log(t) + t$. Then φ is a one to one mapping on $(0,1]$ and on $[1,\infty)$. Let us then consider the following integrals:

$$J^-(\rho,\delta) = \int_0^{1-\delta} e^{\rho(\log(t)-t)}dt, \quad J^+(\rho,\delta) = \int_{1+\delta}^{\infty} e^{\rho(\log(t)-t)}dt.$$

The change of variable $\varphi(t) = x$ gives

$$J^-(\rho,\delta) = \int_{c^-}^{\infty} e^{-\rho x}\left(\frac{1}{\varphi^{-1}(x)} - 1\right)^{-1} dx,$$

with $c^- = \varphi(1-\delta)$. Thus we need estimates on $\varphi^{-1}(x)$. But on $(0, 1-\delta]$, we have $\varphi(t) \leq 1 - \delta - \log(t)$. It implies $\varphi^{-1}(x) \leq e^{1-\delta-x}$. This gives

$$J^-(\rho,\delta) \leq \int_{c^-}^{\infty} \frac{e^{-\rho x}}{e^{x-1+\delta} - 1} dx.$$

A lower bound for $e^{x-1+\delta} - 1$ is

$$e^{x-1+\delta} - 1 \geq \left(e^{-1+\delta} - e^{-c^-}\right)e^x \geq \delta e^{-1+\delta+x}.$$

and hence

$$\begin{aligned} J^-(\rho,\delta) &\leq \int_{c^-}^{\infty} \frac{e^{1-\delta}}{\delta} e^{-(\rho+1)x} dx = \frac{1-\delta}{\delta(\rho+1)}\left((1-\delta)e^{-1+\delta}\right)^{\rho} \\ &\leq \frac{1}{\rho\delta}\left((1-\delta)e^{-1+\delta}\right)^{\rho}. \end{aligned} \tag{A.1}$$

The same change of variable applied to J^+ gives

$$J^+(\rho,\delta) = \int_{c^+}^{\infty} e^{-\rho x}\left(1 - \frac{1}{\varphi^{-1}(x)}\right)^{-1} dx$$

with $c^+ = \varphi(1+\delta)$. On $(1+\delta,\infty)$, we have $\varphi(t) \le t$ and then $\varphi^{-1}(x) \ge x$.

$$J^+(\rho,\delta) \le \int_{c^+}^{\infty} e^{-\rho x}\left(1 - \frac{1}{x}\right)^{-1} dx \le= \frac{c^+}{c^+ - 1}\int_{c^+}^{\infty} e^{-\rho x} dx.$$

Since $\delta < 1$,

$$\frac{c^+}{c^+ - 1} = \frac{\varphi(1+\delta)}{\varphi(1+\delta) - 1} \lesssim \frac{1}{\delta^2}$$

and thus

$$J^+(\rho,\delta) \lesssim \frac{1}{\rho\delta^2}\left((1+\delta)\, e^{-1-\delta}\right)^{\rho}. \tag{A.2}$$

Now,

$$(1-\delta)\, e^{-1+\delta} \le (1+\delta)\, e^{-1-\delta}, \quad \delta^2 \le \delta,$$

and hence the estimates (A.1) and (A.2) give

$$J(\rho,\delta) = J_-(\rho,\delta) + J_+(\rho,\delta) \lesssim \frac{1}{\rho\delta^2}\left((1+\delta)\, e^{-1-\delta}\right)^{\rho}.$$

Also,

$$(1+\delta)\, e^{-\delta} \le e^{-\delta^2/4},$$

so that finally

$$J(\rho,\delta) \lesssim \frac{1}{\rho\delta^2}\exp\left(-\rho\left(1 + \frac{\delta^2}{4}\right)\right).$$

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